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CRITICAL CASE FOR SINGULARLY PERTURBED LINEAR BOUNDARY-VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The conditions under which a unique asymptotic representation of the solution of boundary-value problems exists for singularly perturbed systems of ordinary differential equations are shown in the work. The solution is obtained with the help of boundary functions and pseudo-inverse matrices.

Mathematics Subject Classification: 34B15

Keywords: boundary-value problems, singularly perturbation, asymptotic solution, boundary functions

1. Statement of the problem

We consider the singularly perturbed differential system

$$\varepsilon\dot{x} = Ax + \varepsilon A_1(t)x + \varphi(t), \quad t \in [a,b], \quad 0 < \varepsilon \ll 1, \tag{1.1}$$

$$lx(\cdot) = h, \quad h \in \mathbb{R}^m, \tag{1.2}$$

where the coefficients of system (1.1) and equation (1.2) satisfy the conditions:

(H1) A is a constant $(n\times n)$ matrix. If λ_i are eigenvalues of A, then $\lambda_i = 0$, $i = \overline{1,k}$, $k < n$, $\operatorname{Re}\lambda_i < 0$, $i = \overline{k+1,n}$, as p, $p < k$, linear independent eigenvectors of matrix A correspond to the zero eigenvalue;

(H2) $A_1(t)$ is an $(n \times n)$ matrix, $A_1(t) \in C^\infty[a,b]$, $\varphi(t)$ is an n-dimensional vector-function $\varphi(t) \in C^\infty[a,b]$;

(H3) $l : C[a,b] \to \mathbb{R}^m$ is an m-dimensional linear bounded vector-functional, $l = \operatorname{col}(l^1,\dots,l^m)$;

(H4) The degenerate ($\varepsilon = 0$) system (1.1), $Ax_0 + \varphi(t) = 0$, is solvable with respect to x_0.

We look for an n-dimensional vector-function $x(t,\varepsilon)$: $x(\cdot,\varepsilon) \in C^1[a,b]$, $x(t,\cdot) \in C(0,\varepsilon_0]$, satisfying (1.1), (1.2) and following relation $\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t)$, $t \in (a,b]$.

We shall consider the case $m \neq n$ and $p < k$. We use an asymptotic method of the boundary functions and construct an asymptotic series for the boundary-value problem (1.1), (1.2) with $\det A = 0$ (the critical case [10]).

In the case $m = n$ and $p = k$ an asymptotic solution of the Cauchy problem and two point boundary-value problem for linear and quasilinear systems is studied in [10] on the basis of the method of boundary functions. In the non-critical case $m \neq n$ and $\det A \neq 0$ the system is studied in [5]. When $m \neq n$ and $p = k$, the problem (1.1), (1.2) is considered in [8].

The construction of an asymptotic solution of (1.1), (1.2) in this work $m \neq n$, $p < k$ is represented on the basis of generalized inverse matrices and projectors [1, 4, 7] and central canonical form [2, 3].

We denote by $n_1, n_2, \ldots, n_p$ ($\sum_{i=1}^{p} n_i = k$) the lengths of the Jordan cells. We will consider the case where $n_1 > \cdots > n_s$, $n_{s+1} = n_{s+2} = \cdots = n_{p-1} = n_p = 1$, i. e., the matrix A has a block diagonal representation

$$A = \operatorname{diag}(\overline{A},\ J_1,\ J_2, \ldots, J_s,\ \Theta_{p-s}), \tag{1.3}$$

where $\overline{A}$ is a $((n-k) \times (n-k))$ matrix and has eigenvalues with negative real parts, J_i, $i = \overline{1, s}$, are $(n_i \times n_i)$ Jordan cells, and Θ_{p-s} is the $((p-s) \times (p-s))$ zero matrix.

By $A^\dagger$, we denote the unique Moore–Penrose pseudo-inverse $(n \times n)$ matrix of the matrix A [4, 7]. Denote by P_A and P_{A^*} orthoprojectors $P_A : \mathbb{R}^n \to \ker A$, $P_{A^*} : \mathbb{R}^n \to \ker A^*$, $A^* = A^T$. According to (H1) we find $\operatorname{rank} A = n - p$ and $\operatorname{rank} P_A = \operatorname{rank} P_{A^*} = n - (n - p) = p$. Let P_{A_p} be a $(n \times p)$ matrix with p linear independent columns from the matrix P_A, and let $P_{A_p^*}$ be a $(p \times n)$ matrix with kp linear independent rows of the matrix P_{A^*}.

Let $C = P_{A_p^*} P_{A_p}$ be an $(m \times n)$-constant matrix.

Lemma 1. $\operatorname{rank} C = p - s$.

Proof. The proof is based on the equalities $J_i J_i^\dagger = \operatorname{diag}(1, 1, \ldots, 1, 0)$ and $J_i^\dagger J_i = \operatorname{diag}(0, 1, \ldots, 1, 1)$. Keeping in mind the representation

$$A^\dagger = \operatorname{diag}(\overline{A}^{-1}, J_1^\dagger, J_2^\dagger, \ldots, J_s^\dagger, \Theta_{p-s})$$

and the equalities $P_A = E_n - A^\dagger A$, $P_{A^*} = E_n - AA^\dagger$, we get that $C = P_{A_p^*} P_{A_p} = \operatorname{diag}(0, E_{p-s})$, i. e., $\operatorname{rank} C = p - s$. □

We consider the degenerate differential system

$$C \frac{d}{dt} z(t) = B(t) z(t) + l(t), \quad t \in [a, b], \tag{1.4}$$

where C is the matrix from Lemma 1.1, $B(t) = P_{A_p^*} A_1(t) P_{A_p}$ is $(p \times p)$ matrix, and l is a p-dimensional vector-function, $l(t) \in C^\infty[a, b]$.

Let the matrix $B(t)$ have the block representation

$$\begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix},$$

where the matrices B_{11}, B_{12}, B_{21}, and B_{22} have dimensions $((p-s)\times(p-s))$, $((p-s)\times s)$, $(s\times(p-s))$, and $s\times s$, respectively.

Lemma 2. *System* (1.4) *takes the central canonical form if and only if* $\det B_{11} \neq 0$ $\forall t \in [a,b]$.

