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BLOCK EIGENVALUES AND SOLUTIONS OF DIFFERENTIAL MATRIX EQUATIONS

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ABSTRACT. We present an application of block eigenvalues of the block companion matrix of a matrix polynomial $P(X)$ to obtain a general solution of the differential matrix equation associated with $P(X)$.

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1. INTRODUCTION

The preliminary theory on matrix polynomials revisited next can be found in [4], [1], [6], [3] and [5].

Let

$$P(X) = X^m + A_1 X^{m-1} + \ldots + A_m \tag{1.1}$$

be a monic (right) matrix polynomial of degree m in the indeterminate X with the coefficients $A_1, \ldots, A_m$ being $n \times n$ complex matrices. An $n \times n$ matrix X_1, such that $P(X_1) = 0$, is a (right) solvent of $P(X)$. Furthermore for each nonsingular $n \times n$ matrix formed of leading vectors of Jordan chains of $P(\lambda)$ it is possible to construct one solvent of $P(X)$ and the total number of solvents will be the number of such nonsingular matrices.

The matrix

$$C = \begin{bmatrix} 0_n & I_n & \ldots & 0_n \\ \vdots & & \ddots & \\ & & & I_n \\ -A_m & -A_{m-1} & \ldots & -A_1 \end{bmatrix}, \tag{1.2}$$

(where 0_n and I_n are the null matrix and the identity matrix of order n, respectively) associated with the coefficients of $P(X)$, is said to be the block companion matrix of $P(X)$. Moreover, if $X_1, \ldots, X_m$ are m solvents of $P(X)$, then the respective block

Vandermonde matrix is

$$V(X_1, \ldots, X_m) = \begin{bmatrix} I_n & \ldots & I_n \\ X_1 & \ldots & X_m \\ \vdots & & \vdots \\ X_1^{m-1} & \ldots & X_m^{m-1} \end{bmatrix}$$

and if the matrix $V(X_1, \ldots, X_m)$ is nonsingular, then we say that these m solvents form a complete set of solvents of $P(X)$.

We consider now a differential matrix equation, i. e., homogeneous ordinary differential equation of order m having $n \times n$ matrix coefficients, written by

$$P\left(\frac{d}{dt}\right)x = x^{(m)}(t) + A_1 x^{(m-1)}(t) + \ldots + A_m x(t) = 0. \tag{1.3}$$

An important result on this is that if $X_1, \ldots, X_m$ are a complete set of solvents of the matrix polynomial $P(X)$ associated with $P\left(\frac{d}{dt}\right)x$ (that is, having the same coefficients), then every solution of

$$P\left(\frac{d}{dt}\right)x = 0$$

is of the form

$$x(t) = e^{X_1 t}z_1 + e^{X_2 t}z_2 + \ldots + e^{X_m t}z_m, \tag{1.4}$$

where $z_1, z_2, \ldots, z_m \in \mathbb{C}^n$ ([5], p. 525).

2. Block Eigenvalues

We recall the classical equation of matrix theory, $AX = XB$, where A, B are given complex square matrices (see [2], p. 215). We work here with a variation of this equation that happens when X spans an invariant subspace of A, and only A is given. It appears in the computation of eigenvalues (see [8], p. 587).

Definition 1. Given a matrix A of order p, if a matrix Y of order $q < p$ is such that

$$AW = WY \tag{2.1}$$

for a rectangular matrix W of full rank. We say that Y is a (right) block eigenvalue of A and W is a corresponding (right) block eigenvector of dimension $q \times p$.

This definition can be restricted to block matrices, of order mn, partitioned in blocks of order n. In this case the block eigenvalues are of the same order as the blocks of the block matrix that is n and the corresponding block eigenvector is a vector of blocks of dimension $mn \times n$ (see [7]).

A block eigenvalue has the property that any similar block is also a block eigenvalue, and it is clear that a block eigenvector W spans an invariant subspace of A, since being of full rank is equivalent to having linearly independent columns.

Next we have the strong relationship between a matrix and a block eigenvalue (see also [9], Corollary II).

Theorem 1. *Let A be a matrix, then a matrix Y is a block eigenvalue of A, if and only if the eigenvalues of Y are also eigenvalues of A, and for each common eigenvalue α, the corresponding partial multiplicities $k_1(Y), \ldots, k_n(Y)$ in Y, and $k_1(A), \ldots, k_m(A)$ in A, where the integers k_i are in decreasing order of magnitude, satisfy*

(i) $n \leq m$;

(ii) $k_i(Y) \leq k_i(A)$, $i = 1, \ldots, n$.

