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HOPF BIFURCATION IN MODELS FOR MICRO – AND MACROPARASITICAL DISEASES

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Abstract. The purpose of this article is to establish, via Hopf bifurcation, the occurrence of attracting periodical orbits, in two models for microparasitical and macroparasitical diseases, due to Diekmann and Krezschmar, and described by systems of two ordinary differential equations. This proof is achieved by making use, after a suitable change of coordinates, of a result due to Marsden and McCracken.

Mathematical Subject Classification: 92B05
Keywords: Biomathematics, infectious diseases, host-parasite, S-I system, periodic orbit, Hopf bifurcation.

1. **Introduction**

In this work we will establish the occurrence of attracting periodical orbits, via Hopf bifurcation, in two epidemiological models introduced in [1]; one of them for microparasitical diseases of susceptible-infected S-I and the other for macroparasitical diseases of the host-parasite N-P type.

The first model is described by the system

$$\begin{cases} \dfrac{dS}{dt} = \beta\dfrac{S^2+2\xi IS+\xi^2 IS}{I+S} - \mu S - \dfrac{KI}{c+S+I}S \\ \dfrac{dI}{dt} = \dfrac{KI}{c+S+I}S - \mu I - \alpha I. \end{cases} \tag{1.1}$$

and the macroparasitical model is represented by

$$\begin{cases} \dfrac{dN}{dt} = -\mu N - \alpha P + \beta N\Big(\dfrac{kN}{kN+P(1-\xi)}\Big)^k \\ \dfrac{dP}{dt} = -(\mu+\sigma)P + \dfrac{KPN}{c+N} - \alpha N\Big(\dfrac{P}{N} + \Big(\dfrac{P}{N}\Big)^2\Big(\dfrac{k+1}{k}\Big)\Big). \end{cases} \tag{1.2}$$

where in the model (1.1) (model (1.2)), the parameters $\alpha, \beta, \xi, \mu, \sigma, c, k, K$ stand for :
β: per capita natural birth rate (of hosts)
μ: per capita natural death rate (of hosts), $\mu < \beta$
α: additional mortality rate due to disease (by one parasite)
ξ: parameter describing the reduction of fertility of an infected individual (host) due to the disease (to one parasite), $0 \leq \xi \leq 1$
K: contact rate between infectives and susceptibles (between hosts and infective stages of the parasites, such as eggs, cysts, spores, chrysalis, parasites)
σ: death rate of parasites
k: "clumping" parameter (a small k indicates high clumping, i.e., few hosts carry a large part of the parasites , while a large part of the hosts have very few parasites; for $k \to \infty$, the parasites are randomly distributed over the host population).

The change of variables $x = \dfrac{1}{I+S}$ and $y = \dfrac{I}{I+S}$, transforms (1.1) into

$$\begin{cases} \dfrac{dx}{dt} = x\Big(\mu + \alpha y - \beta(1-(1-\xi)y)^2\Big) \\ \dfrac{dy}{dt} = y\Big(\Big(\dfrac{K}{cx+1} - \alpha\Big)(1-y) - \beta(1-(1-\xi)y)^2 \Big) \end{cases} \tag{1.3}$$

and $x = \dfrac{1}{N}$, $y = \dfrac{P}{N}$, transforms (1.2) into

$$\begin{cases} \dfrac{dx}{dt} = x\Big(\mu + \alpha y - \beta\Big(\dfrac{k}{(1-\xi)y+k}\Big)^k\Big) \\ \dfrac{dy}{dt} = y\left(\dfrac{K}{cx+1} - (\sigma+\alpha) - \dfrac{\alpha}{k}y - \beta\dfrac{k}{(1-\xi)y+k}^{k}\right) \end{cases} \tag{1.4}$$

Each one of these models will be treated with unified notation:

$$\begin{cases} \dot{x} = xF(\xi, y) \\ \dot{y} = yG(K, \xi, x, y) \end{cases}$$

in the corresponding region $M = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ 0 \leq y < b\}$, where b = 1 or ∞ according to the model being (1.3) or (1.4).

The following proposition of [1] synthesizes common properties of both systems and plays an important role in the determination and classification of critical points in both cases. It also ensures the existence of at most one critical point $(\overline{x}, \overline{y})$ in the interior of M.

