A matrix approach for generalizing two curious divisibility properties

Emrah Kılıç

A MATRIX APPROACH FOR GENERALIZING TWO CURIOUS DIVISIBILITY PROPERTIES

EMRAH KILIÇ

Abstract. We shall consider two curious divisibility properties due to [1, 5]. Our main purpose is to generalize these properties for a general second order linear recursion. We use generating matrix approach for our purposes. By using our results, we derive a new recursive identities for two general second order linear recurrences. We give a more general result extending earlier divisibility properties on the similar subject.

1. INTRODUCTION

The Fibonacci numbers have many interesting properties in accordance with their nature and the recursively definition. We shall be interested in their divisibility properties along the paper. We recall that

$$\gcd(F_n, F_m) = F_{\gcd(n,m)}$$

and

$$F_{kn} \text{ is a multiple of } F_n.$$

An extension of these divisibility properties was given showing Matijasevich [4] as that if $n > 2$, the Fibonacci number F_m is a multiple of F_n^2 if and only if m is multiple of nF_n. This fact and its detailed proof was also given in [2] (pp. 294) showning

$$F_{kn} \equiv kF_n F_{n+1}^{k-1} \text{ and } F_{kn+1} \equiv F_{n+1}^k \pmod{F_n^2}.$$

The above fact and the following similar results can also be found in [1, 5] :

i) $F_{kn-1} - F_{n-1}^k$ is divisible by F_n^2 $(n, k = 1, 2, \ldots)$,

ii) $F_{kn-2} - (-1)^{k+1} F_{n-2}^k$ is divisible by F_n^2 $(n, k = 1, 2, \ldots)$.

We mainly consider these two curious properties and then we generalize them to a general second order linear recursion $\{U_n(p)\}$, or briefly U_n, defined for $n > 1$ by

$$U_n(p) = pU_{n-1}(p) + U_{n-2}(p), \qquad (1.1)$$

with initials $U_0(p) = 0$ and $U_1(p) = 1$.

The generalized Lucas sequence $\{V_n(p)\}$, or briefly V_n, is defined by

$$V_n(p) = pV_{n-1}(p) + V_{n-2}(p),$$

with initials $V_0(p) = 2$, $V_1(p) = p$.

The Binet forms of them are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right)/2$.

Throughout the paper we consider the properties

a) $U_{kn-1} - U_{n-1}^k$ is divisible by U_n^2 $(n, k = 1, 2, \ldots)$,

b) $p^{k-1}U_{kn-2} - (-1)^{k+1}U_{n-2}^k$ is divisible by U_n^2 $(n, k = 1, 2, \ldots)$.

Matijasevich and Cavachi proved the properties (i) and (ii) in [2, 5] but the quotients

$$\frac{F_{kn-1} - F_{n-1}^k}{F_n^2} \quad \text{and} \quad \frac{F_{kn-2} - (-1)^{k+1}F_{n-2}^k}{F_n^2}$$

were not considered. It seems that to find the quotients are not easy.

However, we shall focus on quotients arising from (a) and (b) :

$$\frac{U_{kn-1} - U_{n-1}^k}{U_n^2} \quad \text{and} \quad \frac{p^{k-1}U_{kn-2} - (-1)^{k+1}U_{n-2}^k}{U_n^2}$$

for all n and k such that $n, k \geq 1$.

Matrix methods are essential tools for solving problems arising from number theory, for more details see [3].

Our approach is to use matrix methods for deriving these quotients explicitly. We first look at values of the quotients for fixed n and increasing values of k starting from 1 and then find a generating matrix for these values for fixed n and increasing values of k. After we find the generating matrices for these quotients, we will give two explicit statements for them. Using these two explicit statements, we obtain two new recursive identities for the general second order linear recurrences. Finally by considering and extending these two recursive identities, we find new recursive identities for the general second order linear recursions. In order to guarantee the accuracy of matrix computations, all the equalities appearing in this paper have been verified through "Maple" commands.

2. GENERAL CASES

We start with the first quotient with the case $p = 1$, $U_n = F_n$ (nth Fibonacci numbers) and its values for some n and k are shown in Table 1.

We will see that the entries of each row in Table 1 have a generating matrix. For example, the entries of the third row of Table 1, $\{1, 5, 22, 94, 399, 1691, \ldots\}$, could be generated by powers of a matrix. Continuing this process each row entries in each

TABLE 1.

$$\left(F_{kn-1} - F_{n-1}^k\right)/F_n^2$$

$n\backslash k$	1	2	3	4	5	6	7	...
1	0	1	1	2	3	5	8	...
2	0	1	4	12	33	88	232	...
3	0	1	5	22	94	399	1691	...

table can be obtained from the quotients for all n, k and p. We will state this fact in general by matrix methods.

