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A COMMON FIXED POINT THEOREM FOR NONSELF MAPPINGS

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ABSTRACT. A general common fixed point theorem for nonself discontinuous mappings is given. It contains, as particular cases, many classical and recent results in fixed point theory.

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1. INTRODUCTION

Banach's fixed point theorem is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem B. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a strict contraction, i. e., a map satisfying*

$$d(Tx, Ty) \leq \alpha\, d(x, y), \quad \forall\ x, y \in X, \tag{1.1}$$

where $0 < \alpha < 1$ *is a constant. Then* T *has a unique fixed point in* X.

Theorem B, together with its local variants, has many applications in solving nonlinear functional equations, but has one drawback the contraction condition (1.1) forces T to be continuous on the entire X.

In 1968, Kannan [**6**] obtained a fixed point theorem for mappings T that need not be continuous.

Theorem K. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a mapping for which there exists* $a \in \left(0, \frac{1}{2}\right)$ *such that*

$$d(Tx, Ty) \leq a\left[d(x, Tx) + d(y, Ty)\right], \quad \textit{for all} \quad x, y \in X. \tag{1.2}$$

Then T *has a unique fixed point in* X.

Example 1. Let X be the set of reals with the usual norm and $T : X \to X$ given by $Tx = 0$ if $x \in (-\infty, 2]$, and $Tx = -\frac{1}{2}$ if $x \in (2, \infty)$. Then T satisfies (1.2) with $a = \frac{1}{5}$ and T is not continuous.

Following Kannan's theorem, a great many papers were devoted to obtaining fixed point theorems for various contractive conditions that do not require the continuity of T, see, for example, Rus [**11**].

One of the most general contractive conditions obtained in this way, for which the Picard iteration still converges to the unique fixed point, was given by Ćirić [**4**].

Theorem C1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that*

$$d(Tx, Ty) \leq h \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \qquad (1.3)$$

for all $x, y \in X$ and for some constant $0 < h < 1$.

Then T has a unique fixed point in X.

Remarks. **1**° As shown by Rhoades [**10**, Theorem 2], a contractive mapping satisfying (1.3) is still continuous *at the fixed point*.

2° The fixed point theorems for contractive definitions of the form (1.1)–(1.3) were unified by many authors, see for example Berinde [**1**], Rus [**11**]. For a recent comparison of various contractive type conditions we refer to Mészáros [**7**].

3° The set $0_T(x) = \{x, Tx, T^2x, \dots\}$ is called *the orbit* of T relative to x. It is shown in [**12**] that condition (1.3), in fact, ensures that the orbits of T are bounded.

For any $T : X \to X$ and $x, y \in X$, where X is a metric space, let us put

$$B(x, y) = d(x, y);$$
$$K(x, y) = \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right];$$
$$C(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

The following theorem formally unifies Banach's, Kannan's and Ćirić's fixed point theorems.

Theorem G. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq \lambda\, E(x, y) \quad \textit{for all} \quad x, y \in X, \qquad (1.4)$$

where λ is a constant, $0 < \lambda < 1$, and $E(x, y)$ is any of the expressions $B(x, y)$, $K(x, y)$ and $C(x, y)$.

Then T has a unique fixed point.

Remarks. **1**° Theorem G above can be extended by considering a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+$ denotes the set of nonnegative numbers) which preserves some essential properties of the function

$$\varphi(t) = \lambda t, \quad t \in \mathbb{R}_+ \qquad (0 < \lambda < 1) \qquad (1.5)$$

appearing in (1.4) and by replacing condition (1.4) by a more general one:

$$d(Tx, Ty) \leq \varphi(E(x, y)) \quad \text{for all} \quad x, y \in X. \qquad (1.6)$$

2° One of the first results of this kind was obtained by Browder [**3**]. The function φ involved in such fixed point theorems is usually called *comparison function* and is supposed to satisfy at least the following two conditions:

(i_φ) φ is nondecreasing, i. e., $t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;
(ii_φ) The sequence $\{\varphi^n(t)\}$ converges to zero for every $t \in \mathbb{R}_+$, where φ^n stands for the nth iterate of φ.

Example 2. It is easy to check that a comparison function φ needs to be neither linear nor continuous, by considering $\varphi_1(t) = \frac{t}{1+t}$, $t \in \mathbb{R}_+$ and $\varphi_2(t) = \frac{t}{2}$ if $0 \leq t < 1$ and $\varphi_2(t) = t - \frac{1}{3}$ if $t \geq 1$.

