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Z. Szabó

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PARABOLA–PARABOLA COMBINED METHOD

Z. SZABÓ

Abstract. By using the always convergent method of tangential parabolas (but not using any interval-arithmetic tool), a combined root-finding iterative algorithm is given which provides a quadratically convergent descending sequence of compact real intervals J_n containing a simple zero of a twice differentiable real function defined on J_0.

Mathematics Subject Classification: 65H05

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1. Preliminaries, convergence, error estimation

In [1] and [2], we described two combined root-finding algorithms based on Newton's method and the method of tangential parabolas which is always convergent. The information used by the method in [1] comprises 4 Horner units: $f(a_n)$, $f'(a_n)$, $f(b_n)$, and $f'(b_n)$. This number is reduced to 2 in [2]. Keeping this number 2 of Horner's units used, we aim here at finding a faster combined method generating an interval sequence with the known features. The conditions (2), (3), and (6) in [2] will be replaced by the assumption

$$m_2 \le |f''(x)| \quad \text{in} \quad I = [a,b] \subset \mathbb{R}, \quad \text{for some} \quad m_2 > 0, \tag{1.1}$$

provided f'' does not change its sign in I.

In this new algorithm, the tangential straight line in the "Newton–Parabola" combined method described in [2] will be replaced by another tangential parabola lying "outside" the graph of f (the area between the curve of f and the x-axis is considered to be "inside"). For this "outer parabola," we first prove our

Lemma 1. *Assume that the function $f : I = [a,b] \to \mathbb{R}$ is twice differentiable in the compact interval $I \subset \mathbb{R}$, assumption* (1.1) *holds, and $f(x_0) \neq 0$ for some x_0 in I. Then, for the tangential parabola*

$$p(x) = f(x_0) + f'(x_0)(x - x_0) - \frac{1}{2} m_2 \operatorname{sign}(f(x_0))(x - x_0)^2,$$

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the difference $p(x) - f(x)$ *is non-negative (resp., non-positive), provided* $f(x_0)$ *is positive (resp., negative), and* f *is concave from below (resp., above) in* I.

In other words, we claim that $f(x_0)\,(p(x) - f(x)) \geq 0$ in I, provided the assumptions are fulfilled.

Proof of Lemma 1. Let $f(x_0) > 0$. Then we have

$$p(x) = f(x_0) + f'(x_0)(x - x_0) - \frac{1}{2}m_2(x - x_0)^2,$$
$$f(x) = f(x_0) + f'(x_0)(x - x_0) - \frac{1}{2}f''(c)(x - x_0)^2$$

for some $c \in (x_0, x)$, and $0 < m_2 \leq -f''(x)$, $x \in I$. Thus,

$$p(x) - f(x) = \frac{1}{2}(-f''(c) - m_2)(x - x_0)^2 \geq 0, \qquad x \in I,$$

because $-f''(c) = |f''(c)| \geq m_2$. When $f(x_0) < 0$, we have $m_2 \leq f''(x)$, $x \in I$, and the proof is similar, namely, in $p(x) - f(x)$, the term $-f''(c)$ will be replaced by $-m_2$. □

The convexity assumptions in Lemma 1 can be summarized as follows:

$$0 < m_2 \leq -\text{sign}\,(f(x_0))\,f''(x),\ \ x \in I.$$

We need now a more accurate notation for the iteration function of the "tangential parabola" method (briefly, TP-method) described in [1, pp. 581-582]. Let

$$F_{TP}(x, r, M) := \begin{cases} x + sf'(x)/M + r\left(2\,|f(x)|\,/M + (f'(x)/M)^2\right)^{1/2} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ s = f(x_0) \neq 0; \\ x \text{ if } s = 0. \end{cases}$$

In order to define our combined "parabola-parabola" method (PP-method for short), we assume that the following condition is satisfied:*

(A) The nonlinear function $f : I = [a, b] \to \mathbb{R}$ is twice differentiable on I†, the inequalities $|f''(x)| \leq M_2 \neq 0$, $x \in I$, and $f(a)f(b) < 0$ are fulfilled, and f'' does not change its sign on I.‡

