Miskolc Mathematical Notes
Vol. 14 (2013), No 1, pp. 135-142

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2013.471

Fixed point theorem on partial metric spaces involving rational expressions

Erdal Karapinar, Wasfi Shatanawi, and Kenan Tas

Miskolc Mathematical Notes
Vol. 14 (2013), No. 1, pp. 135–142

HU e-ISSN 1787-2413

FIXED POINT THEOREM ON PARTIAL METRIC SPACES INVOLVING RATIONAL EXPRESSIONS

ERDAL KARAPINAR, WASFI SHATANAWI, AND KENAN TAS

Received February 21, 2012

Abstract. We establish a fixed point theorem involving a rational expression in a complete partial metric space. Our result generalizes a well-known result in (usual) metric spaces. Also, we introduce an example to illustrate the usability of our result.

2000 *Mathematics Subject Classification:* 47H10; 54H25; 46J10; 46J15

Keywords: partial metric space, rational contractions, fixed point theory

1. Introduction and Preliminaries

The notion of partial metric space was introduced by Matthews [10,11] in 1992. In fact, a partial metric space is a generalization of usual metric spaces in which $d(x,x)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see e.g. [1–9,12,13]).

The definition of partial metric space is given by Matthews (see e.g.[10, 11]) as follows:

Definition 1. Let X be a nonempty set and let $p : X \times X \to [0,\infty)$ satisfy

$$\begin{array}{ll} (PM1) & x = y \;\Leftrightarrow p(x,x) = p(y,y) = p(x,y) \\ (PM2) & p(x,x) \leq p(x,y) \\ (PM3) & p(x,y) = p(y,x) \\ (PM4) & p(x,y) \leq p(x,z) + p(z,y) \;\; - p(z,z) \end{array}$$

for all x, y and $z \in X$. Then the pair (X, p) is called a partial metric space (in short PMS) and p is called a partial metric on X.

Notice that for a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

is a (usual) metric on X. Observe that each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon)=\{y\in X : p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$. Similarly, closed p-ball is defined as $B_p[x,\varepsilon]=\{y\in X : p(x,y)\le p(x,x)+\varepsilon\}$

Some basic concepts on partial metric spaces are defined as follows:

Definition 2 (See e.g.[10, 11]).

- **(i)** A sequence $\{x_n\}$ in a PMS (X,p) converges to $x\in X$ if and only if $p(x,x)=\lim_{n\to\infty} p(x,x_n)$.
- **(ii)** A sequence $\{x_n\}$ in a PMS (X,p) is called Cauchy if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite).
- **(iii)** A PMS (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to\infty} p(x_n,x_m)$.
- **(iv)** A mapping $f : X\to X$ is said to be continuous at $x_0\in X$, if for every $\varepsilon>0$, there exists $\delta>0$ such that $f(B(x_0,\delta))\subset B(f(x_0),\varepsilon)$.

The following lemmas play important role in the proofs of our main result.

Lemma 1 (See e.g.[10, 11]). (A) *A sequence $\{x_n\}$ is Cauchy in a PMS (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space (X,d_p),*

(B) *A PMS (X,p) is complete if and only if a metric space (X,d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m) \tag{1.2}$$

Lemma 2 (See e.g.[4]). *Let (X,p) be a PMS. Then*

(A) *If $p(x,y)=0$ then $x=y$,*
(B) *If $x\neq y$, then $p(x,y)>0$.*

Lemma 3 (See e.g. [4]). *Let $x_n\to z$ as $n\to\infty$ in a PMS (X,p) where $p(z,z)=0$. Then $\lim_{n\to\infty} p(x_n,y)=p(z,y)$ for every $y\in X$.*

2. THE MAIN RESULT

In this section we establish our main theorem which gives conditions for existence and uniqueness of a fixed point for a certain type operators on partial metric spaces.