Proposition 1: *Consider each one of systems (1.3) and (1.4) in its respective domain M with $0 \leq \xi \leq 1$ and $0 < \mu < \beta$. Suppose further that for system (1.4), $\beta < \alpha + \mu$. Then,*

a) $\dfrac{\partial F}{\partial y} > 0$ *and there is a unique* $\overline{y} = \overline{y}(\xi)$ *in* $[0, b[$ *rendering* $F(\xi, \overline{y}(\xi)) = 0$*;*

b) $\dfrac{\partial G}{\partial x} < 0$ *and there is a unique* $x = g(K, \xi, y)$ *rendering* $G(K,\xi,g(K,\xi,y),y)=0$*;*

c) $\dfrac{\partial G}{\partial K} > 0$ *and* $G(0, \xi, x, y) < 0$ *for all* $(x, y) \in M$.

In Section 2 we will give the preliminary results needed for the study of each one of the systems (1.3) and (1.4) and also introduce the fundamental parameters $\xi = \xi_p$ and $K = K_t(\xi_p)$.

Further, we present some phase portrait simulations , obtained by means of the software MAPLE V, whereby the occurrence of Hopf bifurcation becomes apparent.

In Section 3, guided by a result of [4], we prove the theorem below, which is also stated in [1].

Theorem: *Consider the parameter restrictions* $0 < \mu < \beta, 0 < \xi < 1$ *on systems (1.3), (1.4) and the additional restriction on (1.4):* $\beta < \alpha + \mu$, $0 < \alpha < k\beta$. *Under the assumption that a value* K, $K > K_t(\xi_p)$, *has been fixed it follows that system (1.3) admits a supercritical Hopf bifurcation at the critical point inside* M, *with respect to the parameter* ξ *at the value* ξ_p. *Otherwise, there exists a value* μ_0, $0 < \mu_0 < \beta$, *such that occurs the same for system (1.4) if* $\mu > \mu_0$.

As a consequence we have a Hopf bifurcation for each one of the systems (1.1) and (1.2). Thus, for each value of the parameter ξ in a certain region , there is one periodic (hyperbolic) attracting orbit and, by results of Kooij and Zegeling (see [2], [3]), there are no other periodic orbits.

2. The parameters ξ_p and $K_t(\xi_p)$

Now we consider models (1.3) and (1.4) in their corresponding domains M, with the restrictions on the parameters as in the assumptions of the foregoing theorem.

In this article our attention will be focused solely upon critical points, which are solutions of the system:

$$\begin{cases} F(\xi, y) = 0\,, \\ G(K, \xi, x, y) = 0\,. \end{cases}$$

Given ξ, $0 < \xi < 1$, consider

$$\overline{y}(\xi) = \frac{2(1-\xi)\beta + \alpha - \sqrt{(2(1-\xi)\beta + \alpha)^2 + 4(1-\xi)^2\beta(\mu - \beta)}}{2(1-\xi)^2\beta}$$

the unique value for y in $]0, 1[$ as to render $F(\xi, \overline{y}(\xi)) = 0$.

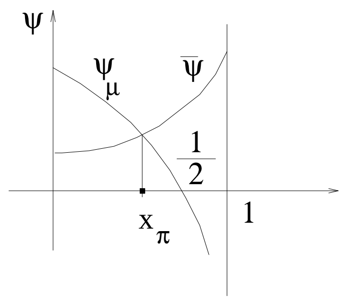
The points $(x, y) \in M$ such that $G(K, \xi, x, y) = 0$ are given by

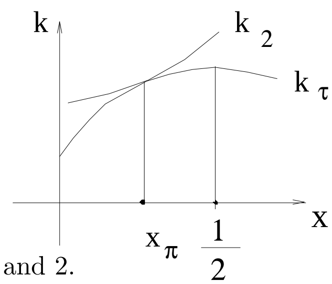
$$x = g(K, \xi, y) = -\frac{1}{c}\left(1 - \frac{K(1-y)}{\beta(1-(1-\xi)y)^2 + \alpha(1-y)}\right).$$

The critical points are obtained by intercepting the straight line $y = \overline{y}(\xi)$ and the graph of $x = g(K, \xi, y)$.

Let ξ_p denote the (unique) value of the parameter ξ for which $\overline{y}(\xi) = y_m(\xi)$, where $y_m(\xi) = 2 - \dfrac{1}{1-\xi}$ is the point of maximum of $x = g(K, \xi, y)$ (see Figure 1): For each fixed ξ, the following values of the parameter K are relevant: $K_t(\xi) = \alpha + 4\beta\xi(1-\xi)$

for which the graph of $x = g(K, \xi, y)$ is tangent to the y-axis and $K_2(\xi)$ which renders





Figures 1 and 2.