Before this process, we need some auxiliary results and definitions.

Define a second order linear sequence $\{A_n(p)\}$, or briefly A_n, for $n > 1$ as

$$A_n = pA_{n-1} + A_{n-2}$$

with initials $A_0 = p$ and $A_1 = p^2 + 3$.

From the definitions of sequences $\{U_n\}$ and $\{A_n\}$, we have that

$$A_n = U_{n+2} + U_n = V_n + U_{n-1}.$$

Denote the first quotient by $s(n, k, p)$, or briefly $s(k, p)$:

$$s(n, k) = \left(U_{kn-1} - U_{n-1}^k\right)/U_n^2.$$

Define two matrices $H(n, p)$, or briefly $H(n)$, and $G(n, k, p)$, or briefly $G(n, k)$, of order 3 as shown:

$$H(n) = \begin{bmatrix} A_{n-1} & -U_{n-1}V_n + (-1)^{n+1} & (-1)^n U_{n-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$G(n, k) = \begin{bmatrix} s(n, k+2) & t(n, k+2) & (-1)^n U_{n-1}s(n, k+1) \\ s(n, k+1) & t(n, k+1) & (-1)^n U_{n-1}s(n, k) \\ s(n, k) & t(n, k) & (-1)^n U_{n-1}s(n, k-1) \end{bmatrix},$$

where

$$t(n, k) = \frac{U_{kn}U_{n-1}^2 - U_{(k-2)n} - U_{2n}U_{n-1}^k}{U_n^3}.$$

Lemma 1. *For $n \geq 1$, the eigenvalues of the matrix $H(n)$ are α^n, β^n and U_{n-1}.*

Proof. The characteristic polynomial of $H(n)$ is

$$x^3 - A_{n-1}x^2 + x\left(U_{n-1}V_n + (-1)^n\right) - (-1)^n U_{n-1}$$

and it is factorized as $(x - \alpha^n)(x - \beta^n)(x - U_{n-1})$, as claimed. $\square$

Now we state one of our main two results:

Theorem 1. *For $n \geq 1$,*

$$H(n)^k = G(n,k).$$

Proof. To prove the claim, it is sufficient to verify the followings:

$$s(n,k+1) A_{n-1} + t(n,k) = s(n,k+2)$$

and

$$s(n,k+1)\left(-U_{n-1}V_n + (-1)^{n+1}\right) + (-1)^n U_{n-1}s(n,k) = t(n,k+2).$$

For proving them, it is enough to consider the definitions of $s(n,k)$ and $t(n,k)$. The above equations give us the following matrix equation:

$$G(n,k-1) H(n) = G(n,k).$$

Since $G(n,1) = H(n)$, we deduce that $G(n,k) = H(n)^k$, as claimed. $\square$

We see that the powers of matrix $H(n)$ generate the first quotient and since the matrix has integer entries, it is clear that the first quotient are integers.

Since all eigenvalues of $H(n)$ are different from each other, we diagonalize it. $H(n)$ is a companion matrix so that it is diagonalized with the Vandermonde matrix

$$V = \begin{bmatrix} \alpha^{2n} & \beta^{2n} & U_{n-1}^2 \\ \alpha^n & \beta^n & U_{n-1} \\ 1 & 1 & 1 \end{bmatrix},$$

as $V^{-1} H(n) V = D$, where D is the diagonal matrix $D = \mathrm{diag}(\alpha^n, \beta^n, U_{n-1})$. Thus we obtain

$$V^{-1} H(n)^k V = D^k \text{ or } V^{-1} G(n,k) V = D^k,$$

and so

$$G(n,k) = V D^k V^{-1}.$$

Consequently, after required computations, we get the following result:

Theorem 2. *For $n,k > 0$*

$$(G(n,k))_{2,1} = s(n,k)$$

$$= \frac{(-1)^n U_{n(k-1)} + U_{n-1}^k U_n - U_{kn} U_{n-1}}{U_n^3}.$$

From the definition of $s(n,k)$ and the above result, we have the following result:

Corollary 1. *For $n,k > 0$*

$$U_{n-1} U_{kn} = U_n U_{kn-1} + (-1)^n U_{n(k-1)}.$$

Now we shall focus on the second quotient

$$\frac{p^{k-1}U_{kn-2} - (-1)^{k+1}U_{n-2}^k}{U_n^2}$$

and we will seek an explicit formula for that.