Theorem 1. *Let (X,p) be a complete partial metric space and $T : X\to X$ be a mapping satisfying*

$$\psi(p(Tx,Ty))\le\psi(M(x,y))-\varphi(M(x,y)), \forall x,y\in X, \tag{2.1}$$

where $M(x,y)$ is given by

$$M(x,y)=\max\left\{p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)},p(x,y)\right\}$$

and $\psi:[0,\infty)\to[0,\infty)$ is a continuous and monotone non-decreasing function with $\psi(t)=0$ if and only if $t=0$ and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with $\varphi(t)=0$ if and only if $t=0$. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. We construct the sequence $\{x_n\}$ in X as follows:

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, 3, \cdots.$$

If there exists n such that $x_n = x_{n+1}$ then x_n is a fixed point of T. Now, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Letting $x = x_{n-1}$ and $y = x_n$ in (2.1), we have

$$\psi(p(x_n, x_{n+1})) = \psi(p(Tx_{n-1}, Tx_n)) \leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n))$$

where

$$M(x_{n-1}, x_n) = \max\left\{p(x_n, Tx_n)\frac{1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x_n)}, p(x_{n-1}, x_n)\right\}.$$

Hence, we obtain

$$\begin{aligned}\psi(p(x_n, x_{n+1})) \leq{} & \psi(\max\{p(x_n, x_{n+1}), p(x_{n-1}, x_n)\}) \\ & - \varphi(\max\{p(x_n, x_{n+1}), p(x_{n-1}, x_n))\}) \end{aligned} \tag{2.2}$$

If $p(x_n, x_{n+1}) > p(x_{n-1}, x_n)$, then from (2.2), we have

$$\psi(p(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1})) - \varphi(p(x_n, x_{n+1})) < \psi(p(x_n, x_{n+1}))$$

which is a contradiction since $p(x_n, x_{n+1}) > 0$ by Lemma 2. So, we have $p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n)$, that is, $\{p(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Thus, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = L \tag{2.3}$$

Suppose that $L > 0$. Taking the upper limit in (2.2) as $n \to \infty$ and using (2.3) and the properties of ψ, φ, we have

$$\psi(L) \leq \psi(L) - \lim_{n\to\infty} \inf \varphi(p(x_{n-1}, x_n)) \leq \psi(L) - \varphi(L) < \psi(L)$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \tag{2.4}$$

Due to (1.1), we have $d_p(x_n, x_{n+1}) \leq 2p(x_n, x_{n+1})$

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0 \tag{2.5}$$

Now, we prove that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Suppose the contrary, that is,

$$\lim_{n,m\to\infty} p(x_n, x_m) \neq 0.$$

Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}, \{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \;\; p(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{2.6}$$

This means that

$$p(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\begin{aligned}\varepsilon \le p(x_{n(k)}, x_{m(k)}) &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\ &< \varepsilon + p(x_{n(k)}, x_{n(k)-1})\end{aligned}$$

Taking $k \to \infty$ and using (2.4), we get

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \varepsilon \tag{2.8}$$

By the triangle inequality, we have

$$\begin{aligned}p(x_{n(k)}, x_{m(k)}) &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &- p(x_{m(k)-1}, x_{m(k)-1}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)})\end{aligned}$$

and

$$\begin{aligned}p(x_{n(k)-1}, x_{m(k)-1}) &\le p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) \\ &\le p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) \\ &\le p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)}, x_{m(k)-1}) \\ &- p(x_{m(k)}, x_{m(k)}) \\ &\le p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)}, x_{m(k)-1})\end{aligned}$$

Taking $k \to \infty$ in the above two inequalities and using (2.4),(2.8), we get

$$\lim_{k\to\infty} p(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon \tag{2.9}$$

Now from (2.1), we have

$$\begin{aligned}\psi(p(x_{m(k)}, x_{n(k)})) &= \psi(p(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\le \psi(M(x_{m(k)-1}, x_{n(k)-1})) - \varphi(p(x_{m(k)-1}), x_{n(k)-1})\end{aligned} \tag{2.10}$$

where

$$M(x_{m(k)-1}, x_{n(k)-1}) =$$

$$= \max\left\{ p(x_{n(k)-1}, x_{n(k)}) \frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}, p(x_{m(k)-1}, x_{n(k)-1}) \right\}$$

By (2.4),(2.8) and (2.9), we have

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.11}$$

Now, passing to the upper limit when $k \to \infty$ in (2.10) and using (2.8),(2.11) and the properties of ψ, φ, we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \lim_{k\to\infty} \inf \varphi(M(x_{m(k)-1}, x_{n(k)-1})) \leq \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon)$$

which is a contradiction. So, we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Since $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and finite, we conclude that (x_n) is a Cauchy sequence in (X, p).