In addition, we assume that (1.1) holds as well. Let, e. g. $f(a) > 0$ and $f''(x) < 0$. Thus, $f(a)f''(x) < 0$. (It does not matter whether $f'(a)$ is negative or non-negative.) Starting with the points $a_0 = a$ and $b_0 = b$, we construct the sequences

$$a_{n+1} = F_{TP}(a_n, 1, M_2), \qquad b_{n+1} = \min\{F_{TP}(a_n, 1, m_2), b\}, \tag{1.2}$$

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*Actually, assumption (A) is the same as (1) in [2].

†At the endpoints of I, the one-sided derivatives are to be considered.

‡It follows from (A) that f' is monotonic on I, f has a unique zero (say, α) in I, and α is a simple zero: $f(\alpha) = 0$, $f'(\alpha) \neq 0$.

for $n = 0, 1, 2, \ldots$. Since α is the only zero of f in (a, b) and, according to [1, pp. 581-582], F_{TP} is an always convergent iteration function: $F_{TP} \in A(f, I)$, and we have $\lim_{n\to\infty} a_n = \alpha$. So, after finitely many (say, N) steps,

$$b_N = \min\{F_{TP}(a_{N-1}, 1, m_2), b\} = F_{TP}(a_{N-1}, 1, m_2) \leq b$$

and, for $n \geq N$, the formulae (1.2) take the form

$$a_{n+1} = F_{TP}(a_n, 1, M_2), \quad b_{n+1} = F_{TP}(a_n, 1, m_2), \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

In this manner, we obtain the sequence of compact real intervals

$$J_n = [a_n, b_n]\,; \quad n = 0, 1, 2, \ldots; \qquad (J_0 = I).$$

It follows that

$$J_{n+1} \subset J_n,\ n = 0, 1, 2, \ldots, \quad \text{and} \quad \bigcap_{n=0}^{\infty} J_n = \alpha \in I, \quad f(\alpha) = 0, \quad f'(\alpha) \neq 0.$$

In general, we start our iteration from the endpoint $E \in \{a, b\}$ for which the inequality

$$f(E) f''(x) < 0 \tag{1.4}$$

is fulfilled. If, by using formulae similar to (1.3), a_n (or b_n) does not belong to I, then we choose the nearest endpoint of I to be a_n (or b_n). This nearest endpoint will be $\hat{E} = \{a, b\} \setminus \{E\}$. In particular, if $f(a)f''(x) < 0$, then $E = a$, $\hat{E} = b$, $r = 1$, and (1.2) remains the same; after a finite number of steps, we reach a point of iteration a_n such that f' keeps its sign in $[a_n, b]$; if $f(b)f''(x) < 0$, then $E = b$, $\hat{E} = a$, $r = -1$, and (1.2) becomes

$$a_{n+1} = \max\{F_{TP}(b_n, -1, m_2), a\}, \quad b_{n+1} = F_{TP}(b_n, -1, M_2), \tag{1.5}$$

for $n = 0, 1, 2, \ldots$. In this case, after finitely many steps, we reach a point b_n such that f' keeps its sign in $[a, b_n]$. On the other hand, we have $r = \operatorname{sign}((a+b)/2 - E) = \operatorname{sign}(a + b - 2E)$. An error estimate for our PP-method is contained in the following

Lemma 2. *Assume that conditions* (1.1) *and* (A) *are fulfilled, and f' keeps its sign in $I = J_0$. Then, for the diameters of the intervals J_n, we have the estimate*

$$d(J_{n+1}) \leq C(d(J_n))^2, \quad n = 0, 1, 2, \ldots,$$

where

$$C = \frac{1}{2}u^2\left(M_2 v^{-3} - m_2\left(u^2 + 2\,|f(E)|\,m_2\right)^{-3/2}\right),$$
$$u = |f'(\hat{E})| = \max\{|f'(a)|, |f'(b)|\},$$
$$v = |f'(E)| = \min\{|f'(a)|, |f'(b)|\}$$

and E, $\hat{E}$ are defined by the relations $\{E, \hat{E}\} = \{a, b\}$ *and* (1.4).