$g(K, \xi, \overline{y}(\xi)) = 0$. Since $K_2(\xi)$ is determined by

$$\begin{cases} \beta(1-(1-\xi)\overline{y}(\xi))^2 + (\alpha - K)(1-\overline{y}(\xi)) = 0 \\ \beta(1-(1-\xi)\overline{y}(\xi))^2 - \alpha\overline{y}(\xi) - \mu = 0 \end{cases}$$

and $\overline{y}'(\xi) > 0$, it follows that $K_2(\xi)$ is derivable and increasing. Further, $K_2(\xi) > K_t(\xi)$ for $\xi \neq \xi_p$, provided that the function $x = g(K, \xi, y)$ is increasing with respect to K, as long as $y < 1$ (see Figure 2).

Thus we may state that for any $K, K > K_t(\xi_p) = K_2(\xi_p)$, there is a neighborhood of ξ_p where $K > K_2(\xi) \geq K_t(\xi)$; therefore, for such values of ξ there is one and only one critical point $(\overline{x}, \overline{y}) = (\overline{x}(\xi), \overline{y}(\xi))$ in the interior of M. Here we denote $\overline{x}(\xi) = g(K, \xi, \overline{y}(\xi))$ (see Figure 6, where, for fixed ξ, near ξ_p the graphs of $x = g(K, \xi, y)$ and $y = \overline{y}(\xi)$ are displayed).

Next we present some simulations of system (1.3), generated by the software MAPLE V. This suggests the existence of Hopf bifurcations.

Let us now regard system (4)

Given ξ between 0 and 1, let $\overline{y}(\xi)$ be the unique value of y in $]0, \infty[$ satisfying $F(\xi, \overline{y}(\xi)) = 0$, that is, $\alpha\overline{y} + \mu = \beta\Big(\dfrac{k}{(1-\xi)\overline{y} + k}\Big)^k$ (see Figure 7). Deriving this equation implicitly, we obtain $\overline{y}'(\xi) > 0$.

The points $(x, y) \in M$ satisfying $G(K, \xi, x, y) = 0$ are:

$$x = g(K, \xi, y) = -\frac{1}{c}\Big(1 - \frac{K}{\phi(\xi, y)}\Big)$$

where $\phi(\xi, y) = \sigma + \alpha + \dfrac{\alpha y}{k} + \beta\,\Big(\dfrac{k}{(1-\xi)y + k}\Big)^k$.

Let $y_m(\xi) = \dfrac{k}{1-\xi}\,\Big[\Big(\dfrac{k\beta(1-\xi)}{\alpha}\Big)^{\frac{1}{k+1}} - 1\Big]$ be the point of minimum of $\phi(\xi, y)$, that is, the point of maximum of $g(K, \xi, y)$. We have $y_m(0) > 0$, since $\alpha < k\beta$, and $y_m(\xi)$ being independent on μ.

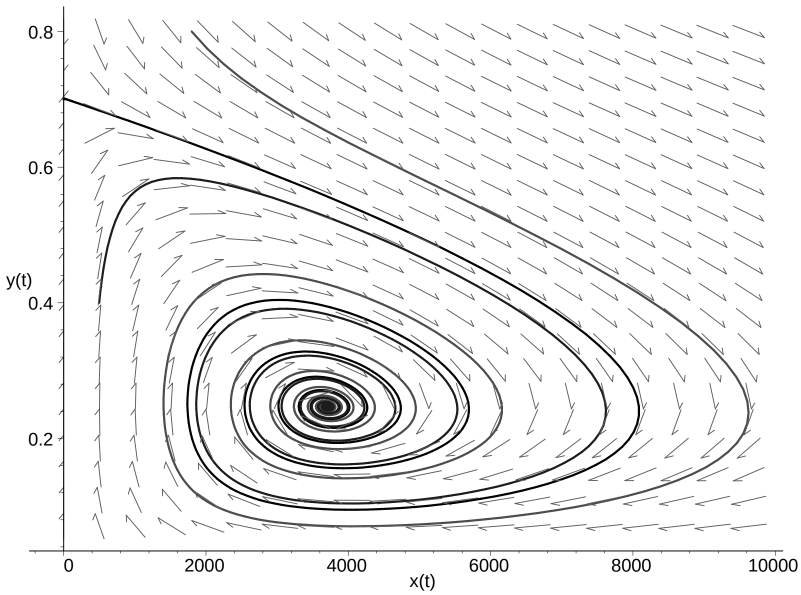


Figure 3. Phase portrait of system (3) with $\mu = 1.2$, $\alpha = 1.0$, $\beta = 1.8$, $\xi = 0.58$, $c = 0.0001$, $\kappa = 4.0$, $\kappa_t = 2.75$, $\kappa_1 = 2.8$, $\kappa_2 = 3.03$

