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BANACH–STEINHAUS TYPE THEOREM IN LOCALLY CONVEX SPACES FOR LINEAR Σ-LOCALLY LIPSCHITZIAN OPERATORS

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ABSTRACT. In the past, all Banach–Steinhaus type results have been established only for some special classes of locally convex spaces, e.g., barrelled spaces ([2], [3], [4]), s-barrelled spaces ([5]), strictly s-barrelled spaces ([6]), etc. Recently, Cui Chengri and Songho Han ([1]) have obtained a Banach–Steinhaus type result for linear bounded operators which is valid in every locally convex space.
In this paper we would like to prove the same result, but for linear locally Lipschitzian operators.

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Let (X, λ) and (Y, μ) be locally convex spaces. Assume that the locally convex topology μ is generated by the family $(q_\beta)_{\beta\in I}$ of semi-norms on Y. Let $B(X_\lambda)$ denote the family of bounded set in (X, λ) and let $\sigma \subset B(X_\lambda)$.

For a linear mapping $T : X \to Y$, a semi-norm p on Y and $C \in \sigma$, set

$$L(p, C)(T) = \sup_{h\in C} p(Th).$$

Let $T : X \to Y$ be a linear operator. T is said to be σ-locally Lipschitzian if

$$\forall C \in \sigma\ \forall \beta \in I \quad L(\beta, C)(T) \equiv L(q_\beta, C)(T) < \infty.$$

By $\operatorname{Lip}(X_\lambda, Y_\mu, \sigma)$ we denote the vector space of σ-locally Lipschitzian operators. Note that $\operatorname{Lip}(X_\lambda, Y_\mu, \sigma)$ is a locally convex space under the locally convex topology $\tau(\mu, \sigma)$ generated by the family of semi-norms $L(\beta, C)$, $\beta \in I$, $C \in \sigma$.

An operator $T : X \to Y$ is said to be sequentially continuous if $\{x_n\}$ is a sequence in X such that $x_n \to x$ then $Tx_n \to Tx$; T is said to be bounded if T sends bounded sets into bounded sets. Clearly, continuous operators are sequentially continuous; sequentially continuous operators are bounded, and linear bounded operators are σ-locally Lipschitzian but in general, converse implications fail. Let X', X^s, X^b and X^L_σ denote the families of continuous linear functionals, sequentially continuous linear functionals, bounded linear functionals and σ-locally Lipschitzian functionals on X, respectively. In general, the inclusions $X' \subset X^s \subset X^b \subset X^L_\sigma$ are strict.

Let $\theta(X, X^L_\sigma)$ denote the topology of uniform convergence on $\sigma(X^L_\sigma, X)$- Cauchy sequences in X^L_σ. Note that if $\sigma = B(X_\lambda)$, then $X^b = X^L_\sigma$ and consequently, $\theta(X, X^L_\sigma) = \theta(X, X^b)$.

Theorem 1. *Let (X, λ), (Y, μ) be locally convex spaces and $T_n : X \to Y$ σ- locally Lipschitzian operators, $n \in N$. If* weak-$\lim_n T_n x = Tx$ *exists at each $x \in X$, then the limit operator T maps $\theta(X, X^L_\sigma)$-bounded sets into bounded sets.*

Proof. Let $y' \in Y'$. Then $\lim_n y'(T_n x) = y'(Tx)$ for each $x \in X$. So $(y' \circ T_n)_{n \in N}$ is a $\sigma(X^L_\sigma, X)$-Cauchy sequence in X^L_σ. Suppose that B is $\theta(X, X^L_\sigma)$-bounded subset of X and $\{x_k\} \subset B$. Then $\frac{1}{k} x_k \to 0$ in $(X, \theta(X, X^L_\sigma))$, so $\lim_k \frac{1}{k} y'(T_n x_k) = 0$ uniformly in $n \in N$.

Now fix $\varepsilon > 0$. There is a $k_0 \in N$ such that $|\frac{1}{k} y'(T_n x_k)| < \frac{\varepsilon}{2}$ for all $n \in N$ and all $k \geq k_0$. Fix a $k \geq k_0$. Since $\lim_n y'(T_n x_k) = y'(Tx_k)$ there is an $n_0 \in N$ such that $|y'(T_{n_0} x_k) - y'(Tx_k)| < \frac{\varepsilon}{2}$. Therefore,

$$|y'(Tx_k)| \leq |y'(Tx_k) - y'(T_{n_0} x_k)| + |y'(T_{n_0} x_k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2k} < \varepsilon.$$

This shows that $\{y'(Tx) : x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is μ-bounded by the classical Mackey theorem. Thus, we obtain the proof. □

Theorem 1 shows that the limit operator T can be bounded even if the sequence $(T_n)_n$ is not bounded.