Proof. Let, e. g., $f(a) > 0$, $f''(x) < 0$. Then $E = a$, $\hat{E} = b$. By using the formulas

$$a_{n+1} = a_n + f'(a_n)/M_2 + (2f(a_n)/M_2 + (f'(a_n)/M_2)^2)^{1/2}$$

and

$$b_{n+1} = a_n + f'(a_n)/m_2 + (2f(a_n)/m_2 + (f'(a_n)/m_2)^2)^{1/2},$$

we get

$$\begin{aligned} d(J_{n+1}) = b_{n+1} - a_{n+1} &= f'(a_n)\,(1/m_2 - 1/M_2) \\ &+ \left(2f(a_n)/m_2 + (f'(a_n)/m_2)^2\right)^{1/2} \\ &- \left(2f(a_n)/M_2 + (f'(a_n)/M_2)^2\right)^{1/2} \\ &= 1/m_2\,(y + F(m_2)) - 1/M_2\,(y + F(M_2)) \end{aligned} \tag{1.6}$$

where $y = f'(a_n) = -|y| < 0$, $q = 2f(a_n)$ and $F(m) = (y^2 + qm)^{1/2}$. We take $F'(m) = \frac{1}{2}q(y^2+qm)^{-1/2}$ and $F''(m) = -\frac{1}{4}q^2(y^2+qm)^{-3/2}$. By using Taylor's formula, we obtain

$$\begin{aligned} F(m) &= F(0) + F'(0)m + \frac{1}{2}F''(w)m^2 \\ &= |y| + qm/(2\,|y|) - \frac{1}{8}q^2m^2(y^2 + qw)^{-3/2}, \end{aligned}$$

where $w \in (0, m)$. Then (1.6) takes the form

$$\begin{aligned} d(J_{n+1}) &= \frac{1}{m_2}\left(y + |y| + q\frac{m_2}{2\,|y|} - \frac{1}{8}q^2m_2^2\left(y^2 + qw\right)^{-3/2}\right) \\ &\quad - \frac{1}{M_2}\left(y + |y| + q\frac{M_2}{2\,|y|} - \frac{1}{8}q^2M_2^2\left(y^2 + qW\right)^{-3/2}\right) \\ &= \frac{q^2}{8}\left(M_2\left(y^2 + qW\right)^{-3/2} - m_2\left(y^2 + qw\right)^{-3/2}\right) \\ &= \frac{1}{2}f^2(a_n)\left(M_2\left(f'^2(a_n) + 2f(a_n)W\right)^{-3/2} - m_2\left(f'^2(a_n) + 2f(a_n)w\right)^{-3/2}\right), \end{aligned}$$

where $w \in (0, m_2)$ and $W \in (0, M_2)$. The factor $f^2(a_n)$ on the right-hand side can be replaced by $f'^2(t_n)(\alpha - a_n)^2$ for some $t_n \in (a_n, \alpha)$ because, by the mean value theorem, $f(\alpha) - f(a_n) = f'(t_n)(\alpha - a_n)$. Thus, one can estimate $d(J_{n+1})$ as follows:

$$\begin{aligned} d(J_{n+1}) &\le \frac{1}{2}f'^2(t_n)(\alpha - a_n)^2\left(M_2/\left|f'(a_n)\right|^3 - m_2\left(f'^2(a_n) + 2f(a_n)m_2\right)^{-3/2}\right) \\ &\le \frac{1}{2}f'^2(b)(\alpha - a_n)^2\left(M_2/\left|f'(a)\right|^3 - m_2\left(f'^2(b) + 2f(a)m_2\right)^{-3/2}\right) \\ &= \frac{1}{2}u^2(\alpha - a_n)^2\left(M_2/v^3 - m_2\left(u^2 + 2\,|f(E)|\,m_2\right)^{-3/2}\right) \\ &\le C(b_n - a_n)^2 = C\,(d(J_n))^2\,. \end{aligned}$$

In the remaining three cases corresponding to the signs of $f(a)$ and $f''(x)$, the proofs are quite similar. □

Remark 1. Assumptions (1.1) and (A) imply that

$$v < u < \left(u^2 + 2\,|f(E)|\,m_2\right)^{1/2},$$

so $C > 0$.