There exists $\mu_0,\ 0 < \mu_0 < \beta$, with $\mu_0 < \mu < \beta \Rightarrow 0 < \overline{y}(0) < y_m(0)$. In fact, if $\alpha y_m(0) \geq \beta\Big(\dfrac{k}{y_m(0)+k}\Big)^k$, $\mu_0 = 0$; otherwise, $\mu_0 = \beta\Big(\dfrac{k}{y_m(0)+k}\Big)^k - \alpha y_m(0)$ (see Figure 8).

In this case there is a unique $\xi = \xi_p \in \Big]0, 1 - \dfrac{\alpha}{k\beta}\Big[$ with $\overline{y}(\xi_p) = y_m(\xi_p)$ (see Figure 9).

For each fixed ξ let $K_t(\xi)$ be the value of K for which the graph of $x = g(K, \xi, y)$ is tangent to the axis y and $K_2(\xi)$ the value giving $g(K, \xi, \overline{y}(\xi)) = 0$. We have, $K_t(\xi) = \phi(\xi, y_m(\xi)) \ \Rightarrow\ K_t'(\xi) = \phi_\xi + \phi_y y' = \phi_\xi = \beta y_m(\xi)\Big(\dfrac{k}{(1-\xi)y_m + k}\Big)^{k+1} > 0$.

Furthermore, $K_2(\xi)$ is derivable and increasing, since it is determined by the system

$$\begin{cases} FK - \sigma - \alpha - \dfrac{\alpha y}{k} - \beta\Big(\dfrac{k}{(1-\xi)y+k}\Big)^k = 0 \\ \alpha y + \mu - \beta\Big(\dfrac{k}{(1-\xi)y+k}\Big)^k = 0 \end{cases}$$

and $\overline{y}'(\xi) > 0$ holds.

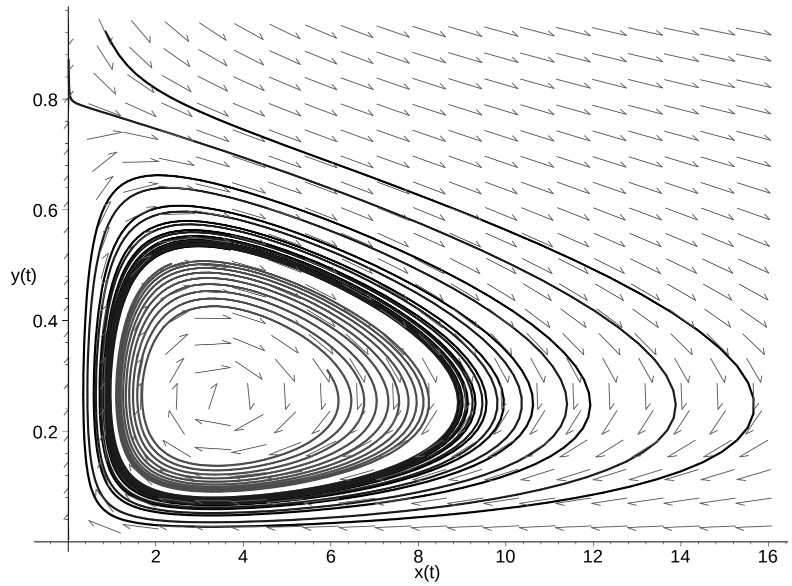


Figure 4. Phase portrait of system (3) with $\mu = 1.2$, $\alpha = 1.0$, $\beta = 1.8$, $\xi = 0.44$, $c = 0.1$, $\kappa = 4$, $\kappa_t = 2.77$, $\kappa_1 = 3.0$, $\kappa_2 = 2.9$

Proceeding in a similar way to what was done in the study of model (3), we get: $K_2(\xi) > K_t(\xi)$ for $\xi \neq \xi_p$ and for K being fixed, $K > K_t(\xi_p)$, we see that there is a neighbourhood of ξ_p where $K > K_2(\xi) > K_t(\xi)$ for $\xi \neq \xi_p$ (see Figure 10).

