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SIGN-CHANGING SOLUTIONS AND W-SOLUTIONS TO SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS

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Abstract. The singular Dirichlet problem $(r(x)x')' = q(t)f(t,x)$, $x(0) = x(T) = 0$, $\lambda \max\{x(t) : 0 \le t \le T\} = -\min\{x(t) : 0 \le t \le T\}$ is considered. Here f is singular at the point $x = 0$ of the phase variable x and λ is a positive parameter. The notions of a solution and a w-solution of the above problem changing its sign exactly once on $(0,T)$ are introduced. Effective conditions for the existence and multiplicity results are presented. Next, the notion of an exceptional n-sign-changing w-solution of our problem with $\lambda = 1$ is given and for such solutions existence and multiplicity results are proved.

Mathematical Subject Classification: 34B15, 34B16
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1. Introduction

Consider the problem

$$(r(x(t))x'(t))' = q(t)f(t,x(t)), \tag{1.1}$$

$$x(0) = 0, \quad x(T) = 0, \tag{1.2}$$

$$\lambda \max\{x(t) : 0 \le t \le T\} = -\min\{x(t) : 0 \le t \le T\}, \tag{1.3}$$

where T is a positive number, λ is a positive parameter and f is singular at the point $x = 0$ of the phase variable x in the following sense

$$\lim_{x\to 0^-} f(t,x) = -\infty, \quad \lim_{x\to 0^+} f(t,x) = \infty \quad \text{for } t \in [0,T]. \tag{1.4}$$

Definition 1.1. We say that $x \in C^1([0,T])$ is a *solution of problem* (1.1) − (1.3) if x has precisely one zero t_0 on $(0,T)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, (1.1) is satisfied for $t \in (0,T) \setminus \{t_0\}$, x fulfils (1.2) and there exists $\lambda_0 \in (0,\infty)$ such that (1.3) holds with $\lambda = \lambda_0$.

Besides a solution of problem (1.1)-(1.3), we introduce in accordance with [15] the notion of a w-solution of problem (1.1)-(1.3).

Definition 1.2. Let $\lambda \in (0, \infty)$. A function $x \in C^0([0,T])$ is called a *w-solution of problem* (1.1) − (1.3) if x has precisely one zero $t_0 \in (0,T)$, $x \in C^1([0,T] \setminus \{t_0\})$, there exist finite $\lim_{t \to t_0^-} x'(t)$ and $\lim_{t \to t_0^+} x'(t)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2), (1.3), and (1.1) holds on $(0,T) \setminus \{t_0\}$.

We note that in contrast to a solution x of problem (1.1)-(1.3) which belongs to the class $C^1([0,T])$ and satisfies (1.3) with a suitable value of λ, a w-solution x of problem (1.1)-(1.3) is continuous on $[0,T]$, has continuous derivative on $[0,t_0) \cup (t_0,T]$ where t_0 is the unique zero of x in $(0,T)$ and (1.3) holds with a given value of λ. Naturaly, any solution of problem (1.1)-(1.3) is also a w-solution of this problem.

In the paper we will use the following assumptions:

(H_1) $r \in C^0(\mathbb{R})$, $r(x) \geq r_0 > 0$ for $x \in \mathbb{R}$;
(H_2) $q \in C^0((0,T))$, $q(t) < 0$ for $t \in (0,T)$ and $Q = \sup\{|q(t)| : 0 \leq t \leq T\} < \infty$;
(H_3) $f \in C^0([0,T] \times D)$, where $D = (-\infty, 0) \cup (0, \infty)$, $f(t, \cdot)$ is nonincreasing on D for $t \in [0,T]$ and

$$0 < f(t,x)\operatorname{sign} x \leq g(x) \quad \text{for } (t,x) \in [0,T] \times D,$$

where $g \in C^0(D)$ and

$$\int^0 g(s)\,ds < \infty, \qquad \int_0 g(s)\,ds < \infty;$$

(H_4) for each $(t_0, x_0, x_1) \in (0,T) \times D \times \mathbb{R}$, there exists a unique solution x of (1.1) satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$ defined in a neighbourhood of $t = t_0$.

Remark 1.3. If f satisfies (H_3) then for each $M > 0$ there exists a positive function $k_M \in C^0([0,T])$ such that

$$0 < k_M(t) \leq f(t,x)\operatorname{sign} x \leq g(x) \quad \text{on } (t,x) \in [0,T] \times ([-M,0) \cup (0,M]).$$

Next under the assumption that f is a locally Lipschitz function on $(0,T) \times D$, assumption (H_4) is satisfied.

In many papers (see, e.g., [1]–[13], [16]–[22] and references therein) only positive (negative) solutions on $(0,T)$ of the Dirichlet boundary value problems with the singularity at the point $x = 0$ of the phase variable x in nonlinearities of considered second-order differential equations have been studied. Solutions were considered either in the class $C^0([0,T]) \cap C^2((0,T))$ ([1]–[3], [7], [11], [12], [18], [19]) or $C^1([0,T]) \cap C^2((0,T))$ ([4]–[6], [12], [13], [16]–[19], [22]) or $C^0([0,T]) \cap AC^1_{\text{loc}}((0,T))$ ([8]–[10], [20], [21]). Here $AC^1_{\text{loc}}((0,T))$ denotes the set of functions having absolutely continuous first derivatives on any compact subintervals of $(0,T)$. The nonlinearities of equations are usually nonpositive ([1], [2], [6]–[8], [11], [12], [16]–[20], [22]), but in [3]–[5], [9], [10], [13] and [21] this assumption is overcome.

For the first time in [14] solutions of singular Dirichlet boundary value problems changing their signs exactly once on $(0,T)$ were considered. Here differential equations of the form

$$(r(x(t))x'(t))' = \mu q(t) f(t, x(t)) \tag{1.5}$$

together with the condition

$$\max\{x(t) : 0 \le t \le T\} \min\{x(t) : 0 \le t \le T\} < 0\,, \tag{1.6}$$

were studied where μ is a positive parameter and f is singular at the point $x = 0$ of the phase variable x. A function $x \in C^1([0,T])$ is called a solution of problem $(1.5), (1.2), (1.6)$ if x has precisely one zero t_0 on $(0,T)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2) and (1.6) and there exists $\mu_0 > 0$ such that (1.5) with $\mu = \mu_0$ is satisfied for $t \in (0,T)\setminus\{t_0\}$. In [14] under assumptions $(H_1)-(H_3)$ it is proved among others that for each $A \in (0,\infty)$ there exists a solution x of problem $(1.5), (1.2), (1.6)$ such that $\max\{x(t) : 0 \le t \le T\} = A$. We see that any solution of problem $(1.5), (1.2), (1.6)$ depends on a value of the parameter μ in equation (1.5) unlike our definition of a solution of problem (1.1)-(1.3) depending on a value of the parameter λ appearing in condition (1.3).

A generalization of the notion of a solution of problem $(1.5), (1.2), (1.6)$ was given in [15]. Here $x \in C^0([0,T])$ is said to be a w-solution of problem (1.5), (1.2), (1.6) if x has precisely one zero t_0 in $(0,T)$, $x \in C^1([0,T]\setminus\{t_0\})$, there exist finite $\lim_{t\to t_{0^-}} x'(t)$, $\lim_{t\to t_{0^+}} x'(t)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2) and (1.6), and finally there exists $\mu_0 > 0$ such that (1.5) with $\mu = \mu_0$ is satisfied for $t \in (0,T) \setminus \{t_0\}$. It is proved among others that under assumptions $(H_1) - (H_3)$ for $A > 0$ and $t_0 \in (0,T)$ problem $(1.5), (1.2), (1.6)$ has just two w-solutions vanishing at t_0 and having their maximum values on $[0,T]$ equal to A.

This paper is a continuation of [15] and in comparison with (1.5) our equation (1.1) does not depend on the parameter μ. By our definitions any solution as well as any w-solution x of problem (1.1)-(1.3) have precisely one zero in $(0,T)$ where they change their signs. Hence any solution and any w-solution of problem (1.1)-(1.3) 'pass through' the singularity of f at a point of the interval $(0,T)$.

The paper is organized as follows. In Section 2 we define functions Λ_+, Φ_+, Λ_- and Φ_- by (2.6)-(2.9) and present some of their important properties. By these functions we prove existence and uniqueness results for w-solutions of problem (1.1)-(1.3) in Section 3 (Theorems 3.1 and 3.2). Section 4 is devoted to the study of existence and multiplicity results for solutions of problem (1.1)-(1.3) (Theorem 4.5). In Section 5 we first give the notion of an exceptional n-sign-changing w-solution x of problem $(1.1), (1.2), (5.1)$ which changes its sign exactly n times on $[0,T]$ and on each maximum subinterval of $[0,T]$ where x keeping its sign the function $|x|$ has the same maximum value. Existence and multiplicity results for the n-sign-changing w-solution of problem $(1.1), (1.2), (5.1)$ are given in Theorem 5.7.

2. Lemmas, notation

Let $0 \le a < b \le T$. In our consideration we will work with the following auxiliary boundary conditions

$$x(a) = x(b) = 0, \quad x(t) > 0 \quad \text{for } t \in (a,b), \tag{2.1}$$

$$x(a) = x(b) = 0, \quad x(t) < 0 \quad \text{for } t \in (a,b) \tag{2.2}$$

and we will use the function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(u) = \int_0^u r(s)\, ds \tag{2.3}$$

with r occurring in (1.1) and satisfying assumption (H_1). Clearly, $H \in C^1(\mathbb{R})$ is increasing on $\mathbb{R}$ and the inverse function to H denoted by H^{-1} is increasing on $\mathbb{R}$.

We say that x is *a solution of problem* (1.1), (j), j $\in \{2.1,\ 2.2\}$ if $x \in C^1([a,b])$, $r(x)x' \in C^1((a,b))$, x satisfies the boundary conditions (j) and (1.1) is fulfilled for $t \in (a,b)$.