Proof. The proof of Lemma 2 is based on Lemma 1 and the work [3]. □

In accordance with Lemma 1 under $p \neq s$ and Lemma 2 $(p\times p)$, matrices $P(t)$ and $Q(t)$ exist such that substituting $z(t) = Q(t)y(t)$ and multiplying by $P(t)$ on the left, the system (1.4) takes central canonical form

$$\begin{pmatrix} E_{p-s} & 0 \\ 0 & \Theta_s \end{pmatrix} \frac{dy(t)}{dt} = \begin{pmatrix} L(t) & 0 \\ 0 & E_s \end{pmatrix} y(t) + \begin{pmatrix} \mu(t) \\ \nu(t) \end{pmatrix}, \tag{1.5}$$

where Θ_s is the $(s\times s)$ zero matrix, $L(t)$ is a $((p-s)\times(p-s))$ matrix, E_{p-s} and E_s are $((p-s)\times(p-s))$ and $(s\times s)$ unit matrices, respectively, and $\mu(t)$ and $\nu(t)$ are $(p-s)$ and s-dimensional vector-functions such that

$$P(t)g(t) = \begin{pmatrix} \mu(t) \\ \nu(t) \end{pmatrix}. \tag{1.6}$$

Let the $(p-s)$-dimensional vector-function $u(t)$ and s-dimensional vector-function $v(t)$ are such that $y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$. Then the system (1.5) takes the form

$$\begin{aligned} \dot{u}_i(t) &= L(t)u_i(t) + \mu_i(t), \\ 0 &= v_i(t) + \nu_i(t). \end{aligned} \tag{1.7}$$

We denote by $\Phi(t)$ a normal fundamental matrix of the solutions of the system $\dot{x} = L(t)x$. Then system (1.7) has a generalized solution

$$\begin{aligned} &u(t) = \Phi(t)\Phi^{-1}(t)\eta + \overline{u}(t), \quad \eta \in \mathbb{R}^{\mathbf{p-s}}, \\ &v(t) = -\nu(t), \end{aligned} \tag{1.8}$$

where $\overline{u}(t) = \Phi(t)\int_a^t \Phi^{-1}(s)\mu(s)ds$.

Let the matrix $Q(t)$ be reduced to the block form $Q(t) = [Q_1(t), Q_2(t)]$, where $Q_1(t)$ is a $(p\times(p-s))$ matrix and $Q_2(t)$ is a $(p\times s)$ matrix. Keeping in mind the substitution $z(t) = Q(t)y(t)$, where $y(t) = [u(t), v(t)]^T$, we obtain

$$z(t) = Q_1(t)u(t) + Q_2(t)v(t).$$

In the last equality we substitute solution (1.8). Thus,

$$z(t) = \Phi(t,a)\eta + \overline{z}(t),\ t \in [a,b],\ \eta \in \mathbb{R}^{p-s}, \tag{1.9}$$

where

$$\overline{\Phi}(t,a) = Q_1(t)\Phi(t)\Phi^{-1}(a) \quad \text{is a } (p\times(p-s)) \text{ matrix and}$$

$$\overline{z}(t) = Q_1(t)\overline{u}(t) - Q_2(t)v(t) \tag{1.10}$$

The following lemma is needed.

Lemma 3. *Let the matrix A satisfy condition* (H1)*, and let the vector-function $f(\tau) \in C[0,+\infty)$ and satisfy the inequality $\|f(\tau)\| < c_1 e^{-\alpha_1\tau}$, where $\tau \geq 0$, $c_1 > 0$, and $\alpha_1 > 0$. Then there exist positive constants c and γ such that the system $dx/d\tau = Ax + f(\tau)$ has a particular solution of the form*

$$x(\tau) = \int_0^\infty K(\tau,s) f(s) ds,$$

satisfying the inequality $\|x(\tau)\| \leq c\exp(-\gamma\tau)$, $\tau \geq 0$, where

$$K(\tau,s) = \begin{cases} X(\tau)PX^{-1}(s) & \text{for } 0 \leq s \leq \tau < \infty, \\ -X(\tau)(I-P)X^{-1}(s) & \text{for } 0 < \tau < s \leq \infty, \end{cases}$$

and P is the spectral projector of the matrix A to the left semi-plane.

The lemma is proved analogously to a similar lemma in [5].

2. Formally asymptotic expansion

We shall seek for a formally asymptotic expansion of the solution of problem (1.1), (1.2) in the form of the regular and singular series

$$x(t,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i (x_i(t) + \Pi_i(\tau)), \quad \tau = \frac{t-a}{\varepsilon}, \tag{2.1}$$

where $x_i(t)$ and $\Pi_i(\tau)$ are unknown n vector functions. By $\Pi_i(\tau)$ (see [10]) we denote the boundary function in a neighbourhood of the point $t = a$. They will be constructed so that when $0 < \varepsilon \leq \varepsilon_0$, the inequalities

$$\|\Pi_i(\tau)\| \leq \gamma_i \exp(-\alpha_i\tau), \tag{2.2}$$

where γ_i and α_i are positive constants for $i = 0,1,2,\ldots$ and $\tau \geq 0$, hold in $[a,b]$.

Formally, by substituting (2.1) in (1.1), (1.2), for $x_i(t)$ we obtain the systems

$$Ax_i(t) = f_i(t), \quad t \in [a,b], \quad i = 0,1,\ldots, \tag{2.3}$$

where

$$f_i(t) = \begin{cases} -\varphi(t) & \text{for } i = 0, \\ L_1(x_{i-1}(t)) & \text{for } i = 1,2,\ldots, \end{cases}$$

and L_1 is the differential operator $L_1(x(t)) = \frac{dx(t)}{dt} - A_1(t)x$. The boundary functions $\Pi_i(\tau)$ are solutions of the boundary problems

$$\frac{d}{d\tau}\Pi_i(\tau) = A\Pi_i(\tau) + \psi_i(\tau), \quad \tau \in [0, \tau_b],\ \tau_b = \frac{b-a}{\varepsilon}, \tag{2.4}$$

$$l(x_i(\cdot)) + l\left(\Pi_i\left(\frac{(\cdot)-a}{\varepsilon}\right)\right) = \begin{cases} h & \text{for } i = 0, \\ 0 & \text{for } i = 1, 2, \dots, \end{cases} \tag{2.5}$$

where

$$\psi_i(\tau) = \begin{cases} 0 & \text{for } i = 0, \\ \sum_{q=i-1}^{0} \frac{1}{q!}\tau^q A_1^{(q)}(a)\Pi_{i-1-q}(\tau) & \text{for } i = 1, 2, \dots, \end{cases} \tag{2.6}$$

We denote the normal fundamental matrix of the solutions of the homogeneous system $\frac{dx}{d\tau} = Ax$, $\tau \in [0, \tau_b]$, by $X(\tau) = \exp(\tau A)$. Let $X_{n-k}(\tau)$ be an $(n \times (n-k))$ matrix with $(n-k)$ columns from the matrix $X(\tau)$, consisting of exponentially small functions (see [8]).