Remark 2. A geometrical representation shows that our PP-method is faster than the combined method described in [2]. (Namely, the "outer parabola" lies below (resp., above) the tangential line of Newton's method if $f(a_n) > 0$ (resp., $f(a_n) < 0$). This fact is also reflected in the error estimates because

$$C = K - \frac{1}{2}u^2 m_2 \left(u^2 + 2\,|f(E)|\,m_2\right)^{-3/2}.$$

Remark 3. If $|f(a_n)| \ll 1$ (i. e., a_n is close to α), then

$$C \approx \frac{1}{2}u^2\left(M_2/v^3 - m_2/u^3\right) = K - \frac{1}{2}m_2/u,$$

where K is the error constant in [2].

Remark 4. For the endpoint $\hat{E}$, we have $f(\hat{E})f''(x) > 0$. According to the Fourier conditions, the Newton method can be used, starting at $\hat{E}$ (like in [1]).

The results above can be summarized in our main

Theorem 1. *If assumptions* (1.1) *and* (A) *hold, then the sequence of compact real intervals* $J_n = [a_n, b_n]$, $n = 0, 1, 2, \ldots$, *generated by the combined PP-method described above has the following properties:*

1° $J_{n+1} \subset J_n$, $n = 0, 1, 2, \ldots$;
2° $\bigcap_{n=0}^{\infty} J_n = \alpha \in I$, $f(\alpha) = 0$, $f'(\alpha) \neq 0$;
3° *if* f' *keeps its sign on* $I = J_0$, *then, for the diameters of the resulting intervals, we have the estimate*

$$d(J_{n+1}) \le C\,(d(J_n))^2, \qquad n = 0, 1, 2, \ldots,$$

where

$$C = \frac{1}{2}u^2\left(M_2 v^{-3} - m_2\left(u^2 + 2\,|f(E)|\,m_2\right)^{-3/2}\right),$$

$$u = |f'(\hat{E})| = \max\{|f'(a)|, |f'(b)|\}, \quad v = |f'(E)| = \min\{|f'(a)|, |f'(b)|\},$$

and E, $\hat{E}$ *are defined by the relations* $\{E, \hat{E}\} = \{a, b\}$ *and* (1.4).

2. Algorithm and some numerical examples

The algorithm for the PP-method can be given by the following Boolean function procedure *PPM*. The input parameters are a, b, eps, $m2$, $M2$, f and $f1$ (corresponding to f'). If they fulfil the requirements $a < b$, eps > 0, $0 < m2 \le M2$ and $f(a)f(b) \le 0$, then *PPM*, the identifier of our Boolean procedure, will have the logical value *true*, otherwise *PPM* will be *false*. The other output parameters are the endpoints of the resulting interval $[A, B]$ including α (the zero of f) and having a diameter less than eps, provided *PPM* = *true*. Otherwise, $[A, B]$ will be the initial interval $[a, b]$.

The subroutines/procedures for the functions f and f' have to be defined by the user.

SUB is a real function procedure with the formal input parameter L. The actual input parameters are m_2 and M_2. This procedure actually calculates the value of $F_{TP}(z, r, L)$, the tangential parabola iterate of z for $L \in \{m_2, M_2\}$, and puts it into the output parameter *SUB*.