Remark 2.1. Let the function $\tilde q : (0,T) \to (-\infty, 0)$ and $\tilde f : [0,T] \times D \to \mathbb{R}$ be defined by

$$\tilde q(t) = q(T-t), \quad \tilde f(t,x) = f(T-t,x).$$

Then

$$0 < \tilde f(t,x) \operatorname{sign} x < g(x), \quad (t,x) \in [0,T] \times D$$

and assumptions (H_2)-(H_4) are satisfied with $\tilde q$ and $\tilde f$ instead of q and f. If we consider the differential equation

$$(r(x(t))x'(t))' = \tilde q(t) \tilde f(t, x(t)), \tag{2.4}$$

we see that a function x is a solution of problem (1.1), (j) with $a = 0$, $b = c\,(< T)$ and j$\in \{2.1,\ 2.2\}$ if the function $\tilde x(t) = x(T-t)$, $t \in [T-c, T]$, is a solution of problem (2.4), (j) with $a = T - c$, $b = T$. Conversely, if $\tilde x$ is a solution of problem (2.4), (j) with $a = T - c\,(> 0)$, $b = T$ and j$\in \{2.1, 2.2\}$, then the function $x(t) = \tilde x(T-t)$, $t \in [0,c]$, is a solution of problem (1.1), (j) with $a = 0$, $b = c$.

Remark 2.2. Let $r^* : \mathbb{R} \to [r_0, \infty)$, $f_* : [0,T] \times D \to \mathbb{R}$ and $g^* : D \to \mathbb{R}$ be defined by the formulas (see [14])

$$r^*(x) = r(-x), \quad f^*(t,x) = -f(t,-x), \quad g^*(x) = g(-x).$$

Then

$$0 < f^*(t,x) \operatorname{sign} x \le g^*(x), \quad (t,x) \in [0,T] \times D$$

and assumptions $(H_1) - (H_4)$ are satisfied with r^*, f^* and g^* instead of r, f and g. It is easily seen that a function x is a solution of problem (1.1), (j), j$\in \{2.1,\ 2.2\}$, if and only if $x^* = -x$ on $[a,b]$ is a solution of problem (2.5), (j), where

$$(r^*(x(t))x'(t))' = q(t) f^*(t, x(t)). \tag{2.5}$$

Lemma 2.3. Let assumptions $(H_1)-(H_3)$ be satisfied. Then for each $a, b \in [0,T]$, $a < b$, there exists a unique solution of problem (1.1), (2.1).

Proof. The assertion of our lemma follows from Theorem 2.1 in [14] with $\mu = 1$. □

Corollary 2.4. Under assumptions of Lemma 2.3, for each $a, b \in [0,T]$, $a < b$, there exists a unique solution of problem (1.1), (2.2).

Proof. Fix $0 \le a < b \le T$. Since assumptions $(H_1)-(H_3)$ are satisfied with the functions r^*, f^* and g^* defined in Remark 2.2 instead of r, f and g, problem (2.5), (2.1) has a unique solution $\tilde{x}$ by Lemma 2.3. Now the function $x = -\tilde{x}$ on $[a,b]$ is the unique solution of problem (1.1), (2.2). □

For each $\alpha \in (0,T]$ and $\beta \in [0,T)$, we denote throughout this paper by u_α and v_β the unique solution of problem (1.1), (2.1) with $a = 0$, $b = \alpha$ and $a = \beta$, $b = T$, respectively. Next by $\overline{u}_\alpha$ and $\overline{v}_\beta$ we denote the unique solution of problem (1.1), (2.2) with $a = 0$, $b = \alpha$ and $a = \beta$, $b = T$, respectively. The existence and uniqueness of u_α, v_β and $\overline{u}_\alpha$, $\overline{v}_\beta$ follow from Lemma 2.3 and Corollary 2.4, respectively.

Lemma 2.5. (Lemma 2.7 in [14]. Let assumptions $(H_1)-(H_3)$ be satisfied and let $0 < \alpha_1 < \alpha_2 \le T$. Then

$$u_{\alpha_1}(t) \le u_{\alpha_2}(t) \quad for\ t \in [0,\alpha_1].$$

By the solutions u_α, v_β, $\overline{u}_\alpha$ and $\overline{v}_\beta$ define the functions $\Lambda_+ : (0,T] \to (0,\infty)$, $\Phi_+ : [0,T) \to (0,\infty)$, $\Lambda_- : (0,T] \to (-\infty,0)$, $\Phi_- : [0,T) \to (-\infty,0)$ by the formulas

$$\Lambda_+(\alpha) = \max\{u_\alpha(t) : 0 \le t \le \alpha\}, \tag{2.6}$$

$$\Phi_+(\beta) = \max\{v_\beta(t) : \beta \le t \le T\}, \tag{2.7}$$

$$\Lambda_-(\alpha) = \min\{\overline{u}_\alpha(t) : 0 \le t \le \alpha\} \tag{2.8}$$

and

$$\Phi_-(\beta) = \min\{\overline{v}_\beta(t) : \beta \le t \le T\}. \tag{2.9}$$

Properties of the functions Λ_+, Φ_+, Λ_- and Φ_- are presented in the following lemmas.

Lemma 2.6. Let assumptions $(H_1)-(H_3)$ be satisfied. Then Λ_+ is continuous nondecreasing on $(0,T]$ and

$$\lim_{\alpha\to 0^+} \Lambda_+(\alpha) = 0.$$

Proof. As a direct consequence of Lemma 2.5 we get that Λ_+ is nondecreasing on $(0,T]$. Suppose that Λ_+ is discontinuous on the right at a point $\alpha_0 \in (0,T)$, i.e. there is a decreasing sequence $\{\alpha_n\} \subset (\alpha_0, T)$ such that $\lim_{n\to\infty} \alpha_n = \alpha_0$ and

$$\lim_{n\to\infty} \Lambda_+(\alpha_n) > \Lambda_+(\alpha_0). \tag{2.10}$$

Consider the sequence $\{u_{\alpha_n}\}$. Since $(r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' = q(t)f(t, u'_{\alpha_n}(t)) < 0$ for $t \in (0, \alpha_n)$, $r(u_{\alpha_n})u'_{\alpha_n}$ is decreasing on $[0, \alpha_n]$ and therefore there exists a (unique) $\xi_n \in (0, \alpha_n)$ such that $u'_{\alpha_n} > 0$ on $[0, \xi_n)$, $u'_{\alpha_n} < 0$ on $(\xi_n, \alpha_n]$ and $u'_{\alpha_n}(\xi_n) = 0$. Integrating the inequalities

$$\begin{aligned} &(r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' r(u_{\alpha_n}(t))u'_{\alpha_n}(t) \\ &\qquad \geq -Qg(u_{\alpha_n}(t))r(u_{\alpha_n}(t))u'_{\alpha_n}(t), \quad t \in (0, \xi_n) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &(r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' r(u_{\alpha_n}(t))u'_{\alpha_n}(t) \\ &\qquad \leq -Qg(u_{\alpha_n}(t))r(u_{\alpha_n}(t))u'_{\alpha_n}(t), \quad t \in (\xi_n, \alpha_n) \end{aligned} \tag{2.12}$$

over $[0, \xi_n]$ and $[\xi_n, \alpha_n]$, we obtain

$$(r(0)u'_{\alpha_n}(0))^2 \leq 2Q \int_0^{\Lambda_+(\alpha_n)} g(s)r(s)\,ds \leq 2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds$$

and

$$(r(0)u'_{\alpha_n}(\alpha_n))^2 \leq 2Q \int_0^{\Lambda_+(\alpha_n)} g(s)r(s)\,ds \leq 2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds,$$

respectively. Hence

$$\begin{aligned} |r(u_{\alpha_n}(t))u'_{\alpha_n}(t)| &\leq r(0)\max\{u'_{\alpha_n}(0),\, |u'_{\alpha_n}(\alpha_n)|\} \\ &\leq \sqrt{2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds}, \quad t \in [0, \alpha_n],\ n \in \mathbb{N} \end{aligned} \tag{2.13}$$

and

$$|u'_{\alpha_n}(t)| \leq \frac{1}{r_0}\sqrt{2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds} \quad \text{for } t \in [0, \alpha_n],\ n \in \mathbb{N}. \tag{2.14}$$

In addition, by Lemma 2.5,

$$u_{\alpha_0}(t) \leq u_{\alpha_n}(t) \leq u_{\alpha_{n+1}}(t) \quad \text{for } t \in [0, \alpha_0],\ n \in \mathbb{N}. \tag{2.15}$$

From (2.14) and (2.15) we deduce that $\{u_{\alpha_n}(t)\}$ is uniformly convergent on $[0, \alpha_0]$ and let $\lim_{n\to\infty} u_{\alpha_n}(t) = u(t)$, $t \in [0, \alpha_0]$. Then $u \in C^0([0, \alpha_0])$, $u(0) = 0$, $u(t) \geq u_{\alpha_0}(t) > 0$ for $t \in (0, \alpha_0)$. Moreover, $u(\alpha_0) = 0$, since in the case that $u(\alpha_0) > 0$ it may be concluded from

$$u(\alpha_0) \leq u_{\alpha_n}(\alpha_0) = u_{\alpha_n}(\alpha_0) - u_{\alpha_n}(\alpha_n) = u'_{\alpha_n}(\eta_n)(\alpha_0 - \alpha_n),$$

where $\eta_n \in (\alpha_0, \alpha_n)$ that

$$\lim_{n\to\infty} u'_{\alpha_n}(\eta_n) \leq \lim_{n\to\infty} \frac{u(\alpha_0)}{\alpha_0 - \alpha_n} = -\infty,$$

contrary to (2.14). As

$$0 < f(t, u_{\alpha_{n+1}}(t)) \le f(t, u_{\alpha_n}(t)), \quad \lim_{n\to\infty} f(t, u_{\alpha_n}(t) = f(t, u(t)), \quad t \in (0, \alpha_0)$$

and (see (2.13))

$$\begin{aligned} 0 &> \int_0^{\alpha_0} q(t) f(t, u_{\alpha_n}(t))\, dt = r(u_{\alpha_n}(\alpha_0)) u'_{\alpha_n}(\alpha_0) - r(0) u'_{\alpha_n}(0) \\ &\ge -2\sqrt{2Q \int_0^{\Lambda_+(\alpha_1)} g(s) r(s)\, ds} \end{aligned}$$

for $n \in \mathbb{N}$, Fatou's and Levi's theorems give $q(\cdot) f(\cdot, u(\cdot)) \in L_1([0, \alpha_0])$ and

$$\lim_{n\to\infty} \int_0^t q(s) f(s, u_{\alpha_n}(s)\, ds = \int_0^t q(s) f(s, u(s)\, ds, \quad t \in [0, \alpha_0].$$

By (2.14), $\{u'_{\alpha_n}(0)\}$ is bounded and we may assume that it is convergent. Let $\lim_{n\to\infty} u'_{\alpha_n}(0) = A$. Letting $n \to \infty$ in

$$H(u_{\alpha_n}(t)) = r(0) u'_{\alpha_n}(0) t + \int_0^t \int_0^s q(v) f(v, u_{\alpha_n}(v))\, dv\, ds \quad \text{for } t \in [0, \alpha_0], \tag{2.16}$$

where H is given by (2.3), we get

$$H(u(t)) = r(0) A t + \int_0^t \int_0^s q(v) f(v, u(v))\, dv\, ds, \quad t \in [0, \alpha_0].$$

Then

$$u(t) = H^{-1}\Big(r(0) A t + \int_0^t \int_0^s q(v) f(v, u(v)\, dv\, ds \Big),$$

and so $u \in C^1([0, \alpha_0])$. Now from $r(u(t)) u'(t) = r(0) A + \int_0^t q(s) f(s, u(s))\, ds$, $t \in [0, \alpha_0]$, and the above proved properties of u, we see that u is a solution of problem (1.1), (2.1) with $a = 0$ and $b = \alpha_0$, and consequently $u = u_{\alpha_0}$ by Lemma 2.3. We have proved that $\lim_{n\to\infty} u_{\alpha_n}(t) = u_{\alpha_0}(t)$ uniformly on $[0, \alpha_0]$, which implies $\lim_{n\to\infty} \Lambda_+(\alpha_n) = \Lambda_+(\alpha_0)$, contrary to (2.10). Hence Λ_+ is continuous on the right on $(0, T)$.