2.1. Obtaining the coefficients $x_0(t)$ and $\Pi_0(\tau)$. Consider systems (2.3)–(2.6) for $i = 0$. Then the degenerate system

$$Ax_0(t) + \varphi(t) = 0 \tag{2.7}$$

is solvable with respect to $x_0(t)$ (according to (H4)) if and only if $P_{A^*}\varphi(t) = 0$ for all $t \in [a, b]$, and it has a solution

$$x_0(t) = P_{A_p}\alpha_0(t) - A^\dagger\varphi(t), \tag{2.8}$$

where $\alpha_0(t)$ is an arbitrary p-dimensional vector-function.

The general solution of system (2.4) has the form

$$\Pi_0(\tau) = X_{n-k}(\tau)c_0, \quad c_0 \in \mathbb{R}^{n-k}. \tag{2.9}$$

We define the vector-function $\alpha_0(t)$ by obtaining of $x_1(t)$. Consider the system $Ax_1(t) = f_1(t)$, where $f_1(t) = L_1(x_0(t))$. The latter system has a solution

$$x_1(t) = P_{A_p}\alpha_1(t) + A^\dagger L_1(x_0(t)) \tag{2.10}$$

if and only if $P_{A_p^*}L_1(x_0(t)) = 0$ for all $t \in [a, b]$. Keeping in mind the representation $x_0(t)$ from (2.8) and L_1, we obtain the differential system for $\alpha_0(t)$,

$$C\frac{d}{dt}\alpha_0(t) = B(t)\alpha_0(t) + g_0(t), \quad t \in [a, b], \tag{2.11}$$

where $g_0(t) = -P_{A_p^*}L_1(A^\dagger\varphi(t))$. System (2.11) coincides with system (1.4) at $l(t) \equiv g_0(t)$, $t \in [a, b]$. Then, according to Lemma 2 and the equality (1.9), we obtain

$$\alpha_0(t) = \Phi(t, a)\eta_0 + \overline{\alpha}_0(t), \quad t \in [a, b], \quad \eta_0 \in \mathbb{R}^{p-s}, \tag{2.12}$$

where $\overline{\alpha}_0(t) = Q_1(t)\overline{u}_0(t) - Q_2(t)\nu_0(t)$,

$$\overline{u}_0(t) = \Phi(t)\int_a^t \Phi^{-1}(s)\mu(s)_0 ds,$$

$$P(t)g_0(t) = \begin{pmatrix}\mu_0(t)\\ \nu_0(t)\end{pmatrix},$$

and $\Phi(t,a)$, $Q(t) = [Q_1(t),\ \ Q_2(t)]$, and $P(t)$ are the matrices from Section 1. The vector-functions $u_0(t)$ and $v_0(t)$ are solutions of the following system (see (1.7)):

$$\begin{aligned}\dot{u}_0(t) &= L(t)u_0(t) + \mu_0(t),\\ 0 &= v_0(t) + \nu_0(t).\end{aligned}$$

We substitute (2.12) into equality (2.8), and for $x_0(t)$ we obtain

$$x_0(t) = P_{A_p}\Phi(t,a)\eta_0 + P_{A_p}\overline{\alpha}_0(t) - A^{\dagger}\varphi(t). \tag{2.13}$$

Finally, for obtaining the functions $x_0(t)$ and $\Pi_0(t)$ it is sufficient to determine the vectors $\eta_0 \in \mathbb{R}^{p-s}$ and $c_0 \in \mathbb{R}^{n-k}$. In this connection, we use the boundary condition (2.5) for $i = 0$, where we substitute (2.13) and (2.9). We obtain the vectors η_0 and c_0 by the system

$$D_0(\varepsilon)c_0 + S_0\eta_o = h_0, \tag{2.14}$$

where $D(\varepsilon) = lX_{n-k}(\cdot)$ is an $(m\times(n-k))$ matrix, $S_0 = lA_p\Phi(\cdot,a)$ is an $(n\times(p-s))$ matrix, $h_0 = h - l(A_p\overline{\alpha}_0(\cdot)) - l(A^{\dagger}\varphi(\cdot))$ is an m-dimensional vector.

Keeping in mind the expression of the matrix $X_{n-k}(\tau)$ and the form of the functional $l(x)$, we assume that $D_0(\varepsilon) = \overline{D}_0 + O(\varepsilon^s\exp(-\alpha/\varepsilon))$, where $\alpha > 0$, $s \in N$, $\overline{D}_0$ is a $(m\times(n-k))$-constant matrix, and $O(\varepsilon^s\exp(-\alpha/\varepsilon))$ we denote a matrix consisting of elements infinitely small with respect to ε. Because the elements of the matrix $\overline{D}_0$ are continuous for all $\varepsilon \in (0,\varepsilon_0]$ and $\lim_{\varepsilon\to 0} D_0(\varepsilon) = \overline{D}_0$, then we determine the matrix $D_0(\varepsilon)$ for $\varepsilon = 0$, putting $D_0(0) = \overline{D}_0$. We neglect the exponentially small elements in the matrix $D_0(\varepsilon)$ and system (2.14) takes the form

$$M\begin{pmatrix}c_0\\ \eta_0\end{pmatrix} = h_0, \tag{2.15}$$

where $M = [\overline{D}_0,\ S_0]$ is a $(m\times(n+p-k-s))$ constant matrix.

Let the following condition hold:

(H5) rank $M = m = n-k+p-s$.

Then $\det M \neq 0$ and system (2.15) is always solvable and

$$\begin{aligned} c_0 &= [M^{-1}]_{n-k}h_0\\ \eta_0 &= [M^{-1}]_{p-s}h_0,\end{aligned} \tag{2.16}$$

where $[M^{-1}]_{n-k}$ and $[M^{-1}]_{p-s}$ are the first $(n-k)$ and last $(p-s)$ rows of the matrix M^{-1}. We should note that in this case $n-m = k-p+s > 0$, i. e., $n > m$.

We substitute (2.16) into (2.13) and (2.9) and get

$$\begin{aligned} x_0(t) &= P_{A_p}\Phi(t,a)[M^{-1}]_{p-s}\, h_0 + \bar{x}_0(t), \\ \Pi_0(\tau) &= X_{n-k}(\tau)[M^{-1}]_{n-k}\, h_0, \end{aligned} \tag{2.17}$$

where $\bar{x}_0(t) = P_{A_p}\bar{\alpha}_0(t) - A^{\dagger}\varphi(t)$.

2.2. Obtaining the coefficients $x_1(t)$ and $\Pi_1(\tau)$. To obtain the coefficient $x_1(t)$ from (2.10), it is sufficient to determine the function $\alpha_1(t)$. This will be realized in terms of the coefficient $x_2(t)$. System (2.3) under $i = 2$ has a solution $x_2(t) = P_{A_p}\alpha_2(t) + A^{\dagger}L_1(x_1(t))$ if and only if

$$P_{A_p^*}L_1(x_1(t)) = 0$$

for all $t \in [a,b]$. In the last equation we substitute $x_1(t)$ from (2.10). Keeping in mind the form of the operator L_1, for determining the function $\alpha_1(t)$, we obtain the degenerate differential system

$$C\frac{d}{dt}\alpha_1(t) = B(t)\alpha_1(t) + g_1(t), \quad t \in [a,b], \tag{2.18}$$

where $g_1(t) = -P_{A_p^*}L_1(A^{\dagger}L_1 x_0(t))$.