To prove our main result we shall need the following Lemma.

Lemma 1. *If φ satisfies* (i_φ) *and* (ii_φ) *and is such that*

$$t \leq \varphi(t) \quad \textit{for a certain } t \in \mathbb{R}_+, \tag{1.7}$$

then $t = 0$.

Proof. Suppose the contrary, i. e., there exists $t > 0$ such that (1.7) is satisfied. Then, by induction, in view of (ii_φ), we get

$$t \leq \varphi^n(t), \qquad n \geq 1.$$

By virtue of (ii_φ), this implies that $t \leq \varphi^n(t) \to 0$ as $n \to \infty$, a contradiction. □

2. Contractions that are not self-mappings

All fixed point theorems stated in the previous section deal with a self-mapping of a metric space. However, in many applications of fixed point theory, either a mapping of a closed subset K of X is not a self-mapping of K or it is very difficult to verify the invariance condition $T(K) \subset K$.

It was thus an open problem for more than 20 years to extend Theorem C1 from self-mappings $T : K \to K$ satisfying (1.3) to the corresponding nonself-mappings $T : K \to X$, where $K \neq X$. Recently, Ćirić [**5**] solved this problem by considering an additional boundary condition, also known as Rothe's boundary condition, which, however, restricts his results to a Banach space setting.

Theorem C2. *Let E be a Banach space, K a nonempty closed subset of E, and ∂K the boundary of K. Let $T : K \to E$ be a nonself-mapping satisfying* (1.3) *for all $x, y \in K$. If*

$$T(\partial K) \subset K, \tag{2.1}$$

then T has a unique fixed point in K.

Very recently, Theorem C2 was extended by Rakočević [**9**] to a common fixed point theorem. Radovanovic [**8**] also considered a similar but more particular contractive condition.

The main aim of this paper is to unify the results of Ćirić and Rakočević, as well as many other related results, in the framework of a very general common fixed point theorem.

3. Main result

Let E be a normed linear space. For $x, y \in E$ we shall denote by

$$\mathrm{seg}[x, y] = \{z \in E : z = (1 - t)x + ty\,, \quad 0 \le t \le 1\}$$

the segment of extremities x and y. The proof of the next lemma is straightforward, see Rakočević [**9**].

Lemma 2. *If $u \in E$ and $z \in \mathrm{seg}[x, y]$, then*

$$\|u - z\| \le \max\{\|u - x\|\,,\ \|u - y\|\}\,.$$

Now we can state the main result of this paper.

Theorem 1. *Let E be a Banach space, K a nonempty closed subset of E and ∂K the boundary of K. Let $S : K \to E$ and $T : E \to E$, $T : K \to K$. Suppose that $\partial K \neq \emptyset$, T is continuous, and that S and T satisfy the following conditions:*

1° *There exists a continuous comparison function φ such that, for every $x, y \in K$,*

$$d(Sx, Sy) \le \varphi(M(x, y))\,, \tag{3.1}$$

where

$$M(x, y) = \max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Tx, Sy), d(Ty, Sx)\}; \tag{3.2}$$

2° *T and S are weakly commutative, i. e.,*

$$d(TSx, STx) \le d(Tx, Sx), \quad \textit{for every} \quad x \in K, \tag{3.3}$$

and, moreover,

$$S(K) \cap K \subset T(K)\,, \tag{3.4}$$

$$S(\partial K) \subset K \tag{3.5}$$

and

$$T(\partial K) \supset \partial K\,. \tag{3.6}$$

Then T and S have a unique common fixed point, provided that T and S have bounded orbits.

Proof. Let $x_0 \in \partial K$. Then $Sx_0 \in K$ by (3.5) and by (3.4) it results that there exists $x_1 \in K$ such that $Tx_1 = Sx_0$.

Consider Sx_1. If $Sx_1 \in K$, (3.4) again implies there exists $x_2 \in K$ such that $Tx_2 = Sx_1$. If $Sx_1 \notin K$, then by (3.6) there exists $x_2 \in \partial K$ such that $Tx_2 \in \partial K \cap \mathrm{seg}\,[Tx_1, Sx_1]$. Hence, by induction, we construct a sequence $\{x_n\}$ of points in K as follows.

If $Sx_n \in K$, then $Tx_{n+1} = Sx_n$ for some $x_{n+1} \in K$, by (3.4). If $Sx_n \notin K$, then, by (3.6), we can pick $x_{n+1} \in \partial K$ such that

$$Tx_{n+1} \in \partial K \cap \operatorname{seg}[Tx_n, Sx_n]\,.$$

We shall prove that both $\{Tx_n\}$ and $\{Sx_n\}$ are Cauchy sequences.