Assume now that Λ_+ is discontinuous on the left at a point $\alpha_0 \in (0, T]$, i.e., there is an increasing sequence $\{\alpha_n\} \subset (0, \alpha_0)$ such that $\lim_{n\to\infty} \alpha_n = \alpha_0$ and

$$\lim_{n\to\infty} \Lambda_+(\alpha_n) < \Lambda_+(\alpha_0). \tag{2.17}$$

Then

$$u_{\alpha_n}(t) \le u_{\alpha_{n+1}}(t) \le u_{\alpha_0}(t) \quad \text{for } t \in [0, \alpha_n],\ n \in \mathbb{N} \tag{2.18}$$

by Lemma 2.5 and as above it can be verified that

$$\begin{aligned} |r(u_{\alpha_n}(t))u'_{\alpha_n}(t)| &\le \sqrt{2Q\int_0^{\Lambda_+(\alpha_0)} g(s)r(s)\,ds}, \\ |u'_{\alpha_n}(t)| &\le \frac{1}{r_0}\sqrt{2Q\int_0^{\Lambda_+(\alpha_0)} g(s)r(s)\,ds} \end{aligned} \tag{2.19}$$

and

$$0 \ge \int_0^t q(s)f(s,u_{\alpha_n}(s))\,ds \ge -2\sqrt{2Q\int_0^{\Lambda_+(\alpha_0)} g(s)r(s)\,ds} \tag{2.20}$$

for $t \in [0,\alpha_n]$ and $n \in \mathbb{N}$. By (2.18) and (2.19), $\{u_{\alpha_n}(t)\}$ is locally uniformly convergent on $[0,\alpha_0)$ and let $\lim_{n\to\infty} u_{\alpha_n}(t) = u(t)$, $t \in [0,\alpha_0)$. Then $u \in C^0([0,\alpha_0))$, $u(0) = 0$, $0 < u(t) \le u_{\alpha_0}(t)$ for $t \in [0,\alpha_0)$ and $\lim_{t\to\alpha_0^-} u(t) = 0$. From the inequalities $f(t,u_{\alpha_n}(t)) \ge f(t,u_{\alpha_{n+1}}(t)) \ge f(t,u_{\alpha_0}(t)) \ge 0$, (2.20), $\lim_{n\to\infty} f(t,u_{\alpha_n}(t)) = f(t,u(t))$ for $t \in (0,\alpha_0)$ and Fatou's theorem we obtain $q(\cdot)f(\cdot,u(\cdot)) \in L_1([0,\alpha_0])$. Define $u^* \in C^0([0,\alpha_0])$ by

$$u^*(t) = \begin{cases} u(t) & \text{for } t \in [0,\alpha_0) \\ 0 & \text{for } t = \alpha_0. \end{cases}$$

Without violating generality, we can assume that $\{u'_{\alpha_n}(0)\}$ is convergent and let $\lim_{n\to\infty} u'_{\alpha_n}(0) = B$. Taking the limit as $n \to \infty$ in (2.16) which now holds on $[0,\alpha_n]$, we obtain

$$H(u^*(t)) = r(0)Bt + \int_0^t \int_0^s q(v)f(v,u^*(v))\,dv\,ds \quad \text{for } t \in [0,\alpha_0].$$

Then $u^* \in C^1([0,\alpha_0])$ and u^* is a solution of problem (1.1), (2.1) with $a = 0$ and $b = \alpha_0$. Hence $u^* = u_{\alpha_0}$ and from $\lim_{n\to\infty} u_{\alpha_n}(t) = u_{\alpha_0}(t)$ locally uniformly on $[0,\alpha_0)$ we deduce that $\lim_{n\to\infty} \Lambda_+(\alpha_n) = \Lambda_+(\alpha_0)$, contrary to (2.17). It follows that Λ_+ is continuous on the left on $(0,T]$. Consequently, Λ_+ is continuous on $(0,T]$.

Finally, assume that $\lim_{\alpha\to 0^+} \Lambda_+(\alpha) = \mu > 0$. Let $\{\alpha_n\} \subset (0,T)$ be a decreasing sequence and $\lim_{n\to\infty} \alpha_n = 0$. Let $\Lambda_+(\alpha_n) = u_{\alpha_n}(\xi_n)$ with a $\xi_n \in (0,\alpha_n)$. Then $u_{\alpha_n}(\xi_n) \ge \mu$ and from $\mu \le u_{\alpha_n}(\xi_n) = u_{\alpha_n}(\xi_n) - u_{\alpha_n}(0) = u'_{\alpha_n}(\tau_n)\xi_n$, where $\tau_n \in (0,\xi_n)$, we have $u'_{\alpha_n}(\tau_n) \ge \mu/\xi_n$ for $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} u'_{\alpha_n}(\tau_n) \ge \lim_{n\to\infty} \mu/\xi_n = \infty$, contrary to (2.14). Hence $\lim_{\alpha\to 0^+} \Lambda_+(\alpha) = 0$. □

Lemma 2.7. Let assumptions $(H_1) - (H_3)$ be satisfied. Then Φ_+ is continuous nonincreasing on $[0,T)$ and

$$\lim_{\beta\to T^-} \Phi_+(\beta) = 0.$$

Proof. By Remark 2.1, for each $\gamma \in (0,T]$ the function $\tilde{u}_\gamma(t) = v_\gamma(T-t)$, $t \in [0,T-\gamma]$, is a (unique) solution of problem (2.4), (2.1) with $a = 0$ and $b = T-\gamma$. Set $\tilde{\Lambda}_+(\gamma) = \max\{\tilde{u}_\gamma(t) : 0 \le t \le \gamma\}$ for $\gamma \in (0,T]$. Applying Lemma 2.6 to equation (2.4), we see that $\tilde{\Lambda}_+$ is continuous and nondecreasing on $(0,T]$ and $\lim_{\gamma\to 0^+} \tilde{\Lambda}_+(\gamma) =$

0. The assertion of our lemma now follows from the equality $\Phi_+(\beta) = \tilde{\Lambda}_+(T-\beta)$ for $\beta \in [0,T)$. □

Lemma 2.8. Let assumptions $(H_1)-(H_3)$ be satisfied. Then Λ_- is continuous nonincreasing on $(0,T]$, Φ_- is continuous nondecreasing on $[0,T)$ and

$$\lim_{\alpha\to 0^+} \Lambda_-(\alpha) = \lim_{\beta\to T^-} \Phi_-(\beta) = 0.$$

Proof. Let $\tilde{\Lambda}_+$ and $\tilde{\Phi}_+$ be associated to problem (2.5), (2.1) analogously as Λ_+ and Φ_+ are to problem (1.1), (2.1). Then $\tilde{\Lambda}_+$ is continuous and nondecreasing on $(0,T]$, $\tilde{\Phi}_+$ is continuous and nonincreasing on $[0,T)$ and $\lim_{\alpha\to 0^+}\tilde{\Lambda}_+(\alpha) = \lim_{\beta\to T^-}\tilde{\Phi}_+(\beta) = 0$ by Lemmas 2.6 and 2.7. The assertions of the lemma follow immediately from the equalities $\Lambda_- = -\tilde{\Lambda}_+$ on $(0,T]$ and $\Phi_- = -\tilde{\Phi}_+$ on $[0,T)$ which we get applying Remark 2.2. □

Lemma 2.9. Let assumptions $(H_1)-(H_4)$ be satisfied. Then for each $\alpha_1,\ \alpha_2 \in (0,T]$, $\alpha_1 < \alpha_2$, the inequality

$$u_{\alpha_1}(t) < u_{\alpha_2}(t) \quad for\ t \in (0,\alpha_1] \tag{2.21}$$

holds.

Proof. Fix $0 < \alpha_1 < \alpha_2 \le T$. Then $0 = u_{\alpha_1}(\alpha_1) < u_{\alpha_2}(\alpha_1)$ and, by Lemma 2.5, $u_{\alpha_1}(t) \le u_{\alpha_2}(t)$ for $t \in (0,\alpha_1]$. If $u_{\alpha_1}(\xi) = u_{\alpha_2}(\xi)$ for some $\xi \in (0,\alpha_1)$, then $u'_{\alpha_1}(\xi) = u'_{\alpha_2}(\xi)$, and consequently $u_{\alpha_1} = u_{\alpha_2}$ in a neighbourhood of $t=\xi$ by assumption (H_4). Repeated application of this result enables us to prove that $u_{\alpha_1} = u_{\alpha_2}$ on $[0,\alpha_1)$, which is impossible. Hence (2.21) holds. □

Lemma 2.10. Under assumptions $(H_1)-(H_4)$, Λ_+ is increasing on $(0,T]$, Φ_+ is decreasing on $[0,T)$, Λ_- is decreasing on $(0,T]$ and Φ_- is increasing on $[0,T)$.

Proof. By Lemma 2.9, for each $0 < \alpha_1 < \alpha_2 \le T$, inequality (2.21) holds and from the definition of Λ_+ we have $\Lambda_+(\alpha_1) < \Lambda_+(\alpha_2)$. Hence Λ_+ is increasing on $(0,T]$. The other three assertions of the lemma can be verified from strict inequalities between solutions $v_{\alpha_1},\ v_{\alpha_2};\ \overline{u}_{\alpha_1},\ \overline{u}_{\alpha_2}$ and $\overline{v}_{\alpha_1},\ \overline{v}_{\alpha_2}$ with different α_1 and α_2. □

3. Existence results for w-solutions of problem (1.1)-(1.3)

Theorem 3.1. Let assumptions $(H_1)-(H_3)$ be satisfied. Then for each $\lambda \in (0,\infty)$ there exist at least two w-solutions of problem (1.1)-(1.3).