Similar to the process for the first quotient, we need some preliminary results and definition. First we define a second order recurrence $\{B_n(p)\}$, or briefly B_n, for $n > 1$ as

$$B_n = pB_{n-1} + B_{n-2}$$

with initials $B_0 = p^2 - 1$ and $B_1 = p^3 + 2p$.

For shortness, we denote the second quotient by $y(n,k)$:

$$y(n,k) = \frac{p^{k-1}U_{kn-2} - (-1)^{k+1}U_{n-2}^k}{U_n^2}.$$

Define two matrices $T(n)$ and $Q(n,k)$ of order 3 as shown:

$$T(n) = \begin{bmatrix} B_{n-1} & C(n) & (-1)^{n+1}p^2U_{n-2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$Q(n,k) = \begin{bmatrix} y(n,k+2) & h(n,k+2) & (-1)^{n+1}p^2U_{n-2}y(n,k+1) \\ y(n,k+1) & h(n,k+1) & (-1)^{n+1}p^2U_{n-2}y(n,k) \\ y(n,k) & h(n,k) & (-1)^{n+1}p^2U_{n-2}y(n,k-1) \end{bmatrix},$$

where $C(n) = p^2\left(D_n + 2(-1)^n\right)$, D_n is defined as for $n > 1$

$$D_n = \left(p^2 + 2\right)D_{n-1} - D_{n-2}$$

with initials $D_0 = 0$ and $D_1 = 1$, and, $h(n,k)$ is given by

$$h(n,k) = \frac{p^{k-1}U_{n-2}^2U_{kn} - p^{k+1}U_{n(k-2)} - p(-1)^k U_{n-2}^k U_{2n}}{U_n^3}.$$

Lemma 2. *For $n \geq 1$, the eigenvalues of $T(n)$ are $p\alpha^n$, $p\beta^n$ and $-U_{n-2}$.*

Proof. The characteristic polynomial of $T(n)$ is $x^3 - B_{n-1}x^2 - C(n)x + (-1)^n p^2 U_{n-2}$ and it is factorized as $(x - p\alpha^n)(x - p\beta^n)(x + U_{n-2})$, as claimed. □

We state our second main result:

Theorem 3. *For $n \geq 1$,*

$$T(n)^k = Q(n,k).$$

Proof. To prove the claim, it is sufficient to verify the followings:

$$y(n,k+1) B_{n-1} + h(n,k+1) = y(n,k+2)$$

and

$$y(n,k+1) C(n) + (-1)^{n+1} p^2 U_{n-2} y(n,k) = h(n,k+2).$$

For proving them, one can consider the definitions of $y(n,k)$ and $h(n,k)$. The above two equations give us the following matrix equation:

$$Q(n,k-1) T(n) = Q(n,k).$$

Since $Q(n,1) = T(n)$, we deduce that $Q(n,k) = T(n)^k$. $\square$

Since all the eigenvalues of $T(n)$ are different from each other, it is diagonalized as

$$V_1^{-1} T(n) V_1 = D_1,$$

where

$$V_1 = \begin{bmatrix} p^2 \alpha^{2n} & p^2 \beta^{2n} & U_{n-2}^2 \\ p\alpha^n & p\beta^n & -U_{n-2} \\ 1 & 1 & 1 \end{bmatrix}.$$

and D_1 is the diagonal matrix $D_1 = \text{diag}(\alpha^n, \beta^n, -U_{n-2})$. Thus we obtain

$$V_1^{-1} T(n)^k V_1 = D_1^k \text{ or } V_1^{-1} Q(n,k) V_1 = D_1^k,$$

and so

$$Q(n,k) = V_1 D_1^k V_1^{-1}.$$

Therefore, by considering $(Q(n,k))_{2,1} = y(n,k)$, we get the following result:

Theorem 4. *For $n,k > 0$*

$$y(n,k) = \frac{p^{k+1}(-1)^n U_{kn} + (-1)^{k+1} U_{n-2}^{k+1} U_n + p^k U_{n-2} U_{n(k+1)}}{U_n^3}$$

From the definition of $y(n,k)$ and the above result, we have the following result:

