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# The Libera generalized integral operator and Hardy spaces

*Georghe Miclăus*

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# THE LIBERA GENERALIZED INTEGRAL OPERATOR AND HARDY SPACES

GHEORGHE MICLĂUS

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ABSTRACT. In this paper, we determine the Hardy spaces to which $L_\gamma^n(f)$ belongs, $n \in \mathbb{N}^*$, if $f \in H^p$, where $L_\gamma^n = \underbrace{L_\gamma \circ L_\gamma \circ \cdots \circ L_\gamma}_{n}$ is the Libera generalized integral operator (2). As a corollary we obtain the Hardy spaces for some classes of analytic functions.

*Mathematics Subject Classification:* 30C35, 30C45

*Keywords:* Hardy spaces, Libera generalized operator

## 1. INTRODUCTION

Let $f$ be an analytic function in the unit disc $U = \{z \,\|z| < 1\}$ and let $H(U)$ denote the set of all analytic functions in $U$. In 1965, R. J. Libera [5] had studied the operator $L : H(U) \to H(U)$ defined by

$$L(f)(z) = \frac{2}{z} \int_0^z f(t)dt \tag{1}$$

showing that $L(S^*) \subset S^*$, where $S^*$ is the class of starlike functions, $L(K) \subset K$, where $K$ is the class of convex functions and $L(C) \subset C$, if $C$ is the class of close-to-convex functions. In 1969, S. D. Benardi generalized these results investigating in [1] a more general integral operator defined by $L_\gamma : H(U) \to H(U)$

$$L_\gamma(f)(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}\, dt\,. \tag{2}$$

He showed that if $\gamma \in \mathbb{N}^*$, then $L_\gamma$ preserves these properties. In this paper we determine the Hardy spaces, to which $L_\gamma^n(f)$ belongs, $n \in \mathbb{N}^*$, if $f \in H^p$, where $L_\gamma^n = \underbrace{L_\gamma \circ L_\gamma \circ \cdots \circ L_\gamma}_{n}$.

## 2. Preliminaries

For $f \in H(U)$ and $z = re^{i\theta}$, we set

$$M_p(r, f) = \begin{cases} \left( \dfrac{1}{2\pi} \int\limits_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{\frac{1}{p}}, & 0 < p < \infty \\ \sup\limits_{0 \le p \le 2\pi} \left| f(re^{i\theta}) \right|, & \text{for} \quad p = \infty . \end{cases}$$

A function $f \in H(U)$ is said to be of Hardy spaces $H^p\,(0 < p < \infty)$ if $M_p(r, f)$ remains bounded as $r \to 1^-$. $H^\infty$ is the class of bounded analytic functions in the unit disc.

We shall need the following well-known lemmas [2].

**Lemma 1.** *If $f' \in H^p$, $0 < p < 1$, then $f \in H^{\frac{p}{1-p}}$. If $f' \in H^p$, $1 \le p$, then $f \in H^\infty$.*

**Lemma 2.** *If $f \in H^p$ and $g \in H^q$, then $fg \in H^\lambda$, $\lambda = \dfrac{pq}{p+q}$.*

**Lemma 3** (Integral theorem of Hardy–Littlewood)**.** *If $f \in H^p$ and $F = \int_0^z f(t)dt$, then $f \in H^{\frac{p}{1-p}}$ for $0 < p < 1$, and $f \in H^\infty$ for $p \ge 1$.*

## 3. Main results

Let $\gamma \in C$, $\mathrm{Re}\gamma > 0$, let $f$ be an analytic function in $U$ and let $L_\gamma$ be the Libera generalized integral operator (Bernardi's operator) defined by (2).

**Theorem 1.** *If $f \in H^p$, then:*

(i) *if $n < \dfrac{1}{p}$, $n \in \mathbb{N}^*$, then $L_\gamma^n(f) \in H^\lambda$, $\lambda = \dfrac{p}{1-np}$;*

(ii) *if $n \ge \frac{1}{p}$, $n \in \mathbb{N}^*$, then $L_\gamma^n(f) \in H^\infty$, where $L_\gamma^n = \underbrace{L_\gamma \circ L_\gamma \circ \cdots \circ L_\gamma}_{n}$;*

(iii) $\left[ L_\gamma(f) \right]' \in H^p$.