Proof. Fix $\lambda \in (0,\infty)$. By Lemmas 2.6 and 2.8, the function $\lambda\Lambda_+ + \Phi_-$ is continuous and nondecreasing on $(0,T)$ and $\lim_{\alpha\to 0^+}(\lambda\Lambda_+(\alpha) + \Phi_-(\alpha)) = \Phi_-(0) < 0$,

$\lim_{\alpha\to T^-}(\lambda\Lambda_+(\alpha)+\Phi_-(\alpha)) = \lambda\Lambda_+(T) > 0$. Hence the equation $\lambda\Lambda_+(\alpha)+\Phi_-(\alpha) = 0$ has at least one solution $\alpha_1 \in (0,T)$. Setting

$$x_1(t) = \begin{cases} u_{\alpha_1}(t) & \text{for } t \in [0,\alpha_1] \\ \overline{v}_{\alpha_1}(t) & \text{for } t \in (\alpha_1,T], \end{cases} \tag{3.1}$$

x_1 is a w-solution of problem (1.1)-(1.3). Analogously, the equation $\lambda\Phi_+(\alpha)+\Lambda_-(\alpha) = 0$ has at least one solution $\alpha_2 \in (0,T)$ since $\lambda\Phi_+ + \Lambda_-$ is continuous and nonincreasing on $(0,T)$ and $\lim_{\alpha\to 0^+}(\lambda\Phi_+(\alpha)+\Lambda_-(\alpha)) = \lambda\Phi_+(0) > 0$, $\lim_{\alpha\to T^-}(\lambda\Phi_+(\alpha)+\Lambda_-(\alpha)) = \Lambda_-(T) < 0$ by Lemmas 2.7 and 2.8. Then setting

$$x_2(t) = \begin{cases} \overline{u}_{\alpha_2}(t) & \text{for } t \in [0,\alpha_2] \\ v_{\alpha_2}(t) & \text{for } t \in (\alpha_2,T], \end{cases} \tag{3.2}$$

x_2 is the second w-solution of problem (1.1)-(1.3). From $x_1 > 0$ on $(0,\alpha_1)$ and $x_2 < 0$ on $(0,\alpha_2)$ we see that $x_1 \neq x_2$. □

Theorem 3.2. Let assumptions $(H_1)-(H_4)$ be satisfied. Then for each $\lambda \in (0,\infty)$ there exist precisely two w-solutions of problem (1.1)-(1.3).

Proof. Fix $\lambda \in (0,\infty)$. It follows from Lemma 2.10 and the properties of the functions Λ_+, Φ_+, Λ_- and Φ_- given in Lemmas 2.6–2.8 that the equations $\lambda\Lambda_+(\alpha)+\Phi_-(\alpha) = 0$ and $\lambda\Phi_+(\alpha)+\Lambda_-(\alpha) = 0$ have in $(0,T)$ the unique solutions α_1 and α_2, respectively. Now x_1 and x_2 defined by (3.1) and (3.2) are unique w-solutions of problem (1.1)-(1.3). □

4. **Existence results for solutions of problem (1.1)–(1.3)**

Let assumptions $(H_1)-(H_4)$ be satisfied. By Theorem 3.2, for each $\lambda \in (0,\infty)$ there exist precisely two w-solutions $x_1(t;\lambda)$ and $x_2(t;\lambda)$ of problem (1.1)-(1.3). If c_λ is the (unique) solution of the equation $\lambda\Lambda_+(c)+\Phi_-(c) = 0$ and α_λ is the (unique) solution of the equation $\lambda\Phi_+(\alpha)+\Lambda_-(\alpha) = 0$, then

$$x_1(t;\lambda) = \begin{cases} u_{c_\lambda}(t) & \text{for } t \in [0,c_\lambda] \\ \overline{v}_{c_\lambda}(t) & \text{for } t \in (c_\lambda,T] \end{cases}$$

and

$$x_2(t;\lambda) = \begin{cases} \overline{u}_{\alpha_\lambda}(t) & \text{for } t \in [0,\alpha_\lambda] \\ v_{\alpha_\lambda}(t) & \text{for } t \in (\alpha_\lambda,T]. \end{cases}$$

Here solutions u_α, v_β, $\overline{u}_\alpha$ and $\overline{v}_\beta$ were defined in Section 2. Of course,

$$\lambda \max\{u_{c_\lambda}(t) : 0 \le t \le c_\lambda\} = -\min\{\overline{v}_{c_\lambda} : c_\lambda \le t \le T\},$$
$$\lambda \max\{v_{\alpha_\lambda}(t) : \alpha_\lambda \le t \le T\} = -\min\{\overline{u}_{\alpha_\lambda} : 0 \le t \le \alpha_\lambda\}$$

and c_λ (resp. α_λ) is the (unique) zero of $x_1(t;\lambda)$ (resp. $x_2(t;\lambda)$) in $(0,T)$.

Lemma 4.1. Let assumptions $(H_1)-(H_4)$ be satisfied. Then *a)* c_λ is continuous and decreasing on $(0,\infty)$,

$$\lim_{\lambda\to\infty} c_\lambda = 0, \quad \lim_{\lambda\to 0^+} c_\lambda = T,$$

b) $u_{c_{\lambda_1}}(t) > u_{c_{\lambda_2}}(t)$ for $t\in(0,c_{\lambda_2}]$ and $0<\lambda_1<\lambda_2$.

Proof. We know (see the proof of Theorem 3.2) that c_λ is the (unique) solution of the equation $\lambda\Lambda_+(c)+\Phi_-(c)=0$. Hence the equality $\lambda\Lambda_+(c_\lambda)+\Phi_-(c_\lambda)=0$ holds for $\lambda\in(0,\infty)$. Let $0<\lambda_1<\lambda_2$. If $c_{\lambda_1}\le c_{\lambda_2}$, then from the properties of the functions Λ_+ and Φ_- given in Lemmas 2.6–2.10 it follows that $0=\lambda_1\Lambda_+(c_{\lambda_1})+\Phi_-(c_{\lambda_1})<\lambda_2\Lambda_+(c_{\lambda_2})+\Phi_-(c_{\lambda_2})$, contrary to $\lambda_2\Lambda_+(c_{\lambda_2})+\Phi_-(c_{\lambda_2})=0$. Therefore c_λ is decreasing on $(0,\infty)$.

Assume that c_λ is discontinuous at a point $\lambda_0\in(0,\infty)$. Then there is a sequence $\{\lambda_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\lambda_n=\lambda_0$ such that $\lim_{n\to\infty}c_{\lambda_n}=\mu_0\ne c_{\lambda_0}$. Letting $n\to\infty$ in the equalities $\lambda_n\Lambda_+(c_{\lambda_n})+\Phi_-(c_{\lambda_n})=0$, $n\in\mathbb{N}$, we get

$$\lambda_0\Lambda_+(\mu_0)+\Phi_-(\mu_0)=0 \tag{4.1}$$

since Λ_+ and Φ_- are continuous. But the equation $\lambda_0\Lambda_+(c)+\Phi_-(c)=0$ has the unique solution $c=c_{\lambda_0}$, contrary to (4.1). Hence c_λ is continuous on $(0,\infty)$.

Suppose $\lim_{\lambda\to\infty}c_\lambda=\mu>0$. Then $\Lambda_+(c_\lambda)\ge\Lambda_+(\mu)>0$ and $\Phi_-(c_\lambda)\le\Phi_-(\mu)<0$ for $\lambda\in(0,\infty)$, and so $\lim_{\lambda\to\infty}(\lambda\Lambda_+(c_\lambda)+\Phi_-(c_\lambda))=\infty$, contrary to

$$\lambda\Lambda_+(c_\lambda)+\Phi_-(c_\lambda))=0 \quad \text{for } \lambda\in(0,\infty). \tag{4.2}$$

Therefore $\lim_{\lambda\to\infty}c_\lambda=0$. If $\lim_{\lambda\to0^+}c_\lambda=\varrho<T$, then $\lim_{\lambda\to0^+}\Lambda_+(c_\lambda)=\Lambda_+(\varrho)>0$, $\lim_{\lambda\to0^+}\Phi_-(c_\lambda)=\Phi_-(\varrho)<0$, and so $\lim_{\lambda\to0^+}(\lambda\Lambda_+(c_\lambda)+\Phi_-(c_\lambda))=\Phi_-(\varrho)<0$, contrary to (4.2). Hence $\lim_{\lambda\to0^+}c_\lambda=T$.

Finally, if $0<\lambda_1<\lambda_2$, then $c_{\lambda_1}>c_{\lambda_2}$ and $u_{c_{\lambda_1}}(t)>u_{c_{\lambda_2}}(t)$ for $t\in(0,c_{\lambda_2}]$ by Lemma 2.9. □

Lemma 4.2. Let assumptions $(H_1)-(H_4)$ be satisfied and let $\{\lambda_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\lambda_n=\lambda_0>0$. Then

$$\lim_{n\to\infty}u_{c_{\lambda_n}}(t)=u_{c_{\lambda_0}}(t) \quad \textit{locally uniformly on } [0,c_{\lambda_0}).$$

Proof. First from (2.14) it follows that

$$|u'_{c_{\lambda_n}}(t)|\le\frac{1}{r_0}\sqrt{2Q\int_0^{\Lambda_+(T)}g(s)r(s)\,ds} \quad \text{for } t\in[0,c_{\lambda_n}],\ n\in\mathbb{N}. \tag{4.3}$$

Now from (4.3) and using the fact that for $\{\lambda_n\}$ decreasing, $\{c_{\lambda_n}\}$ is increasing and

$$u_{c_{\lambda_{n+1}}}(t)>u_{c_{\lambda_n}}(t), \quad t\in[0,c_{\lambda_n}],\ n\in\mathbb{N}$$

and for $\{\lambda_n\}$ increasing, $\{c_{\lambda_n}\}$ is decreasing and

$$u_{c_{\lambda_{n+1}}}(t)<u_{c_{\lambda_n}}(t), \quad t\in[0,c_{\lambda_{n+1}}],\ n\in\mathbb{N}$$

(see Lemma 4.1), we deduce the assertion of our lemma. □

Define the function $S_+ : (0,\infty) \to (-\infty, 0]$ by the formula

$$S_+(\lambda) = u'_{c_\lambda}(c_\lambda),$$

where $u'_{c_\lambda}(c_\lambda)$ denotes the derivative of $u_{c_\lambda}(t)$ on the left at the point $t = c_\lambda$.