Due to (1.1), we have $d_p(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore

$$\lim_{n,m\to\infty} d_p(x_n, x_m) = 0. \tag{2.12}$$

Thus, by Lemma 1, $\{x_n\}$ is a Cauchy sequence in both (X, d_p) and (X, p).

Since (X, p) is a complete partial metric space, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. Since $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, then again by Lemma 1, we have $p(x, x) = 0$. Let us now prove that x is a fixed point of T. Suppose that $Tx \neq x$. From (2.1) and Lemma 3, we have

$$\begin{aligned}
\psi(p(x_n, Tx)) &= \psi(p(Tx_{n-1}, Tx)) \\
&\leq \psi(\max\left\{p(x, Tx)\frac{1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x)}, p(x_{n-1}, x)\right\}) \\
&\quad - \varphi(\max\left\{p(x, Tx)\frac{1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x)}, p(x_{n-1}, x)\right\}) \qquad (2.13)
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and regarding the property of ϕ, ψ, we obtain

$$\psi(p(x, Tx)) \leq \psi(p(x, Tx)) - \varphi(p(x, Tx)) < \psi(p(x, Tx))$$

which is a contradiction. Thus $Tx = x$, that is, x is fixed point of T. Finally, we shall prove the uniqueness. Suppose that y is another fixed point of T such that $x \neq y$. From (2.1),

$$\begin{aligned}
\psi(p(x, y)) = \psi(p(Tx, Ty)) &\leq \psi(M(x, y)) - \varphi(M(x, y)) \\
&=> \psi(p(x, y)) - \varphi(p(x, y)) < \psi(p(x, y))
\end{aligned}$$

which is a contradiction since $p(x, y) > 0$. Hence $x = y$.
The proof is completed. □

Example 1. Let $X=[0,1]$ and $p(x,y)=\max\{x,y\}$ then (X,p) is a PMS. Suppose $T:X\to X$ such that $Tx=\frac{x^2}{1+x}$ for all $x\in X$ and $\phi(t):[0,\infty)\to[0,\infty)$ such that $\psi(t)=\frac{t}{1+t}$ and $\phi(t)=t$. Without loss of generality assume $x\geq y$. Then, we have

$$p(Tx,Ty)=\max\left\{\frac{x^2}{1+x},\frac{y^2}{1+y}\right\}=\frac{x^2}{1+x}$$

On the other hand,

$$\max\left\{p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)},p(x,y)\right\}=\max\left\{y\frac{1+x}{1+x},x\right\}=x$$

Combining the observations above, we get

$$p(Tx,Ty)=\max\left\{\frac{x^2}{1+x},\frac{y^2}{1+y}\right\}=\frac{x^2}{1+x}\leq x-\frac{x}{1+x}=\frac{x^2}{1+x}.$$

Thus, it satisfies all conditions of Theorem 1. Hence, T has a unique fixed point, indeed $x=0$ is the required point.

In Theorem 1, taking $\psi(t)=t$ for all $t\in[0,\infty)$ and $\varphi(t)=(1-k)t$ for all $t\in[0,\infty)$ with $k\in(0,1)$, we get the following result.

Corollary 1. *Let (X,d) be a complete partial metric space and $T:X\to X$ be a mapping satisfying*

$$p(Tx,Ty)\leq k\max\left\{p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)},p(x,y)\right\}\tag{2.14}$$

where $k\in(0,1)$. Then T has a unique fixed point.

Example 2. Let $X=[0,1]$. Define $T:X\times X\to X$ by $Tx=\frac{x}{2}$. Also, define $p:X\times X\to\mathbf{R}^+$ by $p(x,y)=\max\{x,y\}$, then (X,p) is a complete partial metric space and

$$p(Tx,Ty)\leq\frac{1}{2}\max\left\{p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)},p(x,y)\right\}.$$

Thus by Corollary 1, T has a unique fixed point. Here 0 is the unique fixed point of T.