Proof. First note that n and m are the geometric multiplicities of α in Y, and in A, or the number of Jordan blocks, of α in J_Y, and in J_A, the Jordan normal forms of Y and A, respectively. And that the k_i are the orders of these Jordan blocks. Let now $Y = TJ_YT^{-1}$, where T is a nonsingular matrix. Then suppose that $AW = WY$, with W of full rank, thus $AWT = WYT = WTJ_YT^{-1}T = WTJ_Y$. Since WT is still of full rank, it follows that the linearly independent columns of WT are eigenvectors or generalized eigenvectors of A, with respect to the eigenvalues of J_Y, thus the eigenvalues of J_Y (and of Y) are also of A. Furthermore, from $AWT = WTJ_Y$, it follows that J_Y is a submatrix of J_A. Therefore, for each common eigenvalue α, the corresponding geometric multiplicities m in A and n in J_Y, and hence in Y, satisfy $n \leq m$. Also the orders of the Jordan blocks of J_A and of J_Y corresponding to α, satisfy $k_i(Y) \leq k_i(A), i = 1, \ldots, n$. Conversely, suppose that the eigenvalues of Y (and hence of J_Y), are common to A. And supposing (i) and (ii) we can write $AZ = ZJ_Y$, where the columns of Z, eigenvectors or generalized eigenvectors of A, corresponding to the eigenvalues of J_Y, are linearly independent. Hence Z is of full rank, thus $AZT^{-1} = ZJ_YT^{-1} = ZT^{-1}Y$, with ZT^{-1} of full rank, and the conclusion is that Y is a block eigenvalue of A. □

We observe that, for each eigenvalue of a complex matrix, the respective number of partial multiplicities gives the geometric multiplicity, and therefore the number of Jordan blocks of the Jordan normal form of the matrix, for this eigenvalue. These partial multiplicities are the sizes of these Jordan blocks, hence we can conclude that, if we have the partial multiplicities of all the eigenvalues of a complex matrix, we can write its Jordan normal form. Considering that, we define a set of block eigenvalues in which this information can be obtained.

Definition 2. Let A be a matrix, and let $Y_1, \ldots, Y_k$ be a set of block eigenvalues of A. We say that this set is a complete set of block eigenvalues, if the eigenvalues, and respective partial multiplicities, of these block eigenvalues are the eigenvalues, with the same partial multiplicities, of the matrix A.

Theorem 2. *A set of block eigenvalues $Y_1, \ldots, Y_k$ of a matrix A, is a complete set, if and only if there is a set of corresponding block eigenvectors $W_1, \ldots, W_k$, such that the matrix $\begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix}$ is of full rank, and*

$$A\begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} diag(Y_1, \ldots, Y_k)\,, \tag{2.2}$$

where $diag(Y_1, \ldots, Y_k)$ is a block diagonal matrix of the same order of A.

Proof. Let $Y_1, \dots, Y_k$ be a complete set of block eigenvalues, and let

$$D = \mathrm{diag}(Y_1, \dots, Y_k)$$

be a block diagonal matrix. Since the eigenvalues of the $Y_1, \dots, Y_k$, and their partial multiplicities, are the same as those of A. The same happens to D, the direct sum of the $Y_1, \dots, Y_k$. Consequently A and D have the same Jordan normal form, and therefore they are similar, so that there is a nonsingular matrix R, such that $AR = RD$. Writing $R = \begin{bmatrix} R_1 & \cdots & R_k \end{bmatrix}$, with the number of columns of each $R_i, i = 1, \dots, k$ being equal to the order of Y_i, it follows that, $AR_i = R_i Y_i$, for $i = 1, \dots, k$, and it is obvious that each R_i is of full rank, and thus, it is a block eigenvector corresponding to Y_i. Conversely, let $W_1, \dots, W_k$ be a set of right block eigenvectors corresponding to $Y_1, \dots, Y_k$, and let

$$A \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} \mathrm{diag}(Y_1, \dots, Y_k)$$

with the matrix $\begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix}$ being of full rank, then A and $\mathrm{diag}(Y_1, \dots, Y_k)$ are similar, and hence their Jordan normal form, and the partial multiplicities of their eigenvalues, are common. Thus the $Y_1, \dots, Y_k$ are a complete set. □

3. Solutions of matrix differential equations

Now we consider block eigenvalues of the block companion matrix, in order to obtain a general solution to the previously mentioned differential matrix equation.

Theorem 3. *Let $P(X)$ be a matrix polynomial and let C be the associated block companion matrix, if the matrices $Y_1, \dots, Y_k$ are a complete set of block eigenvalues of C, and $W_1, \dots, W_k$ are the corresponding block eigenvectors . Then every solution of $P\left(\frac{d}{dt}\right) x = 0$ is of the form*

$$x(t) = (W_1)_1 e^{Y_1 t} z_1 + \dots + (W_k)_1 e^{Y_k t} z_k\,, \tag{3.1}$$

where $(W_i)_1$ is the top submatrix of n rows of W_i, for $i = 1, 2, \dots k$ and $z_1, \dots, z_k \in \mathbb{C}^n$.