In Figure 11 we display the graph of $x = g(K, \xi, y)$ and $y = \overline{y}(\xi)$, according to the variation of K. Here the value of ξ is kept fixed and close to ξ_p.

3. **The Hopf Bifurcation. Proof of the Theorem**

The Jacobian matrix at the critical point under consideration has for both systems the expression

$$J(\overline{x}, \overline{y}) = \begin{bmatrix} 0 & \overline{x}F_y(\xi, \overline{y}) \\ \overline{y}G_x(K, \xi, \overline{x}, \overline{y}) & \overline{y}G_y(K, \xi, \overline{x}, \overline{y}) \end{bmatrix}, \text{ with } \det J(\overline{x}, \overline{y}) \neq 0.$$

We have also trace $J(\overline{x}, \overline{y}) = -\overline{y}G_x(K, \xi, \overline{x}, \overline{y})g_y(K, \xi, \overline{y})$ since $G(K, \xi, g(K, \xi, y), y), y) = 0 \Rightarrow \; G_x g_y + G_y = 0$.

Thus, for a fixed $K, K > K_t(\xi_p)$, and ξ sufficiently close to ξ_p:
$\xi < \xi_p \Rightarrow g(K, \xi, \overline{y}(\xi)) > 0 \Rightarrow trJ(\overline{x}, \overline{y}) > 0 \Rightarrow (\overline{x}, \overline{y})$ is an unstable spiral point,
$\xi > \xi_p \Rightarrow g(K, \xi, \overline{y}(\xi)) < 0 \Rightarrow trJ(\overline{x}, \overline{y}) < 0 \Rightarrow (\overline{x}, \overline{y})$ is a stable spiral point.

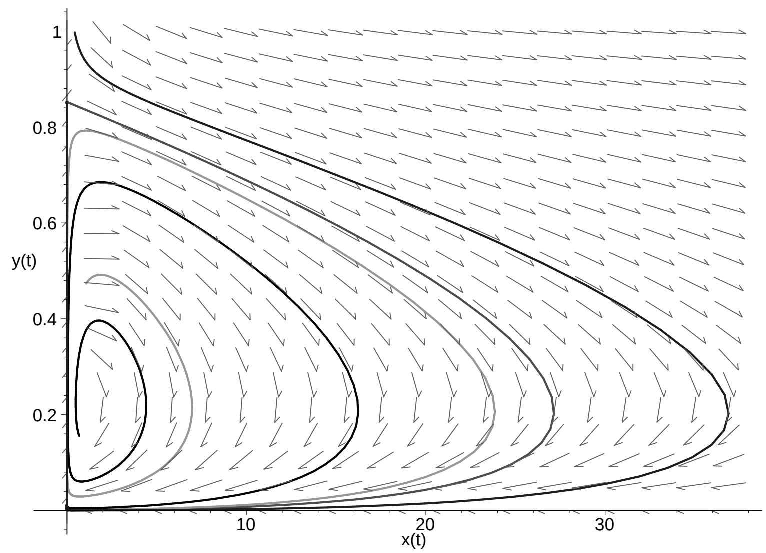


Figure 5. Phase portrait of system (3) with $\mu = 1.2$, $\alpha = 1.0$, $\beta = 1.8$, $\xi = 0.3$, $c = 0.1$, $\kappa = 3.2$, $\kappa_t = 2.51$, $\kappa_1 = 3.0$, $\kappa_2 = 2.83$.