System (2.18) coincides with system (2.3), (1.4) at $l(t) \equiv g_1(t)$, $t \in [a,b]$ and in accordance with Lemma 1.2 and equation (1.9), we obtain

$$\alpha_1(t) = \Phi(t,a)\eta_1 + \bar{\alpha}_1(t), \quad t \in [a,b], \quad \eta_1 \in \mathbb{R}^{p-s}, \tag{2.19}$$

where $\bar{\alpha}_1(t) = Q_1(t)\bar{u}_1(t) - Q_2(t)v_1(t)$,

$$\bar{u}_1(t) = \Phi(t)\int_a^t \Phi^{-1}(s)\mu_1(s)_0 ds,$$

and $P(t)g_1(t) = \binom{\mu_1(t)}{\nu_1(t)}$. The vector-functions $u_1(t)$ and $v_1(t)$ are solutions of system (1.7), where $u(t) = u_1(t)$, $v(t) = v_1(t)$.

We substitute (2.19) into $x_1(t)$ from (2.10) and obtain

$$x_1(t) = P_{A_p}\Phi(t,a)\eta_1 + P_{A_p}\bar{\alpha}_1(t) + A^{\dagger}L_1(x_0(t)). \tag{2.20}$$

In accordance with Lemma 3, the general solution of the system (2.4) at $i = 1$ is

$$\Pi_1(\tau) = X_{n-k}(\tau)c_1 + \int_0^{+\infty} K(\tau,s)\psi_1(s)ds, \quad c_1 \in \mathbb{R}^{n-k}. \tag{2.21}$$

We substitute (2.20) and (2.21) into (2.5) at $i = 1$. The constant vectors η_1 and c_1 are obtained by the system

$$D_0(\varepsilon)c_1 + S_0\eta_1 = h_1(\varepsilon), \tag{2.22}$$

where

$$h_1(\varepsilon) = -l\left(\int_0^{+\infty} K\left(\frac{\cdot - a}{\varepsilon}, s\right)\psi_1(s)ds\right) - l(P_{A_p}\bar{\alpha}_1(\cdot)) - l(A^{\dagger}L_1(x_0(\cdot))).$$

Obviously,

$$h_1(\varepsilon) = h_{10} + O(\varepsilon^{s_1} \exp(-\alpha/\varepsilon)),$$

i. e., $h_1(\varepsilon)$ is with continuous elements for all $\varepsilon \in (\varepsilon_0]$ and $\lim_{\varepsilon\to 0} h_1(\varepsilon) = h_{10}$. Then we determine $h_1(\varepsilon)$ for $\varepsilon = 0$, putting $h_1(0) = h_{10}$. Since $D_0(0) = D_0$, in system (2.22) we neglect the exponentially small elements and obtain

$$M\begin{pmatrix} c_1 \\ \eta_1 \end{pmatrix} = h_{10}, \tag{2.23}$$

where M is the matrix from Section 2.1.

In accordance with condition (H5), the solution of the system (2.23)

$$c_1 = [M^{-1}]_{n-k} h_{10}, \qquad \eta_1 = [M^{-1}]_{p-s} h_{10},$$

we substitute into (2.20) and (2.21). Consequently, the coefficients $x_1(t)$ and $\Pi_1(\tau)$ have the form

$$\begin{aligned} x_1(t) &= P_{A_p}\Phi(t,a)[M^{-1}]_{p-s} h_{10} + \overline{x}_1(t), \\ \Pi_1(\tau) &= X_{n-k}(\tau)[M^{-1}]_{n-k} h_{10} + \overline{\Pi}_1(\tau), \end{aligned} \tag{2.24}$$

where $\overline{x}_1(t) = P_{A_p}\overline{\alpha}_1(t) + A^{\dagger}L_1(x_0(t))$ and $\overline{\Pi}_1(\tau) = \int_0^{+\infty} K(\tau,s)\psi_1(s)ds$.

2.3. Determining the coefficients $x_q(t)$ and $\Pi_q(\tau)$, $q > 1$. The inductive approach shows that the coefficients $x_q(t)$ and $\Pi_q(\tau)$ ($q > 1$) have the form

$$\begin{aligned} x_q(t) &= P_{A_p}\Phi(t,a)[M^{-1}]_{p-s} h_{q0} + \overline{x}_q(t), \\ \Pi_q(\tau) &= X_{n-k}(\tau)[M^{-1}]_{n-k} h_{q0} + \overline{\Pi}_q(\tau), \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} h_{q0} &= \lim_{\varepsilon\to 0} h_q(\varepsilon), \\ h_q(\varepsilon) &= -l\left(\int_0^{+\infty} K\left(\frac{\cdot - a}{\varepsilon}, s\right)\psi_q(s)ds\right) - l(P_{A_p}\overline{\alpha}_q(\cdot)) - l(A^{\dagger}L_1(x_{q-1}(\cdot))), \\ \overline{x}_q(t) &= P_{A_p}\overline{\alpha}_q(t) + A^{\dagger}L_1(x_{q-1}(t)), \\ \overline{\Pi}_q(\tau) &= \int_0^{+\infty} K(\tau,s)\psi_q(s)ds. \end{aligned} \tag{2.26}$$

Assume that the coefficients $x_i(t)$ and $\Pi_i(\tau)$ $i = \overline{1, q-1}$ are determined. System (2.3) for $i = q$ has a solution

$$x_q(t) = P_{A_p}\alpha_q(t) + A^{\dagger}L_1(x_{q-1}(t)) \tag{2.27}$$

if and only if $P_{A_p^*}L_1(x_{q-1}(t)) = 0$ for all $t \in [a,b]$. However, this equality is fulfilled because it is used in obtaining the function $\alpha_{q-1}(t)$. This solution $\alpha_{q-1}(t)$ participates in $x_{q-1}(t)$, which with respect to the induction hypothesis is determined completely.

The function $\alpha_q(t)$ is obtained from the solvability condition $P_{A_p^*}L_1(x_q(t)) = 0$, $\forall t \in [a,b]$ of system (2.3) for $i = q+1$ $Ax_{q+1}(t) = L_1(x_q(t))$. Thus, we obtain the following differential system (see (1.4)):

$$C\frac{d}{dt}\alpha_q(t) = B(t)\alpha_q(t) + g_q(t), \quad t \in [a,b],$$

where $g_q(t) = -P_{A_p^*}L_1(A^\dagger L_1(x_q(t)))$.