Proof. From [5], p. 512, we have

$$x(t) = Pe^{Ct} z,$$

with $P = \begin{bmatrix} I_n & 0_n & \dots & 0_n \end{bmatrix}$ and $z \in \mathbb{C}^{mn}$ is arbitrary. Now let $W_1, \dots, W_k$ be block eigenvectors of C corresponding to the block eigenvalues $Y_1, \dots, Y_k$, thus we have from Theorem 2

$$C \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} \mathrm{diag}(Y_1, \dots, Y_k),$$

now we write $W = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix}$ and it follows that

$$\begin{aligned} x(t) &= Pe^{Ct} z = Pe^{W \mathrm{diag}(Y_1, \dots, Y_k) W^{-1} t} z = \\ &= PWe^{\mathrm{diag}(Y_1, \dots, Y_k) t} W^{-1} z = PW \mathrm{diag}(e^{Y_1 t}, \dots, e^{Y_k t}) W^{-1} z. \end{aligned}$$

Considering that $PW = \begin{bmatrix} (W_1)_1 & \cdots & (W_k)_1 \end{bmatrix}$ and by writing

$$W^{-1}z = \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix},$$

we get

$$\begin{aligned} x(t) &= \begin{bmatrix} (W_1)_1 e^{Y_1 t} & \cdots & (W_k)_1 e^{Y_k t} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} \\ &= (W_1)_1 e^{Y_1 t} z_1 + \ldots + (W_k)_1 e^{Y_k t} z_k. \end{aligned}$$

□

The goal here is to get a general solution when it is not possible to achieve it with solvents. We see this in the following example.

Example 1. Consider the differential equation

$$P\left(\frac{d}{dt}\right)x = x^{(2)}(t) + A_1 x^{(1)}(t) + A_2 x(t),$$

with coefficients given by

$$A_1 = \begin{bmatrix} -98/25 & 108/25 & -112/25 \\ 4/5 & -24/5 & -4/5 \\ 22/25 & 38/25 & -182/25 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 89/25 & -294/25 & 316/25 \\ -7/5 & 42/5 & -8/5 \\ -46/25 & -59/25 & 251/25 \end{bmatrix},$$

the associated matrix polynomial is

$$P(X) = X^2 + A_1 X + A_2,$$

where $m = 2$ and $n = 3$. We have that

$$V_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} -2 & -2 & 5 & 3 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

are the Jordan chains of $P(\lambda)$ and the respective Jordan blocks are

$$J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

It can be verified that there are no nonsingular matrices of order 3 with the leading vectors of V_1 and V_2 (Jordan chains of $P(\lambda)$), hence $P(X)$ has no solvents at all, therefore a general solution of $P\left(\frac{d}{dt}\right)x = 0$ in terms of solvents does not exist. On the other hand, we have $C = S\,\mathrm{diag}(J_1, J_2)S^{-1}$, where

$$S = \begin{bmatrix} V_1 & V_2 \\ V_1 J_1 & V_2 J_2 \end{bmatrix}$$

is nonsigular. Thus we have from Theorem 2 that J_1 and J_2 form a complete set of block eigenvalues of C and

$$\begin{bmatrix} V_1 \\ V_1 J_1 \end{bmatrix} \text{ and } \begin{bmatrix} V_2 \\ V_2 J_2 \end{bmatrix}$$

are the corresponding block eigenvectors. Hence from Theorem 3 it follows that every solution of $P\left(\frac{d}{dt}\right)x = 0$ is of the form

$$x(t) = V_1 e^{J_1 t} z_1 + V_2 e^{J_2 t} z_2 .$$

The considered block eigenvalues are in a canonical form, but in general this is not necessary. In fact, if Y_1 and Y_2 are any matrices similar to J_1 and J_2, respectively, then Y_1 and Y_2 are also block eigenvalues, as pointed out before. Thus if we write $Y_1 = T_1^{-1} J_1 T_1$ and $Y_2 = T_2^{-1} J_2 T_2$, with T_1 and T_2 nonsingular. It follows that

$$\begin{aligned} C &= S\,\mathrm{diag}(T_1 Y_1 T_1^{-1}, T_2 Y_2 T_2^{-1})S^{-1} = \\ &= S\,\mathrm{diag}(T_1, T_2)\mathrm{diag}(Y_1, Y_2)\mathrm{diag}(T_1, T_2)^{-1} S^{-1} = \\ &= U\mathrm{diag}(Y_1, Y_2)U^{-1}, \end{aligned}$$

where $U = S\,\mathrm{diag}(T_1, T_2)$ is nonsingular and so Y_1 and Y_2 are a complete set of block eigenvalues of C (from Theorem 2) with

$$\begin{bmatrix} V_1 T_1 \\ V_1 J_1 T_1 \end{bmatrix} \text{ and } \begin{bmatrix} V_2 T_2 \\ V_2 J_2 T_2 \end{bmatrix}$$

being the corresponding block eigenvectors. Hence from Theorem 3 every solution of $P\left(\frac{d}{dt}\right)x = 0$ can be written in the form

$$x(t) = V_1 T_1 e^{Y_1 t} z_1 + V_2 T_2 e^{Y_2 t} z_2 .$$

Numerical procedures to compute a complete set of block eigenvalues can be found in [7].

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