Corollary 2. *For $n,k > 0$*

$$U_{n-2} U_{nk} = U_{nk-2} U_n - p(-1)^n U_{kn-n}$$

3. New recursive identities

While extending two curious properties in the previous section, we found the following two new identities for general second order recursions:

$$i) \ U_{n-1} U_{kn} = U_n U_{kn-1} + (-1)^n U_{kn-n},$$
$$ii) \ U_{n-2} U_{nk} = U_n U_{kn-2} - p(-1)^n U_{kn-n}.$$

The above recursions suggest us that they could be more generalized. Thus we give our third main result:

Theorem 5. *For all integers r, k and n,*

$$U_{n-r}(p) U_{kn}(p) = U_n(p) U_{kn-r}(p) - (-1)^{r+n} U_r(p) U_{kn-n}(p)$$

and

$$U_{n-r}(p) V_{kn}(p) = U_n(p) V_{kn-r}(p) - (-1)^{r+n} U_r(p) V_{kn-n}(p)$$

Proof. For the first identity consider

$$U_n(p) U_{kn-r}(p) - (-1)^{r+n} U_r(p) U_{kn-n}(p)$$
$$= \frac{\alpha^n - \beta^n}{\alpha - \beta} \frac{\alpha^{kn-r} - \beta^{kn-r}}{\alpha - \beta} - (-1)^{r+n} \frac{\alpha^r - \beta^r}{\alpha - \beta} \frac{\alpha^{kn-n} - \beta^{kn-n}}{\alpha - \beta}$$
$$= (\alpha^{kn+n-r} - (-1)^n \beta^{kn-n-r} - (-1)^n \alpha^{kn-n-r} + \beta^{kn+n-r} - (-1)^{n-r} \alpha^{kn-n+r}$$
$$\quad + (-1)^n \beta^{kn-n-r} + (-1)^n \alpha^{kn-n-r} - (-1)^{n-r} \beta^{kn-n+r}) / (\alpha - \beta)^2$$
$$= (\alpha^{kn+n-r} + \beta^{kn+n-r} - (-1)^{n-r} \alpha^{kn-n+r} - (-1)^{n-r} \beta^{kn-n+r}) / (\alpha - \beta)^2$$
$$= (\alpha^{n-r} - \beta^{n-r})(\alpha^{kn} - \beta^{kn})) / (\alpha - \beta)^2$$
$$= U_{n-r}(p) U_{kn}(p),$$

as claimed. By the Binet formula, the second claim could be similarly proved. □

Our last conclusion is that the last recursive identities are new for general second order linear recurrences $\{U_n\}$ and $\{V_n\}$ according to our literature knowledge.

For the Fibonacci and Lucas cases, we have that

$$i) \quad F_{n-r} F_{kn} = F_n F_{kn-r} - (-1)^{n+r} F_r F_{kn-n},$$
$$ii) \quad F_{n-r} L_{kn} = F_n L_{kn-r} - (-1)^{n+r} F_r L_{kn-n}.$$

4. GENERALIZATION OF THE DIVISIBILITY PROPERTIES

We consider two divisibility properties and then generalize them for the general second order linear recurrence $\{U_n\}$. Now we are ready to present more general case of these properties. We state this general case with the following corollary without proof:

Corollary 3. *For all integers r,*

$$U_r^{k-1} U_{kn-r} - (-1)^{(r-1)(k+1)} U_{n-r}^k \text{ is divisible by } U_n^2 \quad (n, k = 1, 2, \ldots).$$

This corollary gives us all the divisibility properties mentioned in this paper.

References

[1] M. Cavachi, "Some properties of the terms of the Fibonacci sequence," *Gaz. Mat., Bucur.*, vol. 85, pp. 290–293, 1980.

[2] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics: A foundation for computer science*.   Addison-Wesley, 1992.

[3] E. Kiliç and P. Stănică, "A matrix approach for general higher order linear recurrences," *Bull. Malays. Math. Sci. Soc. (2)*, vol. 34, no. 1, pp. 51–67, 2011.

[4] Y. V. Matiyasevich, "Enumerable sets are diophantine," *Sov. Math., Dokl.*, vol. 11, pp. 354–358, 1970.

[5] S. Vajda, *Fibonacci & Lucas numbers, and the golden section. Theory and applications*, ser. Ellis Horwood Books in Mathematics and its Applications.   Chichester, New York: Ellis Horwood Ltd.; Halsted Press, 1989.

Author's address

Emrah Kılıç
TOBB Economics and Technology University, Mathematics Department 06560 Sogutozu Ankara Turkey.

E-mail address: `ekilic@etu.edu.tr`