*Proof. Assertions (i) and (ii).* If $f$ is analytic, $f(z) = z + a_2 z^2 + \ldots$, then there exists a unique function

$$L_\gamma(f)(z) = \frac{\gamma + 1}{z^\gamma} \int\limits_0^z f(t) t^{\gamma - 1}\, dt\,,$$

which is analytic in $U$. Indeed,

$$g(z) = \frac{f(z)}{z} = 1 + a_2 z + \ldots$$

is analytic in $U$. Let $h(z) = z\,[g(z)]^{\frac{1}{\gamma+1}}$ be the branch that is 1 for $z = 0$, and we have that $h(z) = z + b_2 z^2 + \ldots$ is analytic in $U$, $\frac{h(z)}{z} \neq 0$, $[h(z)]^{\gamma+1} = (a_2 z + \ldots)$. Hence

$$\begin{aligned} L_\gamma(f)(z) &= \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}\,dt = \frac{\gamma+1}{z^\gamma}\int_0^z \frac{h^{\gamma+1}(t)}{t}dt = \\ &= \frac{\gamma+1}{z^\gamma}\int_0^z t^\gamma\left(1 + a_2 t + \ldots\right)\,dt = z + \frac{\gamma+1}{\gamma+2}a_2 z^2. \end{aligned}$$

For the integral operator $L_\gamma$ we have $L_\gamma = A \circ B$, where

$$A(f)(z) = \frac{\gamma+1}{z^\gamma}f(z), \quad \gamma \in C, \quad \mathrm{Re}\gamma > 0$$

and

$$B(f)(z)) = \int_0^z f(t)t^{\gamma-1}dt.$$

We determine the Hardy class for $A$ and $B$. If $f \in H^p$, $\gamma = a + ib$, $a > 0$, then

$$M_p(r, A) \leq \frac{\gamma+1}{r^a}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}},$$

and hence $A(f) \in H^p$.

From integral theorem of Hardy–Littlewood (Lemma 3), we have $B(f) \in H^{\frac{p}{1-p}}$ for $f \in H^p$, $p < 1$, and $B(f) \in H^\infty$, for $f \in H^p$, $p \geq 1$. Because $L_\gamma(f) = A(B(f))$ we have for $f \in H^p$, $p < 1$ that $B(f) \in H^{\frac{p}{1-p}}$ and $A(B(f)) \in H^{\frac{p}{1-p}}$. Hence $L_\gamma(f) \in H^{\frac{p}{1-p}}$ for $p < 1$ and $L_\gamma(f) \in H^\infty$, for $p \geq 1$. Suppose that $n < \frac{1}{p}$, $n \in \mathbb{N}^*$and $L_\gamma^k(f) \in H^{\frac{p}{1-kp}}$ for $1 \leq k \leq n-1$. Then

$$L_\gamma^n(f) \in H^\lambda, \quad \lambda = \frac{\frac{p}{1-(n-1)p}}{1 - \frac{p}{1-(n-1)p}} = \frac{p}{1-np}.$$

If $n \geq \frac{1}{p}$, then there is a $k \in \mathbb{N}^*$ such that $k < \frac{1}{p} \leq k+1 \leq n$ and $L_\gamma^k(f) \in H^{\frac{p}{1-kp}}$ and $L_\gamma^{k+1}(f) \in H^\infty$.

*Assertion (iii).* We have

$$\left[L_\gamma(f)(z)\right]' = \frac{1}{z}\left[(\gamma+1)f(z) - \gamma L_\gamma(f)(z)\right]. \tag{3}$$

Applying the Minkowski inequality, we obtain

$$\begin{aligned} M_p^p\left(r,\left[L_\gamma(f)\right]\right) &= \frac{1}{2\pi}\int_0^{2\pi}\left|\left[L_\gamma(f)\left(re^{i\theta}\right)\right]\right|^p d\theta = \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left|re^{i\theta}\right|^p}\left|(\gamma+1)f(re^{i\theta})-\gamma L_\gamma(f)(re^{i\theta})\right|^p d\theta \le \\ &\le \frac{1}{2\pi}\frac{|\gamma+1|^p}{r^p}\int_0^{2\pi}\left|f(re^{i\theta})\right|^p d\theta + \frac{1}{2\pi}\frac{|\gamma|^p}{r^p}\int_0^{2\pi}\left|L_\gamma(f)(re^{i\theta})\right|^p d\theta. \end{aligned}$$

The value $M_p^p\left(r,\left[L_\gamma(f)\right]'\right)$ is bounded for

$$\min\left\{p, \frac{p}{1-p}\right\} = p.$$

Hence, $\left[L_\gamma(f)\right]' \in H^p$. That results are best possible because from $\left[L_\gamma(f)\right]' \in H^q$, $q > p$ applying (3) we should conclude that $f \in H^q$. □

For $n = 1$, these results were obtained in [4].