Lemma 4.3. Let assumptions $(H_1) - (H_4)$ be satisfied. Then S_+ is continuous on $(0,\infty)$ and

$$\lim_{\lambda\to\infty} S_+(\lambda) = 0, \quad \limsup_{\lambda\to 0^+} S_+(\lambda) < 0.$$

Proof. Assume, on the contrary, that S_+ is discontinuous at a point $\lambda_0 \in (0,\infty)$. Then there exist $\varepsilon_0 > 0$ and a sequence $\{\lambda_n\} \subset (\lambda_0/2, 2\lambda_0)$, $\lim_{n\to\infty} \lambda_n = \lambda_0$ such that $|S_+(\lambda_n) - S_+(\lambda_0)| \ge \varepsilon_0$ for $n \in \mathbb{N}$, that is

$$|u'_{c_{\lambda_n}}(c_{\lambda_n}) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \ge \varepsilon_0 \quad \text{for } n \in \mathbb{N}. \tag{4.4}$$

We claim that there exists $\nu > 0$ such that

$$|u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_n}}(c_{\lambda_n})| < \frac{\varepsilon_0}{2} \quad \text{for } t \in [c_{\lambda_n} - \nu, c_{\lambda_n}],\ n \in \mathbb{N}. \tag{4.5}$$

If not, without restriction of generality we can assume that there is a sequence $\{\tau_n\} \subset (0,T)$, $\tau_n < c_{\lambda_n}$, $\lim_{n\to\infty}(\tau_n - c_{\lambda_n}) = 0$ such that

$$|u'_{c_{\lambda_n}}(\tau_n) - u'_{c_{\lambda_n}}(c_{\lambda_n})| = \frac{\varepsilon_0}{2} \quad \text{for } n \in \mathbb{N}. \tag{4.6}$$

If $u'_{c_{\lambda_n}}(\tau_n) \le 0$, then $r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \le 0$ for $t \in [\tau_n, c_{\lambda_n}]$ and integrating the inequality

$$(r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t))' r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \le -Qg(u_{c_{\lambda_n}}(t))r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \tag{4.7}$$

from τ_n to c_{λ_n} we get

$$\begin{aligned} &(0 \le)\, (r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 - (r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \\ &\le -2Q \int_{u_{c_{\lambda_n}}(\tau_n)}^0 g(s)r(s)\,ds = 2Q \int_0^{u_{c_{\lambda_n}}(\tau_n)} g(s)r(s)\,ds. \end{aligned} \tag{4.8}$$

If $u'_{c_{\lambda_n}}(\tau_n) > 0$, then there exists $\xi_n \in (\tau_n, c_{\lambda_n})$ such that $u'_{c_{\lambda_n}}(\xi_n) = 0$ and then integrating the inequality

$$(r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t))' r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \ge -Qg(u_{c_{\lambda_n}}(t))r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t)$$

from τ_n to ξ_n and inequality (4.7) from ξ_n to c_{λ_n}, we have

$$(r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \le 2Q \int_{u_{c_{\lambda_n}}(\tau_n)}^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s)\,ds \tag{4.9}$$

and

$$(r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 \le 2Q \int_0^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s)\,ds. \tag{4.10}$$

Let $\mathbb{N}_+$ be the set of all $n \in \mathbb{N}$ such that $u'_{c_{\lambda_n}}(\tau_n) > 0$. Assume $\mathbb{N}_+$ is infinite. Using (4.3) and the equalities $\lim_{n\to\infty}(\xi_n - c_{\lambda_n}) = 0$, $\lim_{n\to\infty}(\xi_n - \tau_n) = 0$, we have

$$\lim_{n\in\mathbb{N}_+,\, n\to\infty} (u_{c_{\lambda_n}}(\xi_n) - u_{c_{\lambda_n}}(\tau_n)) = 0,$$

$$\lim_{n\in\mathbb{N}_+,\, n\to\infty} u_{c_{\lambda_n}}(\xi_n) = \lim_{n\in\mathbb{N}_+,\, n\to\infty} (u_{c_{\lambda_n}}(\xi_n) - u_{c_{\lambda_n}}(c_{\lambda_n})) = 0,$$

and so (4.9), (4.10) and $r(x) \geq r_0 > 0$ for $x \in \mathbb{R}$ yield

$$\lim_{n\in\mathbb{N}_+,\, n\to\infty} u'_{c_{\lambda_n}}(\tau_n) = \lim_{n\in\mathbb{N}_+,\, n\to\infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = 0,$$

contrary to (4.6). Hence $\mathbb{N}_+$ is finite and there is no loss of generality in assuming $u'_{c_{\lambda_n}}(\tau_n) \leq 0$ for $n \in \mathbb{N}$ and then (see (4.8))

$$(0 \leq)\, (r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 - (r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \leq 2Q \int_0^{u_{c_{\lambda_n}}(\tau_n)} g(s)r(s)\, ds \tag{4.11}$$

for $n \in \mathbb{N}$. From Lemma 1.2 in [14] (with $\mu = 1$) it follows that

$$u'_{c_{\lambda_n}}(c_{\lambda_n}) \leq -\frac{2K}{Vc_{\lambda_0/2}} \quad \text{for } n \in \mathbb{N}\,, \tag{4.12}$$

where

$$V = \max\Big\{r(x) : 0 \leq x \leq \max\{u_T(t) : 0 \leq t \leq T\}\Big\}, \tag{4.13}$$

$$K = \min\Big[\min\Big\{\int_0^{t/2} s|q(s)|k(s)\, ds, \int_{t/2}^{t} (t-s)|q(s)|k(s)\, ds\Big\} : \frac{\lambda_0}{2} \leq t \leq 2\lambda_0\Big]$$

and $k \in C^0([0,T])$ is a positive function such that

$$0 < k(t) \leq f(t,x)\operatorname{sign} x \quad \text{for } (t,x) \in [0,T] \times ([-\|u_T\|, 0) \cup (0, \|u_T\|]) \tag{4.14}$$

with $\|u_T\| = \max\{u_T(t) : 0 \leq t \leq T\}$ (for the function k see Remark 1.3). By (4.3), $\{u'_{c_{\lambda_n}}(c_{\lambda_n})\}$ and $\{u'_{c_{\lambda_n}}(\tau_n)\}$ are bounded, and so going if necessary to subsequences, we can assume that they are convergent, say

$$\lim_{n\to\infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = A, \quad \lim_{n\to\infty} u'_{c_{\lambda_n}}(\tau_n) = B.$$

By virtue of (4.6), we have

$$|A - B| = \frac{\varepsilon_0}{2}. \tag{4.15}$$

In addition, $\lim_{n\to\infty} u_{c_{\lambda_n}}(\tau_n) = \lim_{n\to\infty}(u_{c_{\lambda_n}}(\tau_n) - u_{c_{\lambda_n}}(c_{\lambda_n})) = 0$ since (4.3) holds and $\lim_{n\to\infty}(\tau_n - c_{\lambda_n}) = 0$. Letting $n \to \infty$ in (4.11), we get $0 \leq (r(0))^2(A^2 - B^2) = 0$. Therefore $A^2 - B^2 = 0$ and since $A \leq -2K/(Vc_{\lambda_0/2})$ by (4.12) and $B \leq 0$, we have $A = B$, contrary to (4.15). We have proved that (4.5) holds. Let

$$|u'_{c_{\lambda_0}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \leq \frac{\varepsilon_0}{4} \quad \text{for } t \in [c_{\lambda_0} - \gamma, c_{\lambda_0}]\,, \tag{4.16}$$

where γ is a positive constant. Set $\kappa = \min\{\nu, \gamma\}$. Using (4.4) and (4.5) we have

$$|u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \geq |u'_{c_{\lambda_n}}(c_{\lambda_n}) - u'_{c_{\lambda_0}}(c_{\lambda_0})| - |u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_n}}(c_{\lambda_n})| > \frac{\varepsilon_0}{2}$$

for $t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}]$ and $n \in \mathbb{N}$. Assume that

$$\mathbb{N}^* = \Big\{ n : n \in \mathbb{N},\, u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0}) > \frac{\varepsilon_0}{2} \text{ for } t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}] \Big\}$$

is an infinite set (analogously for $\mathbb{N} \setminus \mathbb{N}^*$ infinite). Then

$$u_{c_{\lambda_n}}(t) = \int_{c_{\lambda_n}}^{t} u'_{c_{\lambda_n}}(s)\,ds \le \Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{2} \Big)(t - c_{\lambda_n}) \tag{4.17}$$

for $t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}]$ and $n \in \mathbb{N}^*$. On the other hand, (4.16) gives

$$\begin{aligned} &\Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{4} \Big)(t - c_{\lambda_0}) \le u_{c_{\lambda_0}}(t) \\ &= \int_{c_{\lambda_0}}^{t} u'_{c_{\lambda_0}}(s)\,ds \le \Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) - \frac{\varepsilon_0}{4} \Big)(t - c_{\lambda_0}) \end{aligned} \tag{4.18}$$

for $t \in [c_{\lambda_0} - \kappa, c_{\lambda_0})$. Since $\lim_{n\to\infty}(c_{\lambda_n} - \kappa) = c_{\lambda_0} - \kappa$, there exists $n_0 \in \mathbb{N}$ such that for $n \in \mathbb{N}^*$, $n \ge n_0$, we have $c_{\lambda_n} - \kappa \le c_{\lambda_0} - \kappa/2$ and then letting $n \to \infty$ in (4.17) and using Lemmas 4.1 and 4.2,

$$u_{c_{\lambda_0}}(t) \le \Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{2} \Big)(t - c_{\lambda_0}), \quad t \in \Big[c_{\lambda_0} - \frac{\kappa}{2}, c_{\lambda_0} \Big],$$

contrary to (4.18). We have proved that S_+ is continuous on $(0,\infty)$.