By Lemma 2 and equation (1.9) we find

$$\alpha_q(t) = \Phi(t,a)\eta_q + \bar{\alpha}_q(t), \quad t \in [a,b], \quad \eta_q \in \mathbb{R}^{p-s}, \tag{2.28}$$

where $\bar{\alpha}_q(t) = Q_1(t)\bar{u}_q(t) - Q_2(t)\nu_q(t)$,

$$\bar{u}_q(t) = \Phi(t)\int_a^t \Phi^{-1}(s)\mu_q(s)_0 ds,$$

and $P(t)g_q(t) = \binom{\mu_q(t)}{\nu_q(t)}$.

The vector-functions $u_q(t)$ and $v_q(t)$ are solutions of system (1.7), where $u(t) = u_q(t)$, $v(t) = v_q(t)$.

We substitute (2.28) into (2.27) and obtain

$$x_q(t) = P_{A_p}\Phi(t,a)\eta_q + P_{A_p}\bar{\alpha}_q(t) + A^\dagger L_1(x_{q-1}(t)), \quad \eta_q \in \mathbb{R}^{p-s}. \tag{2.29}$$

The general solution of system (2.4) for $i = q$ is

$$\Pi_q(\tau) = X_{n-k}(\tau)c_q + \int_0^{+\infty} K(\tau,s)\psi_q(s)ds, \quad c_q \in \mathbb{R}^{n-k}. \tag{2.30}$$

We substitute (2.29) and (2.30) in the boundary condition (2.5) for $i = q$ and get the system

$$D_0(\varepsilon)c_q + S_0\eta_q = h_q(\varepsilon),$$

where

$$h_q(\varepsilon) = -l\left(\int_0^{+\infty} K\left(\frac{\cdot - a}{\varepsilon}, s\right)\psi_q(s)ds\right) - l(P_{A_p}\bar{\alpha}_q(\cdot)) - l(A^\dagger L_1(x_{q-1}(\cdot))).$$

Since

$$D_0(\varepsilon) = \bar{D}_0 + O(\varepsilon^s \exp(-\alpha/\varepsilon)),$$

$\bar{D}_0 = \lim_{\varepsilon\to 0} D_0(\varepsilon)$, $h_q(\varepsilon) = h_{q0} + O(\varepsilon^{s_1}\exp(-\alpha/\varepsilon))$, and $h_{q0} = \lim_{\varepsilon\to 0} h_q(\varepsilon)$, after ignoring the exponentially small elements, the last system takes the form

$$M\begin{pmatrix} c_q \\ \eta_q \end{pmatrix} = h_{q0}$$

with the solution (see (H5))

$$c_q = [M^{-1}]_{n-k}h_{q0}, \quad \eta_q = [M^{-1}]_{p-s}h_{q0}. \tag{2.31}$$

We substitute (2.31) in (2.29) and (2.30) and obtain the equations (2.25), (2.26).

All the boundary functions $\Pi_i(\tau)$ satisfy inequalities (2.2). This follows from Lemma 3 and the inequality

$$\|X_{n-k}(\tau)\| \le c_1 \exp(-\beta_1\tau),$$

where $c_1 > 0, \beta_1 > 0$, and $\tau > 0$. After sequential analysis we get

$$\|\Pi_0(\tau)\| \le \|X_{n-k}(\tau)\|\|[M^{-1}]_{n-k}\|\|h_0\| \le c_1 \exp(-\beta_1\tau)c_2c_3 = \gamma_0 \exp(-\alpha_0\tau),$$

where $\|[M^{-1}]_{n-k}\| \le c_2$, $\|h_0\| \le c_{30}$, $\gamma_0 = c_1c_2c_{30}$, $\alpha_0 = \beta_1$, and

$$\begin{aligned}\|\Pi_1(\tau)\| \le \|X_{n-k}(\tau)\|\|[M^{-1}]_{n-k}\|\|h_{10}\| + \|\overline{\Pi}_1(\tau)\| \le \\ \le c_1 \exp(-\beta_1\tau)c_2c_{31} + \overline{c}_1 \exp(-\overline{\beta}_1\tau) \\ \le (c_1c_2c_{31} + \overline{c}_1)\exp(-\alpha_1\tau) = \gamma_1 \exp(-\alpha_1\tau),\end{aligned}$$

where $|h_{10}\| \le c_{31}$, $\|\overline{\Pi}_1(\tau)\| \le \overline{c}_1 \exp(-\overline{\beta}_q\tau)$, and $\alpha_1 = \max(\beta_1, \overline{\beta}_1)$. Finally,

$$\begin{aligned}\|\Pi_q(\tau)\| \le \|X_{n-k}(\tau)\|\|[M^{-1}]_{n-k}\|\|h_{q0}\| + \|\overline{\Pi}_q(\tau)\| \le \\ \le c_1 \exp(-\beta_1\tau)c_2c_{3q} + \overline{c}_q \exp(-\overline{\beta}_q\tau) \\ \le (c_1c_2c_{3q} + \overline{c}_q)\exp(-\alpha_q\tau) = \gamma_q \exp(-\alpha_q\tau),\end{aligned}$$

where $|h_{q0}\| \le c_{3q}$, $\|\overline{\Pi}_q(\tau)\| \le \overline{c}_q \exp(-\overline{\beta}_q\tau)$, and $\alpha_q = \max(\beta_1, \overline{\beta}_q)$. Thus, the following theorem is true.

Theorem 1. *Let conditions* (H1)–(H5) *hold and let* $\det B_{11}(t) \neq 0$. *Then the boundary-value problems* (1.1), (1.2) *have a formally asymptotic solution of form* (2.1). *The coefficients of the regular and singular series have representations* (2.17) *and* (2.25) *for* $q = 1, 2, \ldots$. *For the boundary functions, the following estimate holds:*

$$\|\Pi_q(\tau)\| \le \gamma_q \exp(-\alpha_q\tau), \quad q = 0, 1, 2, \ldots,$$

where γ_q *and* α_q *are positive constants. Moreover, the equality*

$$\lim_{\varepsilon\to 0} x(t, \varepsilon) = x_0(t)$$

holds for $t \in (a, b]$.

Remark 1. The case where $\operatorname{rank} M = n_1 < \min(m, n - k + p - s)$ and $p = s$ is of independent interest.

3. A bound of the remainder term of the asymptotic series

The solution of the boundary-value problem (1.1), (1.2) we seek in the form

$$x(t, \varepsilon) = X_n(t, \varepsilon) + u_n(t, \varepsilon), \tag{3.1}$$

where $X_n(t, \varepsilon) = \sum_{i=0}^{n} \varepsilon^i(x_i(t) + \Pi_i(\tau))$, $\tau = \frac{t-a}{\varepsilon}$, $t \in [a, b]$.

