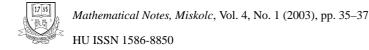


Miskolc Mathematical Notes Vol. 4 (2003), No 1, pp. 35-37 HU e-ISSN 1787-2413 DOI: 10.18514/MMN.2003.6

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[Received: June 18, 2002]

ABSTRACT. The paper is motivated by papers of E. Pap [4] and A. Marková [3]. We give a complete characterization of functions preserving ranks of all matrices, without assuming any regularity conditions.

Mathematics Subject Classification: 15A03

Keywords: rank of matrices

1. INTRODUCTION

The notion of g-calculus has been introduced by E. Pap in [4]. A. Marková in [3] observed that a fundamental role in this calculus is played by functions preserving ranks of matrices.

First of all let us introduce

Definition 1. Function $f : \mathbb{R} \longrightarrow \mathbb{R}$ preserves rank of matrices if and only if the rank of a matrix $A = [a_{ij}]$ is equal to rank of the matrix $f(A) := [f(a_{ij})]$, for every matrix A, where i = 1, 2, ..., n, j = 1, 2, ..., m.

We proved in [1] that a monotonic and continuous function $f : \mathbb{R} \to \mathbb{R}$ with f(0) = 0 preserves ranks of matrices if and only if it is linear, i.e. $f(x) = c \cdot x, x \in \mathbb{R}$, for come constant $c \neq 0$.

In the present paper we show that the characterization holds true even without admitting additional assumptions on function f.

2. MAIN RESULTS

It follows immediately from properties of determinants:

Example 1. The linear function $f(x) = c \cdot x$ on \mathbb{R} with a constant $c \neq 0$ is a function preserving rank of matrices.

Without additional assumptions on f (i.e. continuity or others) the following holds true

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Lemma 1. If rank r(A) = r(f(A)) for all square matrices of degree less or equal to 3, then there exists a constant $c \neq 0$ such that $f(x) = c \cdot x$.

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function preserving rank of square matrices of degree less or equal to 3. For matrix A = [a] of degree 1 it follows that f(a) = 0 if and only if a = 0. Let f(1) = c. From the above it follows that $c \neq 0$. From properties of determinants it follows that $r(A) = r(\frac{1}{c}f(A))$. Consider the function

$$g := \frac{1}{c} \cdot f \quad \text{on} \quad \mathbb{R}, \tag{2.1}$$

so that g(1) = 1. For any square matrix

$$B = \left[\begin{array}{cc} 1 & b \\ a & a \cdot b \end{array} \right],$$

where $a, b \in \mathbb{R}$, we obtain r(B) = 1. Then also r(g(B)) = 1. Thus

$$\det \begin{bmatrix} 1 & g(b) \\ g(a) & g(a \cdot b) \end{bmatrix} = 0$$

or $g(a \cdot b) - g(a) \cdot g(b) = 0$, i.e. g solves the multiplicative Cauchy equation

$$g(x \cdot y) = g(x) \cdot g(y)$$
 for all $x, y \in \mathbb{R}$. (M)

Similarly for any square matrix of the form

$$C = \left[\begin{array}{rrr} 0 & 1 & a \\ 1 & 0 & b \\ 1 & 1 & a+b \end{array} \right]$$

with arbitrary $a, b \in \mathbb{R}$ the rank r(C) = 2. Hence also r(g(C)) = 2 and thus

$$\det \begin{bmatrix} 0 & 1 & g(a) \\ 1 & 0 & g(b) \\ 1 & 1 & g(a+b) \end{bmatrix} = 0,$$

i.e. det C = g(a) + g(b) - g(a + b) = 0. In other words the additive Cauchy functional equation

$$g(x+y) = g(x) + g(y) \quad for \ all \quad x, y \in \mathbb{R}$$
(A)

is fulfilled by *g*. Now, from [2] (Theorem 1, page 356), it follows that the only functions satisfying simultaneously (A) and (M) are g = 0 or g = id. Since g(1) = 1, we see that in our case $g(x) = x, x \in \mathbb{R}$. From Definition 1 we obtain that $f(x) = c \cdot x$ for all $x \in \mathbb{R}$.

The following theorem completely describes all functions preserving rank of matrices.

Theorem 1. Function $f : \mathbb{R} \longrightarrow \mathbb{R}$ preserves rank of matrices if and only if $f(x) = c \cdot x$, where $c \neq 0$ is a constant.

Proof. It follows immediately from Lemma 1 and Example 1.

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