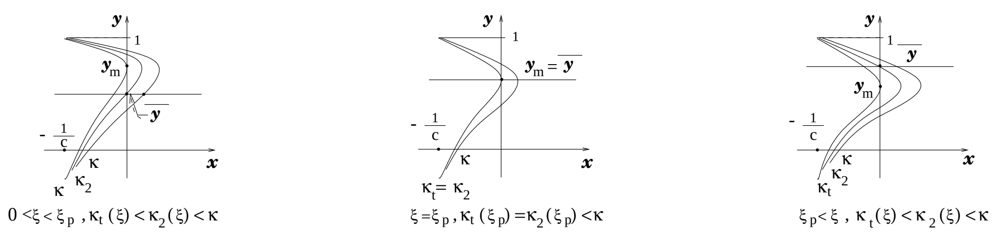
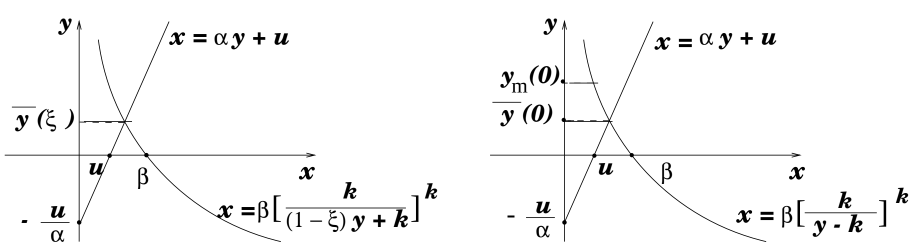
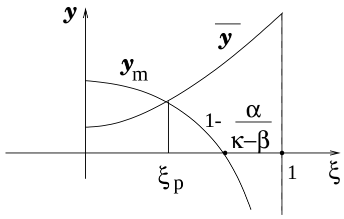
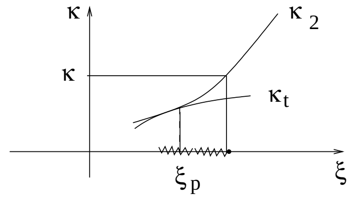


Figure 6.

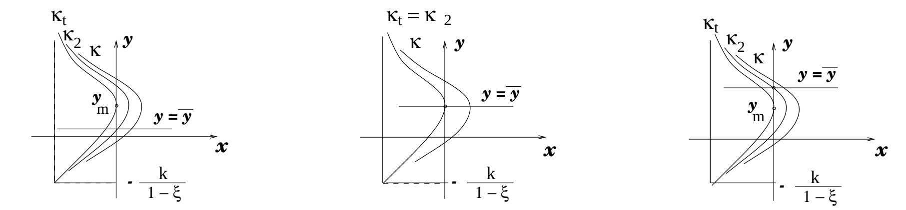


Figures 7 and 8.





Figures 9 and 10.



$0<\xi<\xi_p$, $\kappa_t(\xi)<\kappa_2(\xi)<\kappa$ $\qquad$ $\xi=\xi_p$, $\kappa_t(\xi_p)=\kappa_2(\xi_p)<\kappa$ $\qquad$ $\xi_p=\xi$, $\kappa_t(\xi)=\kappa_2(\xi)<\kappa$

Figure 11.

The eigenvalues of $J(\overline{x},\overline{y})$ are $\lambda(\xi)=\dfrac{\overline{y}G_y(K,\xi,\overline{x},\overline{y})\pm\sqrt{\overline{y}^2G_y^2+4\overline{x}\,\overline{y}F_yG_x}}{2}$, where $\lambda(\xi_p)=\pm i\sqrt{\overline{x}\,\overline{y}F_y(-G_x)}$.

To assure that there is a Hopf bifurcation at ξ_p we will show that $\dfrac{d}{d\xi}\,\mathrm{Re}\,\lambda(\xi)|_{\xi_p}\neq 0$.

In fact,

$$2\frac{d}{d\xi}\mathrm{Re}\lambda(\xi)|_{\xi_p}=\frac{d}{d\xi}\overline{y}G_y(K,\xi,\overline{x},\overline{y})|_{\xi_p}=\overline{y}\{G_{\xi y}+G_{xy}g_\xi+G_{yy}\overline{y}'\}|_{\xi_p}$$

since

$$g_y(K,\xi_p,\overline{y}(\xi_p)=0\;\Rightarrow\left\{\begin{array}{c} G_y(K,\xi_p,\overline{x},\overline{y})=0\\ \overline{x}'(\xi_p)=g_\xi(K,\xi_p,\overline{y}(\xi_p))\end{array}\right. .$$

For system (3):

$$G_x=-(1-y)\frac{Kc}{(cx+1)^2},\qquad G_{xx}=\frac{2c^2K(1-y)}{(cx+1)^3},\qquad G_{xxy}=-\frac{2c^2K}{(cx+1)^3},$$

$$G_y=-\Big(\frac{K}{cx+1}-\alpha\Big)+2\beta(1-\xi)(1-(1-\xi)y),\qquad G_{\xi y}=2\beta((1-\xi)y-1),$$

$$G_{yy}=-2\beta(1-\xi)^2,\qquad G_{xy}=\frac{Kc}{(cx+1)^2},\qquad g_\xi=-\frac{2\beta K(1-y)(1-(1-\xi)y)}{[(\beta(1-(1-\xi)y)^2+\alpha(1-\overline{y})]^2}$$