We shall prove that, for $t \in [a, b]$ and $\varepsilon \in (0, \varepsilon_0]$, the function $u_n(t, \varepsilon)$ satisfies the inequality $\|u_n(t, \varepsilon)\| \le K\varepsilon^{n+1}$, where $K > 0$ and $\lim_{\varepsilon\to 0} x(t, \varepsilon) = x_0(t)$.

Let the smoothness degree of the elements of the matrix $A_1(t)$ and the function $\varphi(t)$ is $n+2$.

If $u_n(t,\varepsilon) = \varepsilon^{n+1}(x_{n+1}(t) + \Pi_{n+1}) + u_{n+1}(t,\varepsilon)$ and we should prove that $\|u_{n+1}(t,\varepsilon)\| \le \bar{K}\varepsilon^{n+1}$, $\bar{K} > 0$, then there would exist a positive constant K such that $\|u_n(t,\varepsilon)\| \le K\varepsilon^{n+1}$.

Substituting $x(t,\varepsilon) = X_{n+1}(t,\varepsilon) + u_{n+1}(t,\varepsilon)$ in problem (1.1), (1.2), for the determination of $u_{n+1}(t,\varepsilon)$, we get the boundary-value problem

$$\varepsilon\frac{du_{n+1}(t,\varepsilon)}{dt} = Au_{n+1}(t,\varepsilon) + G(t,u_{n+1},\varepsilon), \tag{3.2}$$

$$l(u_{n+1}(\cdot,\varepsilon)) = 0. \tag{3.3}$$

The function $G(t,u_{n+1},\varepsilon)$ has the form

$$G(t,u_{n+1}(t,\varepsilon),\varepsilon) = AX_{n+1}(t,\varepsilon) + \varepsilon A_1(t,\varepsilon)[X_{n+1}(t,\varepsilon) + u_{n+1}(t,\varepsilon)] + \\ + \varphi(t) - \varepsilon\frac{dX_{n+1}(t,\varepsilon)}{dt}$$

and satisfies the following conditions:

I. $\|G(t,0,\varepsilon)\| \le \xi\varepsilon^{m+2}$, where $\xi > 0$;

II. For all $\eta > 0$, a exists $\delta = \delta(\eta)$ and $\varepsilon_0 = \varepsilon_0(\eta)$ such that if $\|u'_{n+1}\| \le \delta$ and $\|u''_{n+1}\| \le \delta$, then

$$\|G(t,u'_{n+1},\varepsilon) - G(t,u''_{n+1},\varepsilon)\| \le \eta\|u'_{n+1} - u''_{n+1}\|$$

for $t \in [a,b]$ and $0 < \varepsilon \le \varepsilon_0$.

Let $A = \operatorname{diag}(\bar{A}, \bar{\bar{A}})$, $\bar{\bar{A}} = \operatorname{diag}(J, \Theta_{p-s})$ is a $(k \times k)$ matrix, $J = \operatorname{diag}(J_1, \ldots, J_s)$ is a $((k-p+s)\times(k-p+s))$ matrix. Then we represent u_{n+1} in the form

$$u_{n+1}(t,\varepsilon) = (\omega_1(t,\varepsilon),\ \omega_2(t,\varepsilon), \omega_3(t,\varepsilon))^T,$$

where $\omega_1(t,\varepsilon)$ is a $(n-k)$-dimensional vector, $\omega_2(t,\varepsilon)$ is a $(k-p+s)$-dimensional vector, and $\omega_3(t,\varepsilon)$ is a $(p-s)$-dimensional vector.

We introduce the following notation:

$$A_1(t,\varepsilon) = \begin{pmatrix} A_{111}(t,\varepsilon) & A_{112}(t,\varepsilon) \\ A_{121}(t,\varepsilon) & A_{122}(t,\varepsilon) \end{pmatrix},$$

where $A_{111}(t,\varepsilon)$ is a $((n-k)\times(n-k))$ matrix, $A_{112}(t,\varepsilon)$ is a $((n-k)\times k)$ matrix, $A_{121}(t,\varepsilon)$ is a $(k\times(n-k))$ matrix, $A_{122}(t,\varepsilon)$ is a $(k\times k)$ matrix;

$$A_{112}(t) = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}, \quad A_{121}(t) = \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}, \quad A_{122}(t) = \begin{pmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{pmatrix},$$

where $B_1(t)$ is a $((n-k)\times(k-p+s))$ matrix, $B_2(t)$ is a $((n-k)\times(p-s))$ matrix, $C_1(t)$ is a $((k-p+s)\times(n-k))$ matrix, $C_2(t)$ is a $((p-s)\times(n-k))$ matrix, $D_{11}(t)$ is a $((k-p+s)\times(k-p+s))$ matrix, $D_{12}(t)$ is a $((k-p+s)\times(p-s))$ matrix, $D_{21}(t)$ is a $((p-s)\times(k-p+s))$ matrix, $D_{22}(t)$ is a $((p-s)\times(p-s))$ matrix;

$$G(t,0,0,\varepsilon)=\begin{pmatrix}G_1(t,0,0,0,\varepsilon)\\ G_2(t,0,0,0,\varepsilon)\\ G_3(t,0,0,0,\varepsilon)\end{pmatrix},$$

where $G_1(t,0,0,0,\varepsilon)$ is a $(n-k)$-dimensional vector, $G_2(t,0,0,0,\varepsilon)$ is a $(k-p+s)$-dimensional vector, $G_3(t,0,0,0,\varepsilon)$ is a $(p-s)$-dimensional vector.

System (3.2) takes the form

$$\varepsilon\frac{d\omega_1}{dt}=\bar{A}\omega_1+\varepsilon A_{111}(t)\omega_1+\varepsilon B_1(t)\omega_2+\varepsilon B_2(t)\omega_3+G_1(t,0,0,0,\varepsilon), \tag{3.4}$$

$$\varepsilon\frac{d\omega_2}{dt}=(J+\varepsilon D_{11}(t))\,\omega_2+\varepsilon D_{12}(t)\omega_3+\varepsilon C_1(t)\omega_1+G_2(t,0,0,0,\varepsilon), \tag{3.5}$$

$$\varepsilon\frac{d\omega_3}{dt}=\varepsilon D_{21}(t)\omega_2+\varepsilon D_{22}(t)\omega_3+\varepsilon C_2(t)\omega_1+G_3(t,0,0,0,\varepsilon). \tag{3.6}$$

Obviously, the inequalities $\|G_i(t,0,0,0,\varepsilon)\|\le c_{1i}\varepsilon^{n+2}$, $c_{1i}>0$, $i=1,2,3$, hold on $[a,b]$.

Let $W(t,s,\varepsilon)$ and $V(t,s)$ be the fundamental matrices for the homogeneous systems $\varepsilon\dot{x}=\bar{A}x$ and $\dot{x}=D_{22}x$. Here, $W(s,s,\varepsilon)=E_{n-k}$ and $V(s,s)=E_{p-s}$ are the unit matrices.