Let $\{\lambda_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \lambda_n = \infty$. Then $\lim_{n\to\infty} c_{\lambda_n} = 0$ by Lemma 4.1, and there exists $\{\xi_n\}$, $0 < \xi_n < c_{\lambda_n}$, such that $u'_{c_{\lambda_n}}(\xi_n) = 0$ and $r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) < 0$ on $(\xi_n, c_{\lambda_n}]$ and $n \in \mathbb{N}$. Integrating (4.7) from ξ_n to c_{λ_n} we get

$$(r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 \le 2Q \int_0^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s)\,ds \quad \text{for } n \in \mathbb{N}. \tag{4.19}$$

By Lemma 1.2 in [14] (with $\mu = 1$), $u_{c_{\lambda_n}}(t) \le L_n$ for $t \in [0, c_{\lambda_n}]$ where $L_n > 0$ is an arbitrary constant satisfying the inequality

$$1 \le \frac{2\Big(\displaystyle\int_0^{L_n} r(s)\,ds \Big)^2}{(c_{\lambda_n})^2 Q \displaystyle\int_0^{L_n} g(s)r(s)\,ds}.$$

From the last inequality we see that L_n can be chosen such that $\lim_{n\to\infty} L_n = 0$ and then (4.19) yields $\lim_{n\to\infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = 0$. Hence $\lim_{\lambda\to\infty} S_+(\lambda) = 0$.

Let $\{\lambda_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \lambda_n = 0$. Then $\lim_{n\to\infty} c_{\lambda_n} = T$ by Lemma 4.1, and from Lemma 1.2 in [14] (with $\mu = 1$) we deduce that for each $n \in \mathbb{N}$ such that $c_{\lambda_n} \ge T/2$,

$$u'_{c_{\lambda_n}}(c_{\lambda_n}) \le -\frac{2K_1}{VT}, \quad n \in \mathbb{N},$$

where V is given by (4.13) and

$$K_1 = \min\Big[\min\Big\{ \int_0^{t/2} s|q(s)|k(s)\,ds,\, \int_{t/2}^{t} (t-s)|q(s)|k(s)\,ds \Big\} : \frac{T}{2} \le t \le T \Big]$$

with $k \in C^0([0,T])$ satisfying (4.14). Hence $\limsup_{\lambda \to 0^+} S_+(\lambda) \le -2K_1/(VT)$. $\square$

Define the functions $S_- : (0,\infty) \to (0,\infty)$, $Z_+ : (0,\infty) \to (0,\infty)$ and $Z_- : (0,\infty) \to (-\infty,0)$ by the formulas

$$S_-(\lambda) = \overline{u}'_{\alpha_\lambda}(\alpha_\lambda),$$
$$Z_+(\lambda) = v'_{c_\lambda}(\alpha_\lambda),$$
$$Z_-(\lambda) = \overline{v}'_{\alpha_\lambda}(c_\lambda).$$

We observe that c_λ (resp. α_λ) is the (unique) solution of the equation $\lambda\Lambda_+(c) + \Phi_-(c) = 0$ (resp. $\lambda\Phi_+(\alpha) + \Lambda_-(\alpha) = 0$). From the properties of the functions Λ_+, Λ_-, Φ_+, Φ_-, using Remarks 2.1 and 2.2 and applying procedures analogical to those in the proofs of Lemmas 4.1–4.3, we can show properties of the functions S_-, Z_+ and Z_- which are given in the following lemma.

Lemma 4.4. Let assumptions $(H_1) - (H_4)$ be satisfied. Then the functions S_-, Z_+ and Z_- are continuous on $(0,\infty)$ and

$$\liminf_{\lambda\to 0^+} S_-(\lambda) > 0, \quad \lim_{\lambda\to\infty} S_-(\lambda) = 0,$$
$$\lim_{\lambda\to 0^+} Z_+(\lambda) = 0, \quad \liminf_{\lambda\to\infty} Z_+(\lambda) > 0,$$
$$\lim_{\lambda\to 0^+} Z_-(\lambda) = 0, \quad \limsup_{\lambda\to\infty} Z_-(\lambda) < 0.$$

Theorem 4.5. Let assumptions $(H_1) - (H_4)$ be satisfied. Then problem (1.1)-(1.3) has at least two solutions.

Proof. Define the function k, $p : (0,\infty) \to \mathbb{R}$ by

$$k(\lambda) = S_+(\lambda) - Z_-(\lambda), \quad p(\lambda) = S_-(\lambda) - Z_+(\lambda).$$

By Lemmas 4.3 and 4.4, the functions k and p are continuous on $(0,\infty)$ and

$$\limsup_{\lambda\to 0^+} k(\lambda) < 0, \quad \liminf_{\lambda\to\infty} k(\lambda) > 0,$$
$$\liminf_{\lambda\to 0^+} p(\lambda) > 0, \quad \limsup_{\lambda\to\infty} p(\lambda) < 0.$$

Hence there exist λ_1, $\lambda_2 \in (0,\infty)$ such that $k(\lambda_1) = 0$ and $p(\lambda_2) = 0$, that is $S_+(\lambda_1) = Z_-(\lambda_1)$ and $S_-(\lambda_2) = Z_+(\lambda_2)$. Then the functions

$$x_1(t) = \begin{cases} u_{c_1}(t) & \text{for } t \in [0,c_1] \\ \overline{v}_{c_1}(t) & \text{for } t \in (c_1,T] \end{cases}$$

and

$$x_2(t) = \begin{cases} \overline{u}_{\alpha_1}(t) & \text{for } t \in [0,\alpha_1] \\ v_{\alpha_1}(t) & \text{for } t \in (\alpha_1,T] \end{cases}$$

are solutions of problem (1.1)-(1.3), where c_1 (resp. α_1) is the (unique) solution of the equation $\lambda_1\Lambda_+(c) + \Phi_-(c) = 0$ (resp. $\lambda_2\Phi_+(\alpha) + \Lambda(\alpha) = 0$). Clearly, $x_1 \neq x_2$. $\square$

5. Exceptional n-sign-changing w-solutions of problem (1.1), (1.2), (5.1)

Let $c \in (0, T]$. In this Section we will use the following conditions

$$\max\{x(t) : 0 \le t \le T\} = -\min\{x(t) : 0 \le t \le T\}, \tag{5.1}$$

$$x(0) = 0, \quad x(c) = 0 \tag{5.2}$$

and

$$\max\{x(t) : 0 \le t \le c\} = -\min\{x(t) : 0 \le t \le c\}. \tag{5.3}$$

We note that (5.1) is (1.3) with $\lambda = 1$.

Definition 5.1. Let $n \in \mathbb{N}$, $n \ge 2$. We say that x is an *n-sign-changing w-solution of problem* (1.1), (5.2), (5.3) if x has precisely $n-1$ zeros $t_1 < t_2 < \cdots < t_{n-1}$ in $(0, c)$, $x \in C^0([0, c]) \cap C^1([0, c] \setminus \{t_1, t_2, \cdots, t_{n-1}\})$, there exist finite $\lim_{t \to t_i^-} x'(t)$, $\lim_{t \to t_i^+} x'(t)$ for $i = 1, 2, \cdots, n-1$, $r(x)x' \in C^1([0, c] \setminus \{t_1, t_2, \cdots, t_{n-1}\})$, x satisfies (5.2), equality (1.1) holds on $(0, c) \setminus \{t_1, t_2, \cdots, t_{n-1}\}$ and finally

$$\max\{x(t) : t_i \le t \le t_{i+2}\} \min\{x(t) : t_i \le t \le t_{i+2}\} < 0$$

for $i = 0, 1, \cdots, n-2$ with $t_0 = 0$ and $t_n = c$.

If, in addition,

$$\max\{|x(t)| : 0 \le t \le t_1\} = \max\{|x(t)| : t_j \le t \le t_{j+1}\}$$

for $j = 1, 2, \cdots, n-1$, we say that x is an *exceptional n-sign-changing w-solution of problem* (1.1), (5.2), (5.3). In case of $c = T$, x is called an *exceptional n-sign-changing w-solution of problem* (1.1), (1.2), (5.1).

Remark 5.2. We observe, that the notion of the w-solution of problem (1.1)-(1.3) with $\lambda = 1$ stated in Section 1 corresponds to the notion of exceptional 2-sign-changing w-solution of problem (1.1), (1.2), (5.1).

Before we give existence results for exceptional n-sign-changing w-solutions of problem (1.1), (1.2), (5.1), we will define a function Δ_+ whose properties are important in our next considerations.

Let assumptions $(H_1) - (H_4)$ be satisfied. Then, by Theorem 3.2 (with $\mu = 1$) and its proof, for each $c \in (0, T]$ there exists the unique w-solution of problem (1.1), (5.2), (5.3), which is positive in the right neighbourhood of $t = 0$. This solution we will denote throughout this Section by w_c. Using w_c we define the function $\Delta_+ : (0, T] \to (0, \infty)$ by

$$\Delta_+(c) = \max\{w_c(t) : 0 \le t \le c\}.$$

Lemma 5.3. Let assumptions $(H_1) - (H_4)$ be satisfied. Then Δ_+ is continuous and increasing on $(0, T]$ and

$$\lim_{c \to 0^+} \Delta_+(c) = 0. \tag{5.4}$$

Proof. To prove that Δ_+ is increasing on $(0, T]$ we assume, on the contrary, that $\Delta_+(a) \geq \Delta_+(b)$ for some $0 < a < b \leq T$. Let $w_a(t_a) = 0$ and $w_b(t_b) = 0$ with unique $t_a \in (0, a)$ and $t_b \in (0, b)$. From Lemma 2.9 and our assumption $\Delta_+(a) \geq \Delta_+(b)$ we deduce that $t_a \geq t_b$. We claim that

$$w_b(t) < w_a(t) \quad \text{for } t \in (t_a, a]. \tag{5.5}$$

If not, since $0 = w_a(t_a) \geq w_b(t_a)$ and $0 = w_a(a) > w_b(a)$ we have either $w_a(\xi) = w_b(\xi)$ for some $\xi \in (t_a, a)$ and $w_b(t) \leq w_a(t)$ for $t \in [t_a, a]$ or there exist $t_a \leq \nu < \tau < a$ such that $w_a(\nu) = w_b(\nu)$, $w_a(\tau) = w_b(\tau)$ and $w_b(t) > w_a(t)$ for $t \in (\nu, \tau)$. In the first case $w_a'(\xi) = w_b'(\xi)$ and $w_a = w_b$ in a neighbourhood of $t = \xi$ by (H_4), and then by repeated application of (H_4) we get $w_a = w_b$ on (t_a, a), which is impossible. In the second case, we have $r(w_a(\nu))w_a'(\nu) \leq r(w_b(\nu))w_b'(\nu)$ and $f(t, w_a(t)) \geq f(t, w_b(t))$ for $t \in (\nu, \tau]$. Hence

$$\Big(\int_{w_a(t)}^{w_b(t)} r(s)\, ds \Big)'' = q(t)(f(t, w_b(t)) - f(t, w_a(t))) \geq 0, \quad t \in (\nu, \tau],$$

and so $\Big(\int_{w_a(t)}^{w_b(t)} r(s)\, ds \Big)'$ is nondecreasing on $[\nu, \tau]$ and then the equalities $w_a(\nu) = w_b(\nu)$, $w_a(\tau) = w_b(\tau)$ imply $\int_{w_a(t)}^{w_b(t)} r(s)\, ds = 0$ for $t \in [\nu, \tau]$, contrary to $w_b > w_a$ on (ν, τ). Now (5.5) yields $\min\{w_b(t) : 0 \leq t \leq b\} < \min\{w_a(t) : 0 \leq t \leq a\}$, hence $\Delta_+(b) > \Delta_+(a)$, contrary to our assumption $\Delta_+(a) \geq \Delta_+(b)$. We have proved that Δ_+ is increasing on $(0, T]$.