Example 3. Let $X=[0,\infty)$. Define $T:X\times X\to X$ by $Tx=2x$. Also, define $p:X\times X\to\mathbf{R}^+$ by $p(x,y)=\max\{x,y\}$, then (X,p) is a complete partial metric space. It is clear that Matthew's Theorem (analog of Banach Contraction Mapping principle) does not work. Indeed, without loss of generality, we may assume that $x\leq y$.

Then

$$p(Tx,Ty) = 2y > ky = kp(x,y)$$

for any $k \in [0,1)$.

However, for $k = \frac{1}{3}$, we have

$$2y = p(Tx,Ty) \leq \frac{1}{3}2y\frac{1+2x}{1+x} = k\max\left\{p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}, p(x,y)\right\}.$$

Thus by Corollary 1, T has a unique fixed point. Here 0 is the unique fixed point of T.

REFERENCES

[1] M. Abbas, T. Nazir, and S. Romaguera, "Fixed point results for generalized cyclic contraction mappings in partial metric spaces," *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, vol. 106, no. 1, pp. 287–297, 2012.

[2] T. Abdeljawad, E. Karapınar, and K. Taş, "Existence and uniqueness of a common fixed point on partial metric spaces," *Appl. Math. Lett.*, vol. 24, no. 11, pp. 1900–1904, 2011.

[3] I. Altun, F. Sola, and H. Simsek, "Generalized contractions on partial metric spaces," *Topology Appl.*, vol. 157, no. 18, pp. 2778–2785, 2010.

[4] D. M. Erhan, E. Karapinar, and T. D., "Different types Meir-Keeler contractions on partial metric spaces," *J. Comput. Anal. Appl.*, vol. 14, no. 6, pp. 1000–1005, 2012.

[5] E. Karapinar, "Generalizations of Caristi Kirk's Theorem on partial metric spaces," *Fixed Point Theory and. Appl.*, vol. 2011, no. 4, 2011.

[6] E. Karapinar, "A note on common fixed point theorems in partial metric spaces," *Math. Notes, Miskolc*, vol. 12, no. 2, pp. 185–191, 2011.

[7] E. Karapinar, "Weak ϕ-contraction on partial contraction and existence of fixed points in partially ordered sets," *Mathematica Aeterna*, vol. 1, no. 4, pp. 237–244, 2011.

[8] E. Karapinar, "Weak ϕ-contraction on partial metric spaces," *J. Comput. Anal. Appl.*, vol. 14, no. 2, pp. 206–210, 2012.

[9] E. Karapınar and D. M. Erhan, "Fixed point theorems for operators on partial metric spaces," *Appl. Math. Lett.*, vol. 24, no. 11, pp. 1894–1899, 2011.

[10] S. G. Matthews, "Partial metric topology," Dept. of Computer Science, University of Warwick, Research Report 212, 1992.

[11] S. G. Matthews, "Partial metric topology," in *Papers on general topology and applications*, ser. Papers from the 8th summer conference at Queens College, New York, NY, USA, June 18–20, 1992, S. Andima, Ed. New York: The New York Academy of Sciences, 1994, vol. 728, pp. 183–197.

[12] S. Oltra and O. Valero, "Banach's fixed point theorem for partial metric spaces," *Rend. Ist. Mat. Univ. Trieste*, vol. 36, no. 1-2, pp. 17–26, 2004.

[13] O. Valero, "On Banach fixed point theorems for partial metric spaces," *Appl. Gen. Topol.*, vol. 6, no. 2, pp. 229–240, 2005.

Authors' addresses

Erdal Karapınar

Department of Mathematics, Atılım University, 06836, İncek, Ankara, Turkey

E-mail address: erdalkarapinar@yahoo.com *or* ekarapinar@atilim.edu.tr

Wasfi Shatanawi
Department of Mathematics, Hashemite University, Zarqa, Jordan
E-mail address: swasfi@hu.edu.jo

Kenan Tas
Çankaya University, Department of Mathematics and Computer Science, Ankara, TURKEY
E-mail address: kenan@cankaya.edu.tr