**Remark 1.** *Since the results for Hardy classes do not depend on $\gamma$, the results remain the same for $L = L_{\gamma 1} \circ L_{\gamma 2} \circ \cdots \circ L_{\gamma n}$, where $Re\gamma_i > 0$, $i = \overline{1, n}$.*

For $\gamma = 1$, $L_\gamma = L$ is the Libera operator (1), and we have the following

**Corollary 1.** *If $f$ is an analytic function in $U$, $\gamma \in \mathbb{C}$, $Re\gamma_i > 0$ and $f \in H^p$ then:*

(i) *if $n < \dfrac{1}{p}$, $n \in \mathbb{N}^*$, then $L^n(f) \in H^\lambda$, $\lambda = \dfrac{p}{1-np}$;*

(ii) *if $n \ge \dfrac{1}{p}$, $n \in \mathbb{N}^*$, then $L^n(f) \in H^\infty$;*

(iii) $[L^n(f)]' \in H^p$.

**Corollary 2.** *If $f \in K$, $f \neq \dfrac{z}{1-ze^{i\tau}}$, $\tau \in \mathbb{R}$ then $L_\gamma(f)$ is a bounded function.*

It is well-known that if $f \in K$ (a convex function) then $f \in H^{1+\varepsilon}$, $\varepsilon = \varepsilon(f) > 0$. Hence, we obtain from Theorem 1 that $L_\gamma(f) \in H^\infty$.

Therefore, $L_\gamma$ transforms $K$ into its subclass of bounded functions, excepting extremal functions.

**Corollary 3.** *If $f \in S^*$, $f(z) \neq \dfrac{z}{(1+ze^{i\tau})^2}$, then*

(i) $L_\gamma(f) \in H^1$;

(ii) $L_\gamma^2(f) \in H^\infty$ *if* $L_\gamma(f) \neq \dfrac{z}{(1+ze^{i\tau})^2}$, $\tau \in \mathbb{R}$.

It is known that if $f \in S^*$, then $f \in H^{\frac{1}{2}+\varepsilon}$, $\varepsilon = \varepsilon(f) > 0$. Hence $L_\gamma(f) \in H^1$. If

$$L_\gamma(f) = \frac{z}{(1+ze^{i\tau})},$$

then $L_\gamma^2(f) \in H^\infty$.

In other words, $L_\gamma$ transforms $S^*$ into its subclass and $L_\gamma^2$ transforms $S^*$ into a subclass of bounded functions, with the exception of extremal functions.

**Corollary 4.** *If* $f \in \mathbb{C}$, $f(z) \neq \dfrac{p(z)}{(1+ze^{i\tau})^2}$, $Rep(z) > 0$, *then*

(i) $L_\gamma(f) \in H^1$*;*

(ii) $L_\gamma^2(f) \in H^\infty$ *if* $L_\gamma(f) \neq \dfrac{p(z)}{(1+ze^{i\tau})}$, $Rep(z) > 0$.

It is known that if $f \in \mathbb{C}$ and $f(z) \neq \dfrac{p(z)}{(1+ze^{i\tau})^2}$, $\mathrm{Re}\,p(z) > 0$ then $f \in H^{\frac{1}{2}+\varepsilon}$, $\varepsilon = \varepsilon(f) > 0$. Hence, $L_\gamma(f) \in H^1$. If $L_\gamma(f) \neq \dfrac{p(z)}{(1+ze^{i\tau})}$, $\mathrm{Re}\,p(z) > 0$ from Theorem 1 we obtain $L_\gamma^2(f) \in H^\infty$.

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## *Author's Address*


**Gheorghe Miclăus:**

Department of Mathematics and Computer Science, North University of Baia Mare, Romania

*E-mail address*: `miclaus5@yahoo.com`