Suppose that Δ_+ is discontinuous on the right at a point $c_0 \in (0, T)$, i.e., there is a decreasing sequence $\{c_n\} \subset (c_0, T)$ such that $\lim_{n \to \infty} c_n = c_0$ and

$$\lim_{n \to \infty} \Delta_+(c_n) = \mu > \Delta_+(c_0). \tag{5.6}$$

Let $w_{c_n}(t_n) = 0$ for the (unique) $t_n \in (0, c_n)$, $n \in \mathbb{N} \cup \{0\}$. Since $\mu < \Delta_+(c_{n+1}) < \Delta_+(c_n)$ for $n \in \mathbb{N}$, Lemma 2.9 shows that $t_0 < t_{n+1} < t_n$ for $n \in \mathbb{N}$. There is no loss of generality in assuming $t_1 < c_0$. Moreover, $\Delta_+(c_n) = \Lambda_+(t_n)$ for $n \in \mathbb{N} \cup \{0\}$ and from $\Delta_+(c_0) < \mu < \Lambda_+(t_n)$ and the continuity of Λ_+ by Lemma 2.6, we see that $\lim_{n \to \infty} t_n = t_* > t_0$. Applying the procedure as in the proof of Lemma 2.6 (now on $[t_n, c_n]$), we get

$$|r(w_{c_n}(t))w_{c_n}'(t)| \leq \sqrt{2Q \int_0^{\Lambda_+(t_n)} g(s)r(s)\, ds} \leq \sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s)r(s)\, ds} \tag{5.7}$$

for $t \in [t_n, c_n]$, $n \in \mathbb{N}$, and then

$$0 \le \int_{t_n}^{t} q(s) f(s, w_{c_n}(s))\, ds = r(w_{c_n}(t)) w'_{c_n}(t) - r(0) w'_{c_n}(t_n) \le 2\sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s) r(s)\, ds} \tag{5.8}$$

for $t \in [t_n, c_n]$ and $n \in \mathbb{N}$. By (5.7),

$$|w'_{c_n}(t)| \le S \quad \text{for } t \in [t_n, c_n],\ n \in \mathbb{N}, \tag{5.9}$$

where

$$S = \frac{1}{r_0} \sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s) r(s)\, ds}. \tag{5.10}$$

From (5.9), $0 \ge w_{c_n}(t) \ge -\Lambda_+(t_1)$ for $t \in [t_n, c_n]$, $n \in \mathbb{N}$, and the Arzelà–Ascoli theorem, we deduce that there exists a subsequence of $\{w_{c_n}\}$, which we denote by $\{w_{c_n}\}$ again, such that $\lim_{n\to\infty} w_{c_n}(t) = w(t)$ locally uniformly on $(t_*, c_0]$. Then $w \in C^0((t_*, c_0])$, $w \le 0$ on $(t_*, c_0]$ and $w(c_0) = 0$ since in the case that $w(c_0) < 0$ from the relations

$$\frac{w(c_0)}{2} \ge w_{c_n}(c_0) = w_{c_n}(c_0) - w_{c_n}(c_n) = w'_{c_n}(\xi_n)(c_0 - c_n)\,,$$

which are satisfied for sufficiently large n and where $\xi_n \in (c_0, c_n)$, we obtain

$$\lim_{n\to\infty} w'_{c_n}(\xi_n) \ge \lim_{n\to\infty} \frac{w(c_0)}{2(c_0 - c_n)} = \infty,$$

contrary to (5.9). We are going to show that $w(t) < 0$ on (t_*, c_0) and

$$\lim_{t\to t_*^+} w(t) = 0. \tag{5.11}$$

By (H_3) (see Remark 1.3), there exists a positive function $k \in C^0([0,T])$ such that

$$k(t) \le f(t,x)\operatorname{sign} x \qquad \text{for } (t,x) \in [0,T] \times [-\Delta_+(c_1), 0) \cup (0, \Delta_+(c_1)].$$

Now using our Remark 2.2 and Lemma 1.2 in [14] with $\mu = 1$, we get (for $n \in \mathbb{N}$)

$$w_{c_n}(t) \le \begin{cases} H^{-1}\Big(-\dfrac{2K_n(t - t_n)}{c_n - t_n}\Big) & \text{for } t \in \Big[t_n, \dfrac{t_n + c_n}{2}\Big] \\ H^{-1}\Big(-\dfrac{2K_n(c_n - t)}{c_n - t_n}\Big) & \text{for } t \in \Big(\dfrac{t_n + c_n}{2}, c_n\Big], \end{cases} \tag{5.12}$$

where

$$K_n = \min\Big\{ \int_{t_n}^{(t_n+c_n)/2} (s - t_n)|q(s)|k(s)\, ds,\ \int_{(t_n+c_n)/2}^{c_n} (c_n - s)|q(s)|k(s)\, ds \Big\}$$

and H^{-1} is the inverse to H given by (2.3). Let $t_n \le (3t_* + c_0)/4$ and $t_n + c_n \le (t_* + 3c_0)/2$ for $n \ge n_1$ with an $n_1 \in \mathbb{N}$. Then (for $n \ge n_1$)

$$\int_{t_n}^{(t_n+c_n)/2} (s - t_n)|q(s)|k(s)\, ds \ge \int_{(3t_*+c_0)/4}^{(t_*+c_0)/2} \Big(s - \frac{3t_* + c_0}{4}\Big)|q(s)|k(s)\, ds,$$

$$\int_{(t_n+c_n)/2}^{c_n}(c_n-s)|q(s)|k(s)\,ds \ge \int_{(t_*+3c_0)/4}^{c_0}(c_0-s)|q(s)|k(s)\,ds$$

and from (5.12) it follows that

$$w_{c_n}(t) \le \begin{cases} H^{-1}\Big(-\dfrac{2K(t-t_n)}{c_n-t_n}\Big) & \text{for } t\in\Big[t_n,\dfrac{t_n+c_n}{2}\Big] \\ H^{-1}\Big(-\dfrac{2K(c_n-t)}{c_n-t_n}\Big) & \text{for } t\in\Big(\dfrac{t_n+c_n}{2},c_n\Big], \end{cases}$$

where

$$K=\min\Big\{\int_{(3t_*+c_0)/4}^{(t_*+c_0)/2}\Big(s-\frac{3t_*+c_0}{4}\Big)|q(s)|k(s)\,ds,\ \int_{(t_*+3c_0)/4}^{c_0}(c_0-s)|q(s)|k(s)\,ds\Big\}.$$

Consequently,

$$w(t)=\lim_{n\to\infty}w_{c_n}(t)\le\begin{cases} H^{-1}\Big(-\dfrac{2K(t-t_*)}{c_0-t_*}\Big)<0 & \text{for } t\in\Big(t_*,\dfrac{t_*+c_0}{2}\Big] \\ H^{-1}\Big(-\dfrac{2K(c_0-t)}{c_0-t_*}\Big)<0 & \text{for } t\in\Big(\dfrac{t_*+c_0}{2},c_0\Big), \end{cases}$$

and we see that $w<0$ on (t_*,c_0). If (5.11) is not true, then there exist $\delta<0$ and a decreasing sequence $\{\nu_n\}\subset(t_*,c_0)$ such that $\lim_{n\to\infty}\nu_n=t_*$ and $w(\nu_n)\le\delta$ for $n\in\mathbb{N}$. Now let

$$\nu_{n_*}<t_*-\frac{r_0\delta}{4\sqrt{2Q\int_0^{\Lambda_+(t_1)}g(s)r(s)\,ds}}$$

for some $n_*\in\mathbb{N}$. Then there exists $n_2\in\mathbb{N}$ such that $w_{c_n}(\nu_{n_*})<\delta/2$ and

$$\frac{\delta}{2}>w_{c_n}(\nu_{n_*})=w_{c_n}(\nu_{n_*})-w_{c_n}(t_n)=w'_{c_n}(\varphi_n)(\nu_{n_*}-t_n)$$

for $n\ge n_2$, where $\varphi_n\in(t_n,\nu_{n_*})$. Therefore

$$w'_{c_n}(\varphi_n)<\frac{\delta}{2(\nu_{n_*}-t_n)}<\frac{\delta}{2(\nu_{n_*}-t_*)}<-\frac{2}{r_0}\sqrt{2Q\int_0^{\Lambda_+(t_1)}g(s)r(s)\,ds}$$

for $n\ge n_2$, contrary to (5.9).