Let us first prove that

$$Tx_{n+1} \neq Sx_n \Rightarrow Tx_n = Sx_{n-1}\,. \tag{3.7}$$

Suppose the contrary, that is, $Tx_n \neq Sx_{n-1}$. Then $x_n \in \partial K$ and (3.5) implies $Sx_n \in K$, i. e. $Tx_{n+1} = Sx_n$, a contradiction. This proves (3.7).
By setting

$$\begin{aligned} B(n,k) &= \{Tx_j, Sx_j : n \leq j \leq n+k\},\\ b(n,k) &= \operatorname{diam}(B(n,k))\,,\\ B(n) &= \{Tx_j, Sx_j : n \leq j\}\,,\\ b(n) &= \operatorname{diam}(B(n))\,, \end{aligned}$$

we obtain that $b(n,k) \uparrow b(n)$ as $k \to \infty$ and $\{b(n)\}$ is a decreasing sequence with positive terms, hence $b = \lim_{n\to\infty} b(n)$ exists.

In order to prove that $\{Tx_n\}$ and $\{Sx_n\}$ are Cauchy sequences we must show that $b = 0$. We claim that

$$b(n,k) \leq \varphi(b\,(n-2, k+2))\,,\ n \geq 2\,, k \geq 0\,, \tag{3.8}$$

and consider the following three cases.

Case 1. *$b(n,k) = d(Tx_i, Sx_j)$ with $n \leq i, j \leq n+k$.*

If $Tx_i = Sx_{i-1}$, then, by (3.1), we get

$$b(n,k) = d(Sx_{i-1}, Sx_j) \leq \varphi(M\,(x_{i-1}, x_j)) \leq \varphi(b\,(n-2, k+2))$$

because φ is monotonically increasing.

If $Tx_i \neq Sx_{i-1}$, then $Tx_{i-1} = Sx_{i-2}$ and

$$Tx_i \in \operatorname{seg}\,[Tx_{i-1}, Sx_{i-1}] = \operatorname{seg}\,[Sx_{i-2}, Sx_{i-1}]\,.$$

Thus,

$$\begin{aligned} b(n,k) = d(Tx_i, Sx_j) &\leq \max\{d(Sx_{i-2}, Sx_j), d(Sx_{i-1}, Sx_j)\}\\ &\leq \max\{\varphi(M\,(x_{i-2}, x_j)), \varphi(M\,(x_{i-1}, x_j))\}\\ &= \varphi\Big(\max\{M(x_{i-2}, x_j), M(x_{i-1}, x_j)\}\Big) \leq \varphi(b\,(n-2, k+2))\,. \end{aligned}$$

Case 2. *$b(n,k) = d(Tx_i, Tx_j)$ with $n \leq i, j \leq n+k$.*

If $Tx_j = Sx_{j-1}$, then Case 2 reduces to Case 1. If $Tx_j \neq Sx_{j-1}$, then as in Case 1 we have $Tx_{j-1} = Sx_{j-2}$ and

$$Tx_j \in \partial K \cap \operatorname{seg}[Sx_{j-2}, Sx_{j-1}]\,.$$

Hence,

$$b(n,k) = d(Tx_i, Tx_j) \leq \max\{d(Tx_i, Sx_{j-2}), d(Tx_i, Sx_{j-1})\}$$

and so Case 2 also reduces to Case 1.

Case 3. *$b(n,k) = d(Sx_i, Sx_j)$, with $n \leq i, j \leq n+k$.*

Then

$$b(n,k) = d(Sx_i, Sx_j) \leq \varphi(M(x_i, x_j)) \leq \varphi(b(n,k)),$$

which by Lemma 1 implies $b(n,k) = 0$. Hence, due to $Sx_i = Tx_i$, $b(n,k) = d(Sx_j, Tx_i)$, which means Case 1. Therefore, (3.8) is proved.

Now, having in view the continuity of φ, we let $k \to \infty$ in (3.8), and obtain

$$b(n) \leq \varphi(b(n-2)).$$

Letting $n \to \infty$ in the previous inequality we obtain $b \leq \varphi(b)$ which, by Lemma 1, implies $b = 0$. This shows that both $\{Tx_n\}$ and $\{Sx_n\}$ are Cauchy sequences.