Let the Cauchy problem for the homogeneous system $\varepsilon\dot{x}=(J+\varepsilon D_{11}(t))x$ have only a trivial solution, and system (3.4) has the particular solution

$$\omega_2(t,\varepsilon)=\int_a^b K(t,s,\varepsilon)\left[\varepsilon D_{12}(s)\omega_3+\varepsilon C_1(s)\omega_1+G_2(s,0,0,0,\varepsilon)\right]ds,\quad t\in[a,b],$$

where

$$K(t,s,\varepsilon)=\begin{cases}\frac{1}{\varepsilon}\bar{X}(t,\varepsilon)\bar{X}^{-1}(s,\varepsilon), & \tau_{i-1}\le s\le t,\\ 0, & \tau_{i-1}\le t\le s,\end{cases}$$

if the eigenvalues of the matrix $J+\varepsilon D_{11}(t)$ are purely imaginary and

$$K(t,s,\varepsilon)=\begin{cases}\frac{1}{\varepsilon}\bar{X}(t,\varepsilon)P\bar{X}^{-1}(s,\varepsilon), & \tau_{i-1}\le s\le t,\\ \\ -\frac{1}{\varepsilon}\bar{X}(t,\varepsilon)(I-P)\bar{X}^{-1}(s,\varepsilon), & \tau_{i-1}\le t\le s,\end{cases}$$

if the eigenvalues are with a positive or negative real part. The matrix P is a spectral projector of the matrix $J+\varepsilon D_{11}(t)$ on the left half-plane, and $\bar{X}(t,\varepsilon)$ is a normal fundamental matrix for the system $\varepsilon\dot{x}=(J+\varepsilon D_{11}(t)x$.

Obviously, $\int_a^b\|K(t,s,\varepsilon)\|ds\le\xi_1,\ \xi_1>0$, for $t\in[a,b]$, $\varepsilon\in(0,\varepsilon_0]$.

Lemma 4 ([6, 10]). *For the matrix $W(t,s,\varepsilon)$, when $a < s \le t \le b$, $0 < \varepsilon \le \varepsilon_0$, the exponential estimate*

$$\|W(t,s,\varepsilon)\| \le \beta \exp\left(-\alpha\left(\frac{t-s}{\varepsilon}\right)\right), \quad a \le s \le t \le b,$$

is fulfilled, where $\alpha > 0$, $\beta > 0$.

It is clear that $\|V(t,s,\varepsilon)\| \le \beta_1$, where $a \le s \le t \le b$, $\beta_1 > 0$.

Lemma 5. *Any continuous solution of system* (3.4)–(3.6) *is a solution of the system of integral equations*

$$\begin{aligned}\omega_1(t,\varepsilon) = {} & W(t,a,\varepsilon)\omega_1(a,\varepsilon) + \int_a^t \frac{1}{\varepsilon}\left[\varepsilon A_{111}(s)\omega_1(s,\varepsilon) + \right.\\ & \left. + \varepsilon B_1(s)\omega_2(s,\varepsilon) + \varepsilon B_2(s)\omega_3(s,\varepsilon) + G_1(s,0,0,0,\varepsilon)\right] ds, \end{aligned} \tag{3.7}$$

$$\begin{aligned}\omega_2(t,\varepsilon) = {} & \int_a^b K(t,s,\varepsilon)\left[\varepsilon D_{12}(s)\omega_3(s,\varepsilon) + \varepsilon C_1(s)\omega_1(s,\varepsilon)\right.\\ & \left. + G_2(s,0,0,0,\varepsilon)\right] ds, \end{aligned} \tag{3.8}$$

$$\begin{aligned}\omega_3(t,\varepsilon) = {} & V(t,a)\omega_3(a,\varepsilon) + \int_a^t V(t,s)\frac{1}{\varepsilon}\left[\varepsilon D_{21}(t)\omega_2(s,\varepsilon) + \right.\\ & \left. + \varepsilon D_{22}(s)\omega_3 + \varepsilon C_2(s)\omega_1(s,\varepsilon) + G_3(s,0,0,0,\varepsilon)\right] ds. \end{aligned} \tag{3.9}$$

We substitute $u_{n+1}(t,\varepsilon) = (\omega_1(t,\varepsilon), \omega_2(t,\varepsilon), \omega_3(t,\varepsilon))^T$ into the boundary condition (3.3) and obtain

$$\overline{l}_1\omega_1((\cdot),\varepsilon) + \overline{l}_2\omega_2((\cdot),\varepsilon) + \overline{l}_3\omega_3((\cdot),\varepsilon) = 0,$$

where $\overline{l}_i$, $i = 1,2,3$ are linear m-dimensional bounded functionals. After transformations using (3.7)–(3.9), we obtain

$$\omega_1(t,\varepsilon) = W_i(t,a,\varepsilon)\omega_1(a,\varepsilon) + V_i(t,a,\varepsilon)\omega_3(a,\varepsilon) + S_i(t,\omega_1,\omega_3,a,\varepsilon), \tag{3.10}$$

$i = 1,2,3$, where $W_1(t,a,\varepsilon) = W(t,a,\varepsilon)$, and W_i, $i = 1,2$, V_i, S_i, $i = 1,2,3$, are functions such that, for all $t \in [a,b]$ and $\varepsilon \in (0,\varepsilon_0]$,

$$\begin{gathered}\|W_i(t,a,\varepsilon)\| \le \varepsilon k_i, \quad k_i > 0,\ i = 1,2,\\ \|V_i(t,a,\varepsilon) \le \varepsilon d_i, \quad d_i > 0,\ i = 1,2,\\ \|V_3(t,a,\varepsilon)\| \le \beta_2 + \varepsilon d_3, \quad \beta_2 > 0,\ d_3 > 0,\\ \|S_i(t,0,0,0,a,\varepsilon)\| \le c_i\varepsilon^{n+1}, \quad c_i > 0,\ i = 1,2,3,\end{gathered} \tag{3.11}$$

and

$$\begin{aligned}&\|S_i(t,\omega_1^2,\omega_3^2,a,\varepsilon) - S_i(t,\omega_1^1,\omega_3^1,a,\varepsilon)\| \le \\ &\qquad \le \varepsilon r_i \max_{t\in[a,b]}\left(\|\omega_1^2(t,\varepsilon) - \omega_1^1(t,\varepsilon)\| + \|\omega_3^2(t,\varepsilon) - \omega_3^1(t,\varepsilon)\|\right), \end{aligned} \tag{3.12}$$

where $r_i > 0$, $i = 1,2,3$. It follows from relation (3.10) that the vector $\omega(a,\varepsilon) = (\omega_1(a,\varepsilon), \omega_3(a,\varepsilon))^T$ is determined by the equation