Define $w_*:[t_*,c_0]\to(-\infty,0]$ by

$$w_*(t)=\begin{cases} w(t) & \text{for } t\in(t_*,c_0] \\ 0 & \text{for } t=t_*. \end{cases}$$

Then $w_*\in C^0([t_*,c_0])$, $w_*(t_*)=w_*(c_0)=0$ and $w_*<0$ for $t\in(t_*,c_0)$. We are now in a position to show that w_* is a solution of problem (1.1), (2.2) with $a=t_*$ and $b=c_0$. For this, we define for each $n\in\mathbb{N}$ the function $p_n:[t_*,c_0]\to(-\infty,0]$ by

$$p_n(t)=\begin{cases} f(t,w_{c_n}(t)) & \text{for } t\in(t_n,c_0] \\ 0 & \text{for } t\in[t_*,t_n]. \end{cases}$$

Since

$$0 \le \int_{t_*}^{c_0} q(t)p_n(t)\,dt = \int_{t_n}^{c_0} q(t)f(t, w_{c_n}(t))\,dt \le 2\sqrt{2Q\int_0^{\Lambda_+(t_1)} g(s)r(s)\,ds}$$

by (5.8) and $\lim_{n\to\infty} p_n(t) = f(t, w_*(t))$ for $t \in (t_*, c_0)$, Fatou's theorem gives $q(\cdot)f(\cdot, w_*(\cdot)) \in L_1([t_*, c_0])$. Fix $\beta \in (t_1, c_0)$. Going if necessary to a subsequence, we can assume that $\{r(w_{c_n}(\beta))w'_{c_n}(\beta)\}$ is convergent, $\lim_{n\to\infty} r(w_{c_n}(\beta))w'_{c_n}(\beta) = A$. Letting $n \to \infty$ in

$$\int_{w_{c_n}(\beta)}^{w_{c_n}(t)} r(s)\,ds = r(w_{c_n}(\beta))w'_{c_n}(\beta)(t-\beta) + \int_\beta^t \int_\beta^s q(v)f(v, w_{c_n}(v))\,dv\,ds,\ t \in [t_n, c_n]$$

and using the Lebesgue dominated theorem, we get

$$\int_{w_*(\beta)}^{w_*(t)} r(s)\,ds = A(t-\beta) + \int_\beta^t \int_\beta^s q(v)f(v, w_*(v))\,dv\,ds, \quad t \in [t_*, c_0].$$

Whence

$$w_*(t) = H^{-1}\Big(H(w_*(\beta)) + A(t-\beta) + \int_\beta^t \int_\beta^s q(v)f(v, w_*(v))\,dv\,ds\Big), \quad t \in [t_*, c_0]$$

and we see that $w_* \in C^1([t_*, c_0])$ and $(r(w_*(t))w'_*(t))' = q(t)f(t, w_*(t))$ for $t \in (t_*, c_0)$. Therefore w_* is a solution of problem (1.1), (2.2) with $a = t_*$ and $b = c_0$. Finally, applying Remarks 2.1 and 2.2 to Lemma 2.9, we have $w_{c_0}(t) < w_*(t)$ for $t \in [t_*, c_0)$, and consequently

$$-\Delta_+(c_0) = \min\{w_{c_0}(t) : t_0 \le t \le c_0\} < \min\{w_*(t) : t_* \le t \le c_0\} = -\mu,$$

contrary to (5.6). Hence Δ_+ is continuous on the right on $(0, T)$.

The continuity of Δ_+ on the left on $(0, T]$ can be proved similarly. □

Now, define the 'dual' function $\Delta_- : (0, T] \to (0, \infty)$ to Δ_+ by the formula

$$\Delta_-(c) = \min\{\overline{w}_c(t) : 0 \le t \le c\},$$

where $\overline{w}_c$ is the unique solution of problem (1.1), (5.2), (5.3) such that $\overline{w}_c < 0$ in the right neighbourhood of $t = 0$.

Lemma 5.4. Let assumptions $(H_1) - (H_4)$ be satisfied. Then Δ_- is continuous decreasing on $(0, T]$ and

$$\lim_{c\to 0^+} \Delta_-(c) = 0.$$

Proof. Since the proof of the lemma is similar to that of Lemma 5.3, we will omit it. □

Lemma 5.5. Let assumptions $(H_1) - (H_4)$ be satisfied. Then there exist exactly two exceptional 3-sign-changing w-solutions of problem (1.1), (1.2), (5.1).

Proof. By Lemmas 2.7, 2.10 and 5.3, Φ_+ is continuous decreasing on $[0,T)$, Δ_+ is continuous increasing on $(0,T]$ and $\lim_{c\to T^-}\Phi_+(c) = \lim_{c\to 0^+}\Delta_+(c) = 0$. Set $p_+(c) = \Delta_+(c) - \Phi_+(c)$ for $c \in (0,T)$. Then $p_+ \in C^0((0,T))$, $\lim_{c\to 0^+} p_+(c) = -\Phi_+(0) < 0$, $\lim_{c\to T^-} p_+(c) = \Delta_+(T) > 0$ and since p_+ is increasing on $(0,T)$, there is the unique $c_+ \in (0,T)$ such that $p_+(c_+) = 0$. Hence the function

$$x_+(t) = \begin{cases} w_{c_+}(t) & \text{for } t \in [0, c_+] \\ v_{c_+}(t) & \text{for } t \in (c_+, T] \end{cases}$$

is an exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1), which is positive in the right neighbourhood of $t = 0$. Assume that $\overline{x}_+$ is an additional exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1) having positive values in the right neighbourhood of $t = 0$ and let $\overline{x}_+(t_j) = 0$, $j = 1, 2$, with $0 < t_1 < t_2 < T$. Since $\overline{x}_+ \neq x_+$, it is necessary $t_2 \neq c_+$, say $t_2 > c_+$. Then $\Delta_+(t_2) - \Phi_+(t_2) > 0$ and from this inequality we deduce that

$$\max\{\overline{x}_+(t) : 0 \le t \le t_2\} = \Delta_+(t_2) > \Phi_+(t_2) = \max\{\overline{x}_+(t) : t_2 \le t \le T\},$$

contrary to the definition of an exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1).

By Lemmas 2.8, 2.10 and 5.4, Φ_- is continuous increasing on $[0,T)$, Δ_- is continuous decreasing on $(0,T]$ and $\lim_{c\to T^-}\Phi_-(c) = \lim_{c\to 0^+}\Delta_-(c) = 0$. Set $p_-(c) = \Delta_-(c) - \Phi_-(c)$ for $c \in (0,T)$. Then $p_- \in C^0((0,T))$, $\lim_{c\to 0^+} p_-(c) = -\Phi_-(0) > 0$, $\lim_{c\to T^-} p_-(c) = \Delta_-(T) < 0$ and since p_- is decreasing on $(0,T)$, there is the unique $c_- \in (0,T)$ such that $p_-(c_-) = 0$. The function

$$x_-(t) = \begin{cases} \overline{w}_{c_-}(t) & \text{for } t \in [0, c_-] \\ \overline{v}_{c_-}(t) & \text{for } t \in (c_-, T] \end{cases}$$

is an exceptional 3-sign-changing w-solution of problem in the right neighbourhood of $t = 0$ and x_- is the unique exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1) having negative value in the right neighbourhood of $t = 0$. □

Remark 5.6. Let assumptions $(H_1) - (H_4)$ be satisfied and let $c \in (0,T]$. Then $(H_1) - (H_4)$ are satisfied with c instead of T, and so there exist exactly two exceptional 3-sign-changing w-solutions of problem (1.1), (5.2), (5.3) by Lemma 5.5.

Theorem 5.7. Let assumptions $(H_1) - (H_4)$ be satisfied. Then for each $n \in \mathbb{N}$, $n \ge 2$, there exist exactly two exceptional n-sign-changing w-solutions of problem (1.1), (1.2), (5.1).

Proof. We proceed by induction. By Theorem 3.2 (with $\lambda = 1$) and Lemma 5.5, there exist exactly two exceptional j-sign-changing w-solutions of problem (1.1), (1.2), (5.1), $j = 2, 3$. In addition (see Remark 5.6), for each $c \in (0,T]$ there exists exactly two exceptional 3-sign-changing w-solutions of problem (1.1), (5.2), (5.3). Assuming that this result holds for $n = k \ge 3$, we will prove it for $n = k + 1$. Since the proof is similar to that of Lemma 5.5, we give only its main ideas.

First, we assume that for each $c \in (0,T]$ there exist exactly two exceptional k-sign-changing w-solutions w_{kc} and $\overline{w}_{kc}$ of problem $(1.1),(5.2),(5.3)$ such that w_{kc} is positive and $\overline{w}_{kc}$ is negative in the right neighbourhood of $t=0$. Now define $\Delta_+^k : (0,T] \to (0,\infty)$ and $\Delta_-^k : (0,T] \to (0,\infty)$ by

$$\Delta_+^k(c) = \max\{w_{kc}(t) : 0 \le t \le c\}$$

and

$$\Delta_-^k(c) = \max\{\overline{w}_{kc}(t) : 0 \le t \le c\}.$$

Further, we can proceed as in the proof of Lemmas 5.3 and 5.4 to verify that Δ_+^k and Δ_-^k are continuous increasing on $(0,T]$, $\lim_{c\to 0^+}\Delta_+^k(c) = \lim_{c\to 0^+}\Delta_-^k(c) = 0$. Finally, if k is an even positive integer, set

$$p_k^+(c) = \Delta_+^k(c) - \Phi_+(c), \quad p_k^-(c) = \Delta_-^k(c) + \Phi_-(c) \quad \text{for } c \in (0,T).$$

Since p_k^+ and p_k^- are continuous increasing on $(0,T)$, $\lim_{c\to 0^+} p_k^+(c) = -\Phi_+(0) < 0$, $\lim_{c\to T^-} p_k^+(c) = \Delta_+^k(T) > 0$, $\lim_{c\to 0^+} p_k^-(c) = \Phi_-(0) < 0$ and $\lim_{c\to T^-} p_k^-(c) = \Delta_-^k(T) > 0$, there exists the unique solution c_+ (resp. c_-) of the equation $p_k^+(c) = 0$ (resp. $p_k^-(c) = 0$). Then

$$x_+(t) = \begin{cases} w_{kc_+}(t) & \text{for } t \in [0,c_+] \\ v_{c_+}(t) & \text{for } t \in (c_+,T], \end{cases}$$

$$x_-(t) = \begin{cases} \overline{w}_{kc_-}(t) & \text{for } t \in [0,c_-] \\ \overline{v}_{c_-}(t) & \text{for } t \in (c_-,T] \end{cases}$$

are the unique two exceptional $(k+1)$-sign-changing w-solutions of problem (1.1), (1.2), (5.1).

If k is an odd positive integer, we now set

$$p_k^+(c) = \Delta_+^k(c) + \Phi_-(c), \quad p_k^-(c) = \Delta_-^k(c) - \Phi_+(c) \quad \text{for } c \in (0,T).$$

Then p_k^+ and p_k^- are continuous increasing on $(0,T)$, the equations $p_k^+(c) = 0$ and $p_k^-(c) = 0$ have the unique solutions c_+ and c_-, respectively, and

$$x_+(t) = \begin{cases} w_{kc_+}(t) & \text{for } t \in [0,c_+] \\ \overline{v}_{c_+}(t) & \text{for } t \in (c_+,T], \end{cases}$$

and

$$x_-(t) = \begin{cases} \overline{w}_{kc_-}(t) & \text{for } t \in [0,c_-] \\ v_{c_-}(t) & \text{for } t \in (c_-,T] \end{cases}$$

are the unique two exceptional $(k+1)$-sign-changing w-solutions of problem (1.1), (1.2), (5.1). □

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