Now we have the following useful result.

Theorem 2. *For a locally convex space X the following conditions are equivalent.*

(1) *For every locally convex space Y and for every sequence T_n of σ- locally Lipschitzian linear operators from X into Y such that* weak-$\lim_n T_n x = Tx$ *exists at each $x \in X$, the limit operator T is σ- locally Lipschitzian.*

(2) $(X^L_\sigma, \ \sigma(X^L_\sigma, X))$ *is sequentially complete.*

Proof. (1)⇒ (2). Let $\{f_n\}$ be a $\sigma(X^L_\sigma, X)$- Cauchy sequence in X^L_σ. Then, there exists a linear functional f such that for every $x \in X$ $\lim_n f_n(x) = f(x)$. Consequently, $f \in X^L_\sigma$ by (1).

(2)⇒ (1). Let Y be a locally convex space and $\{T_n\}$ a sequence of σ- locally Lipschitzian linear operators from X into Y such that weak-$\lim_n T_n x = Tx$ exists at each $x \in X$. Let $y' \in Y'$, $C \in \sigma$. Then $\lim_n y'(T_n x) = y'(Tx)$ at each $x \in X$. Since $y' \circ T_n \in X^L_\sigma$ for all $n \in N$, $y' \circ T \in X^L_\sigma$ by (2). Therefore $\{y'(Tx) : \ x \in C\}$ is bounded and hence $T(C)$ is μ-bounded by the classical Mackey theorem. Thus, T is σ- locally Lipschitzian. □

Remark 1. The proof of [1, Theorem 4] seems to be incorrect and, hence, the result is not correct because even if $\lim_n y'(T_n x) = y'(Tx)$ for each $x \in X$, $\{y' \circ T_n : n \in N\}$

is not necessarily conditionally $\sigma(X^b, X)$-sequentially compact. Indeed, $y' \circ T$ is not necessarily in X^b. Consequently, $y' \circ T_n$ do not converge to $y' \circ T$ in $(X^b, \sigma(X^b, X))$.

Let X^θ denote the space of linear $\theta(X, X^L_\sigma)$-bounded functionals on X. By $\eta(X, X^\theta)$ we denote the topology of uniform convergence on conditionally $\sigma(X^\theta, X)$-sequentially compacts sets of X^θ.

Now we have a useful proposition as follows.

Theorem 3. *Let* (X, λ), (Y, μ) *be locally convex spaces and* $T_n : X \to Y$ *σ-locally Lipschitzian operators,* $n \in N$. *If* weak-$\lim_n T_n x = Tx$ *exists at each* $x \in X$, *then the limit operator* T *maps* $\eta(X, X^\theta)$*-bounded sets into bounded sets.*

Proof. Let $y' \in Y'$. Then $\lim_n y'(T_n x) = y'(Tx)$ for each $x \in X$. It follows from Theorem 2 that $T \in X^\theta$. Consequently, $y' \circ T \in X^\theta$. On the other hand, since $(y' \circ T_n)_{n \in N}$ is $\sigma(X^L_\sigma, X)$-Cauchy sequence in X^L_σ, then $y' \circ T_n \in X^\theta$. Therefore, $\{y' \circ T_n : n \in N\}$ is conditionally $\sigma(X^\theta, X)$-sequentially compact.

Suppose that B is a $\eta(X, X^\theta)$-bounded subset of X and $\{x_k\} \subset B$. Then $\frac{1}{k} x_k \to 0$ in $(X, \eta(X, X^\theta))$, so $\lim_k \frac{1}{k} y'(T_n x_k) = 0$ uniformly in $n \in N$.

Now fix $\varepsilon > 0$. There is a $k_0 \in N$ such that $|\frac{1}{k} y'(T_n x_k)| < \frac{\varepsilon}{2}$ for all $n \in N$ and all $k \geq k_0$. Fix a $k \geq k_0$. Since $\lim_n y'(T_n x_k) = y'(T x_k)$ there is an $n_0 \in N$ such that $|y'(T_{n_0} x_k) - y'(T x_k)| < \frac{\varepsilon}{2}$. Therefore,

$$|y'(Tx_k)| \leq |y'(Tx_k) - y'(T_{n_0} x_k)| + |y'(T_{n_0} x_k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2k} < \varepsilon.$$

This shows that $\{y'(Tx) : x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is μ-bounded by the classical Mackey theorem. Thus, we achieve the proof. □

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