$$R(\varepsilon)\omega(a,\varepsilon) = q(\varepsilon,\omega_1,\omega_3), \tag{3.13}$$

where $R(\varepsilon) = [R_1(\varepsilon) \quad R_2(\varepsilon)]$ is an $(m\times(n+p-k-s))$ matrix, $R_1(\varepsilon) = \overline{l_1}W_1((\cdot),a,\varepsilon)+\overline{l_2}W_2((\cdot),a,\varepsilon)+\overline{l_3}W_3((\cdot),a,\varepsilon)$ is an $(m\times(n-k))$ matrix, $R_2(\varepsilon) = \overline{l_1}V_1((\cdot),a,\varepsilon)+\overline{l_2}V_2((\cdot),a,\varepsilon)+\overline{l_3}V_3((\cdot),a,\varepsilon)$ is an $(m\times(p-s))$ matrix, and

$$q(\varepsilon,\omega_1,\omega_3) = -\overline{l_1}S_1(\cdot,\omega_1,\omega_3,a,\varepsilon) - \overline{l_2}S_2(\cdot,\omega_1,\omega_3,a,\varepsilon) - \overline{l_3}S_3(\cdot,\omega_1,\omega_3,a,\varepsilon)$$

is an m-dimensional vector. Also, one has $\|q(\varepsilon,0,0)\| \le c_4\varepsilon^{n+1}$, $c_4 > 0$, and

$$\|q(\varepsilon,\omega_1^2,\omega_3^2) - q(\varepsilon,\omega_1^1,\omega_3^1)\| \le \varepsilon r_4 \max_{t\in[a,b]}\left(\|\omega_1^2-\omega_1^1\| + \|\omega_3^2-\omega_3^1\|\right),$$

where $r_4 > 0$. Since

$$R(\varepsilon) = R_0 + O\left(\exp\left(-\frac{\alpha}{\varepsilon}\right)\right),$$

where R_0 is a constant matrix, then the following condition is fulfilled:

(H6) $m = n+p-k-s$; $\det R(\varepsilon) \neq 0$ $\forall\varepsilon\in[0,\varepsilon_0]$.

System (3.13) is always solvable and

$$\begin{aligned} \omega_1(a,\varepsilon) &= [R^{-1}]_{n-k}\, q(\varepsilon,\omega_1,\omega_3),\\ \omega_3(a,\varepsilon) &= [R^{-1}]_{p-s}\, q(\varepsilon,\omega_1,\omega_3). \end{aligned} \tag{3.14}$$

We shall substitute (3.14) into (3.7)–(3.9) and obtain a system which will be solved by the method of successive approximations. Let

$$\begin{aligned} &\omega_i^0(t,\varepsilon) = 0,\\ &\omega_i^{s+1}(t,\varepsilon) = W_i(t,a,\varepsilon)[R^{-1}]_{n-k}\, q(\varepsilon,\omega_1^s,\omega_3^s)+\\ &\qquad + V_i(t,a,\varepsilon)[R^{-1}]_{p-s}\, q(\varepsilon,\omega_1^s,\omega_3^s) + S_i(t,\omega_1^s,\omega_3^s,a,\varepsilon), \quad i=1,2,3, \end{aligned} \tag{3.15}$$

be the Picard successive approximations.

Theorem 2. *Let the conditions of Theorem 1 and assumption* (H6) *be fulfilled. If* $\|R^{-1}\| \le c_R$, *then there exists a positive constant* K *such that the asymptotic solution of the boundary-value problem* (1.1), (1.2) *has representation* (3.1), *where* $u_n(t,\varepsilon)$ *satisfies the inequality*

$$\|u_n(t,\varepsilon)\| \le K\varepsilon^{n+1}.$$

Moreover, $x(t,\varepsilon)$ *approaches the generating system when* $\varepsilon\to 0$ *and* $t\in(a,b]$.

Proof. By virtue of (3.10), (3.11), and (3.12), for the first approximation, we have

$$\max_{t\in[a,b]}\|\omega_i^1(t,\varepsilon) - \omega_i^0(t,\varepsilon)\| \le K_{i1}, \quad K_{i1} > 0,$$

where the constant $K_{i1}\varepsilon^{n+1}$ is determined by the constants c_R, k_i, d_i, c_i, and r_i.

Let $K^1 = \max_i(K_{i1})$ and $K^1\varepsilon^{n+1} = \delta$. For the last approximation we have

$$\max_{t\in[a,b]} \|\omega_i^2(t,\varepsilon) - \omega_i^1(t,\varepsilon)\| \le \varepsilon K_{i2}\delta, \quad K_{i2} > 0, \ i = 1,2,3.$$

Let $\varepsilon_0 = \frac{1}{2}\min_i(1/K_{i2})$. Then

$$\max_{t\in[a,b]} \|\omega_i^2(t,\varepsilon) -_i^1 (t,\varepsilon)\| \le \frac{1}{2}\delta = \frac{1}{2^2}2\delta.$$

Inductively we obtain

$$\max_{t\in[a,b]} \|\omega_i^{k+1}(t,\varepsilon) - \omega_i^k(t,\varepsilon)\| \le \frac{1}{2^{k+1}}2\delta.$$

This reveals that in the segment $[a,b]$, when ε is sufficiently small, the successive approximations (3.15) are absolutely and uniformly convergent. In addition, we have

$$\|\omega_i^{k+1}(t,\varepsilon)\| \le \sum_{j=1}^{k+1} \|\omega_i^j(t,\varepsilon) - \omega_i^{j-1}(t,\varepsilon)\| \le \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k}\right)\delta \le$$
$$\le \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} + \ldots\right)\delta = 2\delta.$$

Let

$$\lim_{k\to\infty} \omega_i^k(t,\varepsilon) = \omega_i(t,\varepsilon)$$

satisfy (3.10) identically. Then, on the interval $[a,b]$, for $\varepsilon \to 0$, the inequality

$$\|\omega_i(t,\varepsilon)\| \le 2\delta$$

is fulfilled. Consequently, system (3.10) has an unique continuous solution, which does not escape from the domain $\{(t,\omega) \mid a \le t \le b, \|\omega\| \le 2\delta\}$. Then, for all $t \in [a,b]$ and $\varepsilon \in (0,\varepsilon_0]$,

$$\|u_{n+1}(t,\varepsilon)\| \le \sum_{i=1}^{3} \|\omega_i(t,\varepsilon)\| \le 6\delta = 6K^1\varepsilon^{n+1} ,$$

i. e., there exists a positive constant K such that the inequality

$$\|u_n(t,\varepsilon)\| \le K\varepsilon^{n+1}$$

is fulfilled and

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t)$$

for all $t \in [a,b]$. □

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