For system (4):

$$G_y = \frac{\alpha}{k} + \beta(1-\xi)\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+1},$$

$$G_{\xi y} = \beta((1-\xi)y-1)\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+2},\ G_{xy} = 0,$$

$$G_{yy} = -\beta(1-\xi)^2\frac{k+1}{k}\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+2},$$

$$g_\xi = -\frac{K}{c}\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+1}\frac{\beta y}{\Big(\sigma+\alpha+\frac{\alpha}{k}y+\beta\Big(\frac{k}{(1-\xi)y+k}\Big)^k\Big)^2}.$$

Therefore, in both cases:

$$2\frac{d}{d\xi}\mathrm{Re}\lambda(\xi)|_{\xi_p} = \overline{y}(G_{\xi y}+\overline{y}G_{xy}g_\xi + G_{yy}\overline{y}')|_{\xi_p} < 0.$$

Next we engage in showing, in this following closely [4], that the critical point is a vague attractor at $\xi = \xi_p$. Thus we will conclude the existence of attracting periodical orbits, for ξ sufficiently close to ξ_p , $\xi < \xi_p$ (see Figure 12).

Let us consider the system for a fixed $K > K_t(\xi_p)$, and $\xi = \xi_p$, and let us, for the sake of simplicity, omit the notation for those parameters.

Initially we apply a translation, carrying the critical point to the origin, thereby transforming any of the systems in

$$\begin{cases} \dot{\tilde{X}} = \tilde{X}F(\tilde{Y}+\overline{y}) + \overline{x}F(\tilde{Y}+\overline{y}) \\ \dot{\tilde{Y}} = \tilde{Y}G(\tilde{X}+\overline{x},\tilde{Y}+\overline{y}) + \overline{y}G(\tilde{X}+\overline{x},\tilde{Y}+\overline{y}) \end{cases} \tag{3.1}$$

whose Jacobian matrix at the origin is

$$J(0,0) = \begin{bmatrix} 0 & \overline{x}F'_y(\overline{y}) \\ \overline{y}G_x(\overline{x},\overline{y}) & 0 \end{bmatrix}.$$

Applying the coordinate change:

$$X = \tilde{X}, Y = \frac{\overline{x}F'(\overline{y})\tilde{Y}}{\sqrt{-\overline{x}\,\overline{y}F'(\overline{y})G_x(\overline{x},\overline{y})}}$$

system (3.1) becomes

$$\begin{cases} \dot{X} = Z_1(X,Y) = XF(aY+\overline{y}) + xF(aY+\overline{y}) \\ \dot{Y} = Z_2(X,Y) = YG(X+\overline{x},aY+\overline{y}) + \dfrac{\overline{y}}{a}G(X+\overline{x},aY+\overline{y}) \end{cases} \tag{3.2}$$

where

$$a = \frac{\sqrt{-\overline{x}\overline{y}F'(\overline{y})G_x(\overline{x},\overline{y})}}{\overline{x}F'(\overline{y})}.$$

By showing that $V'''(0) < 0$ where $V(X) = P(X) - X, P(X)$ being the local Poincaré transformation at $(0,0)$ we may conclude that $(0,0)$ is a vague attractor for (6) whereas $(\overline{x},\overline{y})$ is a vague attractor for (3) or (4).

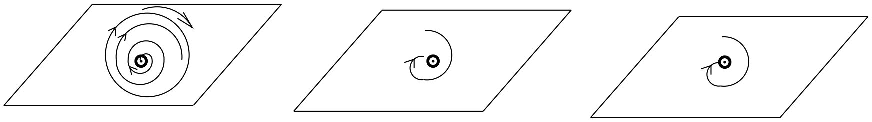


Figure 12.

To this end we use the following expression, taken from [4]:

$$V'''(0) = \frac{3\pi}{4|\lambda(\xi_p)|}\Big(\frac{\partial^3 Z_1}{\partial X^3} + \frac{\partial^3 Z_1}{\partial X \partial Y^2} + \frac{\partial^3 Z_2}{\partial X^2 \partial Y} + \frac{\partial^3 Z_2}{\partial Y^3}\Big) + \\ + \frac{3\pi}{4|\lambda(\xi_p)|^2}\Big(-\frac{\partial^2 Z_1}{\partial X^2}\frac{\partial^2 Z_1}{\partial X \partial Y} + \frac{\partial^2 Z_2}{\partial Y^2}\frac{\partial^2 Z_2}{\partial X \partial Y} + \frac{\partial^2 Z_2}{\partial X^2}\frac{\partial^2 Z_2}{\partial X \partial Y} - \\ - \frac{\partial^2 Z_1}{\partial Y^2}\frac{\partial^2 Z_1}{\partial X \partial Y} + \frac{\partial^2 Z_1}{\partial X^2}\frac{\partial^2 Z_2}{\partial X^2} - \frac{\partial^2 Z_1}{\partial Y^2}\frac{\partial^2 Z_2}{\partial Y^2}\Big).$$