As $Tx_n \in K$ and K is a closed subset of the Banach space E, we conclude that

$$\lim_{n\to\infty} Tx_n = p \in K.$$

Since

$$d(Tx_n, Sx_n) \leq b(n) \to 0 \quad \text{as} \quad n \to \infty$$

we also have $\lim Sx_n = p$. As T is continuous, we obtain

$$\lim_{n\to\infty} T(Sx_n) = T(\lim_{n\to\infty} Sx_n) = Tp \in K,$$

and in view of the weak commutativity (3.3), we have

$$d(STx_n, Tp) \leq d(STx_n, TSx_n) + d(TSx_n, Tp) \leq$$
$$\leq d(Tx_n, Sx_n) + d(TSx_n, Tp) \to 0, \text{ as } n \to \infty. \tag{3.9}$$

This shows that

$$\lim_{n\to\infty} (ST)(x_n) = Tp, \tag{3.10}$$

and therefore, by (3.9) and (3.10), we have

$$M(Tx_n, p) \to d(Tp, Sp) \quad \text{as} \quad n \to \infty$$

and

$$d(Tp, Sp) \leq \varphi(d(Tp, Sp)),$$

which, again by Lemma 1, yields $d(Tp, Sp) = 0$, i. e.,

$$Tp = Sp. \tag{3.11}$$

We shall prove that Sp (and also Tp) is a common fixed point for S and T. Indeed, by (3.11) and (3.3) it results that

$$TSp = STp = SSp. \tag{3.12}$$

Now, by (3.1), (3.11), and (3.12), we have

$$d(SSp, Sp) \leq \varphi(M(Sp, p)) = \varphi(d(SSp, Sp)),$$

which yields $SSp = Sp$. It follows from (3.12) that Sp is a fixed point of T as well. To prove the uniqueness, relation (3.1) is used. □

Remarks. **1**° For $T = 1_E$ (the identity map) and φ given by (1.5), from Theorem 1 we obtain Theorem C2 of Ćirić.

2° For $\varphi(t) = \lambda t$ with $0 < \lambda < 1$, Theorem 1 implies Theorem 2 of Rakočević [**9**]. It is known (see Lemma 4.3.1 in [**11**]) that if T is a generalized strict φ-contraction, i. e., T satisfies (1.6) with $E(x, y) \equiv C(x, y)$, and φ is a strict comparison function, then T has bounded orbits.

It is, however, an open question whether or not two mappings S and T satisfying (3.1) have bounded orbits.

3° By considering other comparison functions in Theorem 1, we obtain various related fixed point theorems as well as common fixed point theorems. Moreover, all the results in Rakočević [**9**] can be extended in a similar way. We restrict our study to Theorem 3.

4° The continuity assumption of T in Theorem 1 can be weakened to obtain a more general result similar to Theorem 3 of Rakočević [**9**].

Theorem 2. *Let E be a Banach space, K a nonempty closed subset of E, and $\partial K \neq \emptyset$ the boundary of K. Let $S : K \to E$, $T : E \to E$, and $T : K \to K$.*

Suppose that T^m is continuous for some fixed positive integer m, S and T satisfy (3.1), (3.4), (3.5), (3.6)*, and, moreover, T and S commute, i. e.,*

$$TSx = STx \quad \text{for every} \quad x \in K.$$

Then S and T have a unique common fixed point in K, provided that T and S have bounded orbits.

Proof. Let $\{x_n\}$, Sx_n and Tx_n be constructed as in the proof of Theorem 1. Hence,

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = p \in K.$$

For every $n \geq 1$, we have

$$\begin{aligned} d(T^m Sx_n, ST^{m-1}p) &= d(ST^m x_n, ST^{m-1}p) \leq \\ &\leq \varphi(M(T^m x_n, T^{m-1}p)) \\ &= \varphi\Big(\max\Big\{d(T^m Tx_n, T^m p), d(T^m Tx_n, T^m Sx_n), \\ &\qquad d(T^m p, ST^{m-1}p), d(T^m Tx_n, ST^{m-1}p), d(T^m p, T^m Sx_n)\Big\}\Big). \end{aligned}$$

Then, by the continuity of T^m,

$$d(T^m p, ST^{m-1}p) \leq \varphi(d(T^m p, ST^{m-1}p)),$$

and hence $T^m p = ST^{m-1}p$, by Lemma 1. So, similarly to the method used in the proof of Theorem 1, it can be shown that $T^m p$ is a common fixed point of T and S. The uniqueness is proved by using the contraction condition (3.1). □

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