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ON FUNCTIONS PRESERVING RANK OF MATRICES

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ABSTRACT. The paper is motivated by papers of E. Pap [4] and A. Marková [3]. We give a complete characterization of functions preserving ranks of all matrices, without assuming any regularity conditions.

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1. INTRODUCTION

The notion of g-calculus has been introduced by E. Pap in [4]. A. Marková in [3] observed that a fundamental role in this calculus is played by functions preserving ranks of matrices.

First of all let us introduce

Definition 1. *Function $f : \mathbb{R} \longrightarrow \mathbb{R}$ preserves rank of matrices if and only if the rank of a matrix $A = [a_{ij}]$ is equal to rank of the matrix $f(A) := [f(a_{ij})]$, for every matrix A, where $i = 1, 2 \ldots, n$, $j = 1, 2, \ldots, m$.*

We proved in [1] that a monotonic and continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ preserves ranks of matrices if and only if it is linear, i.e. $f(x) = c \cdot x$, $x \in \mathbb{R}$, for come constant $c \neq 0$.

In the present paper we show that the characterization holds true even without admitting additional assumptions on function f.

2. MAIN RESULTS

It follows immediately from properties of determinants:

Example 1. *The linear function $f(x) = c \cdot x$ on $\mathbb{R}$ with a constant $c \neq 0$ is a function preserving rank of matrices.*

Without additional assumptions on f (i.e. continuity or others) the following holds true

Lemma 1. *If rank $r(A) = r(f(A))$ for all square matrices of degree less or equal to 3, then there exists a constant $c \neq 0$ such that $f(x) = c \cdot x$.*

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function preserving rank of square matrices of degree less or equal to 3. For matrix $A = [a]$ of degree 1 it follows that $f(a) = 0$ if and only if $a = 0$. Let $f(1) = c$. From the above it follows that $c \neq 0$. From properties of determinants it follows that $r(A) = r(\frac{1}{c} f(A))$. Consider the function

$$g := \frac{1}{c} \cdot f \quad \text{on} \quad \mathbb{R}, \tag{2.1}$$

so that $g(1) = 1$. For any square matrix

$$B = \begin{bmatrix} 1 & b \\ a & a \cdot b \end{bmatrix},$$

where $a, b \in \mathbb{R}$, we obtain $r(B) = 1$. Then also $r(g(B)) = 1$. Thus

$$\det \begin{bmatrix} 1 & g(b) \\ g(a) & g(a \cdot b) \end{bmatrix} = 0$$

or $g(a \cdot b) - g(a) \cdot g(b) = 0$, i.e. g solves the multiplicative Cauchy equation

$$g(x \cdot y) = g(x) \cdot g(y) \quad for\ \ all \quad x, y \in \mathbb{R}. \tag{M}$$

Similarly for any square matrix of the form

$$C = \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & b \\ 1 & 1 & a+b \end{bmatrix}$$

with arbitrary $a, b \in \mathbb{R}$ the rank $r(C) = 2$. Hence also $r(g(C)) = 2$ and thus

$$\det \begin{bmatrix} 0 & 1 & g(a) \\ 1 & 0 & g(b) \\ 1 & 1 & g(a+b) \end{bmatrix} = 0,$$

i.e. $\det C = g(a) + g(b) - g(a+b) = 0$. In other words the additive Cauchy functional equation

$$g(x + y) = g(x) + g(y) \quad for\ \ all \quad x, y \in \mathbb{R} \tag{A}$$

is fulfilled by g. Now, from [2] (Theorem 1, page 356), it follows that the only functions satisfying simultaneously (A) and (M) are $g = 0$ or $g = id$. Since $g(1) = 1$, we see that in our case $g(x) = x$, $x \in \mathbb{R}$. From Definition 1 we obtain that $f(x) = c \cdot x$ for all $x \in \mathbb{R}$. □

The following theorem completely describes all functions preserving rank of matrices.

Theorem 1. *Function $f : \mathbb{R} \longrightarrow \mathbb{R}$ preserves rank of matrices if and only if $f(x) = c \cdot x$, where $c \neq 0$ is a constant.*

Proof. It follows immediately from Lemma 1 and Example 1. □

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