For both systems:

$$\frac{\partial Z_1}{\partial X}(0,0) = F(\overline{y}) = 0\ , \quad \frac{\partial^2 Z_1}{\partial X^2}(0,0) = \frac{\partial^3 Z_1}{\partial X^3}(0,0) = 0,$$
$$\frac{\partial Z_1}{\partial Y}(0,0) = a\overline{x}F(\overline{y})\ , \quad \frac{\partial^2 Z_1}{\partial Y^2}(0,0) = \overline{x}a^2F''(\overline{y}),$$
$$\frac{\partial^3 Z_1}{\partial X \partial Y^2}(0,0) = a^2F''(\overline{y}) \quad \frac{\partial^2 Z_1}{\partial X \partial Y}(0,0) = aF'(\overline{y}),$$
$$\frac{\partial Z_2}{\partial Y}(0,0) = 0 \quad \frac{\partial^2 Z_2}{\partial Y^2}(0,0) = a\overline{y}G_{yy}(\overline{x},\overline{y}),$$

$$\frac{\partial Z_2}{\partial Y}(0,0) = 0, \quad \frac{\partial^2 Z_2}{\partial Y^2}(0,0) = a\overline{y}G_{yy}(\overline{x},\overline{y}),$$
$$\frac{\partial^3 Z_2}{\partial Y^3}(0,0) = 3a^2G_{yy}(\overline{x},\overline{y})\ , \quad \frac{\partial^2 Z_2}{\partial X \partial Y}(0,0) = G_x(\overline{x},\overline{y}) + \overline{y}G_{xy}(\overline{x},\overline{y}),$$
$$\frac{\partial Z_2}{\partial X}(0,0) = \frac{\overline{y}G_x(\overline{x},\overline{y})}{a}\ , \quad \frac{\partial^2 Z_2}{\partial X^2}(0,0) = \frac{\overline{y}}{a}G_{xx}(\overline{x},\overline{y}),$$
$$\frac{\partial^3 Z_2}{\partial X^3}(0,0) = \frac{\overline{y}G_{xxx}(\overline{x},\overline{y})}{a}\ , \quad \frac{\partial^3 Z_2}{\partial X^2 \partial Y}(0,0) = G_{xx}(\overline{x},\overline{y}) + \overline{y}G_{yxx}(\overline{x},\overline{y}).$$

Thus, taking into account:

$$a = \frac{|\lambda(\xi_p)|}{\overline{x}F'(\overline{y})}\,, \frac{aF'(\overline{y}) + \overline{y}aG_{yy}(\overline{x},\overline{y})}{|\lambda(\xi_p)|} = \frac{1}{x} + \frac{\overline{y}G_{yy}(\overline{x},\overline{y})}{\overline{x}F'(\overline{y})},$$
$$G_{yxx}(\overline{x},\overline{y}) - \frac{G_{xx}(\overline{x},\overline{y})G_{xy}(\overline{x},\overline{y})}{G_x(\overline{x},\overline{y})} = 0$$

we obtain

$$V'''(0) = \frac{3\pi}{4|\lambda(\xi_p)|}\Big(-\frac{2\overline{y}G_x(\overline{x},\overline{y})G_{yy}(\overline{x},\overline{y})}{\overline{x}F'(\overline{y})} + \frac{\overline{y}^2G_{yy}(\overline{x},\overline{y})G_{xy}(\overline{x},\overline{y})}{\overline{x}F'(\overline{y})} - \frac{\overline{y}a^2F''(\overline{y})G_{yy}(\overline{x},\overline{y})}{F'(\overline{y})}\Big).$$

Finally, we get from this:

$$\left.\begin{array}{c} G_x(\overline{x},\overline{y}) < 0 \; G_{yy}(\overline{x},\overline{y}) < 0\,, \\ F'(\overline{y}) > 0\,, G_{xy}(\overline{x},\overline{y}) \geq 0\,, \\ F''(\overline{y}) < 0 \end{array}\right\} \Rightarrow V'''(0) < 0.$$

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