The integer variable k contains the information on convexity of the graph of f. If $k = 1$, then $(f(a) + f(b))/2 > f((a + b)/2)$ and f is convex; if $k = -1$, then f is concave. Thus, $k = \text{sign}(f(a) + f(b) - 2f((a + b)/2))$. (We have $k \neq 0$, i. e., f is non-linear because $m_2 > 0$.)

```
Boolean procedure PPM (a, b, eps, m2, M2, A, B);
begin integer k, r, s; real a, b, eps, m2, M2, A, B, F, H, P, y, z;
    real procedure f(x); real x;        (user should define f)
    real procedure f1(x); real x;       (user should define f1 = f′)
    real procedure SUB(L); real L;
    begin real T, S; T := 2 ⋆ abs (F)/L; S := H/L;
    SUB := z + s ⋆ S + r ⋆ sqr (T + S ⋆ S) end SUB;
        A := a; B := b;
        if  a < b and eps > 0 and 0 < m2 and m2 ≤ M2 then goto gd;
    bd: PPM := false; goto E;
    gd: PPM := true;
        F := f(a); if F = 0 then begin B := a; goto E end;
        P := f(b); if P = 0 then begin A := b; goto E end;
        if F ⋆ P > 0  then goto bd; k := sign (F + P − 2 ⋆ f((a + b)/2));
        if F ⋆ k < 0  then begin z := a; r := 1;  end
        else begin  z := b; F := P; r := −1;  end;
        H := f1(z); s := sign (F); y := SUB(m2);
        if a > y  then y := a;  if y > b  then y := b;
    W: z := SUB(M2); if abs (z − y) < eps  then goto T;
        F := f(z); H := f1(z); y := SUB(m2); goto W;
    T: if r > 0  then begin A := z; B := y;  end
        else begin  A := y; B := z;  end;
E: end  PPM;
```

Some numerical examples can be found in the following tables, where the correct digits are underlined.

Example 1. $f(x) = x - e^{-x}$, $a = 0$, $b = 1(= E)$, $M_2 = 1$, $m_2 = 0.35$;

n	a_n	b_n	$b_n - a_n$
0	0	1	1
1	0.56238349331149966899	0.59719164168881961091	$3 \cdot 10^{-2}$
2	0.56708373561334769584	0.56727015271662188072	$2 \cdot 10^{-4}$
3	0.56714328929501556755	0.56714329263260011212	$3 \cdot 10^{-9}$
4	0.56714329040978387265	0.56714329040978387368	$1 \cdot 10^{-18}$
5	0.56714329040978387300	0.56714329040978387300	$1 \cdot 10^{-37}$

Example 2. $f(x) = \tan^{-1} x - 2.6 + \sqrt{x}$, $a = 1$, $b = 4(= E)$, $M_2 = 0.75$, $m_2 = 0.035$;

n	a_n	b_n	$b_n - a_n$
0	1	4	3
1	1.89970378394449937319	2.96088085705371547709	1
2	2.06567277560842922080	2.36679176536415266599	$3 \cdot 10^{-1}$
3	2.13894682376643847337	2.16810797543185165243	$3 \cdot 10^{-2}$
4	2.14658693492719685263	2.14689875120987042863	$3 \cdot 10^{-4}$
5	2.14666632870554397592	2.14666636586609630990	$4 \cdot 10^{-8}$
6	2.14666633811284909659	2.14666633811284962657	$5 \cdot 10^{-16}$
7	2.14666633811284923074	2.14666633811284923074	$1 \cdot 10^{-31}$

Example 3. $f(x) = 1 - x - \sin x$, $a = 0.01$, $b = 1(= E)$, $M_2 = 0.842$, $m_2 = 0.0099$;

n	a_n	b_n	$b_n - a_n$
0	0.01	1	1
1	0.45465326096563166766	0.51736453936087952833	$6 \cdot 10^{-2}$
2	0.51096815380042764464	0.51097723467313242901	$9 \cdot 10^{-6}$
3	0.51097342938671630865	0.51097342938993405418	$3 \cdot 10^{-12}$
4	0.51097342938856910952	0.51097342938856910952	$4 \cdot 10^{-25}$

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Author's Address

Z. Szabó:

Sultan Qaboos University, Department of Mathematics and Statistics, P. O. Box 36., Al-Khodh P. C. 123, Sultanate of Oman

E-mail address: szabo@squ.edu.om