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Quanguo Chen

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THE MASCHKE-TYPE THEOREM OF SMASH PRODUCTS OF GENERALIZED QUANTUM COMMUTATIVE ALGEBRAS OVER WEAK HOPF ALGEBRAS

Q. G. CHEN

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Abstract. The paper is concerned with the semisimplicity of smash products of generalized quantum commutative algebras in weak Hopf algebra setting. Let H be a weak Hopf algebra over a field k and A any semisimple and generalized quantum commutative weak Yetter-Drinfeld H-module algebra. It is shown that $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module. Applying results to quasitriangular (weak) Hopf algebras is considered.

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1. INTRODUCTION

In [10], Yang and Wang have proved the following statement:
Suppose that H is a finite dimensional quasitriangular Hopf algebra acting on an algebra A and A is quantum commutative. If A is semisimple, then $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module.

The statement above is extended to weak Hopf algebras setting by Zhai and Zhang in [11].

Weak Hopf algebras were introduced by Böhm et al. in [1] as an important generalization of ordinary Hopf algebras and groupoid algebras besides quasi-Hopf algebras, multiplier Hopf algebras, Hopf quasigroups, etc ([5, 6, 9]). The axioms are the same as the ones for a Hopf algebra, except that the coproduct of the unit, the product of the counit and the antipode condition are replaced by weaker properties. The initial motivation to study weak Hopf algebras comes from the fact that some classical theory and lots of basic properties of ordinary Hopf algebras have "weak" analogues (see [3, 4, 7, 8]).

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Given a quasitriangular weak Hopf algebra (H,R), a weak H-module algebra A can be endowed with a suitable coaction associated to R to make A be a weak quantum Yetter-Drinfeld H-module algebra which is similar to one studied by Caenepeel et al. in the setting of Hopf algebras [2].

The purpose of this paper is to discuss the validity of the statement such as the start under the more general assumption.

The paper is organized as follows.

In Section 2, we recall basic definitions and give a summary of fundamental properties concerning weak Hopf algebras and quasitriangular weak Hopf algebras and weak Yetter-Drinfeld H-module algebras. In Section 3, as the main content, we discuss the semisimplicity of smash products of generalized quantum commutative algebras in weak Hopf algebra setting. Finally, the application of our results is considered.

2. Preliminaries

Throughout the paper k is a fixed field. Unless otherwise stated, all vector spaces are over k and all maps are k-linear. We will use the Heyneman-Sweedler notation $\Delta(c)=c_{(1)}\otimes c_{(2)}$ for coproduct (summation understood). $\otimes$ mean $\otimes_k$ unless otherwise specified, etc.

2.1. *Weak Hopf algebras*

Recall from Böhm et al. ([1]) that a *weak Hopf algebra* (H,Δ,ε,S) is both an associative algebra and a coalgebra with an antipode $S:H\to H$ satisfying the following conditions (1)-(4):

$$\begin{cases} (1) & \Delta(xy)=\Delta(x)\Delta(y) \text{ for all } x,y\in H;\\ (2) & \Delta^2(1)=(\Delta(1)\otimes 1)(1\otimes\Delta(1)),\ \ \Delta^2(1)=(1\otimes\Delta(1))(\Delta(1)\otimes 1);\\ (3) & \varepsilon(xyz)=\varepsilon(xy_{(1)})\varepsilon(y_{(2)}z),\ \ \varepsilon(xyz)=\varepsilon(xy_{(2)})\varepsilon(y_{(1)}z) \text{ for all } x,y,z\in H;\\ (4) & x_{(1)}S(x_{(2)})=\varepsilon(1_{(1)}x)1_{(2)},\ \ S(x_{(1)})x_{(2)}=1_{(1)}\varepsilon(x1_{(2)}),\\ & \qquad\qquad\qquad\qquad S(x_{(1)})x_{(2)}S(x_{(3)})=S(x), \text{for all } x\in H. \end{cases}$$

Let H be a weak Hopf algebra. Then we define the maps $\varepsilon_t,\varepsilon_s:H\to H$ by the formulas

$$\varepsilon_t(x)=\varepsilon(1_{(1)}x)1_{(2)},\ \ \varepsilon_s(x)=1_{(1)}\varepsilon(x1_{(2)})$$

and denote by H_t the image $\varepsilon_t(H)$ and denote by H_s the image $\varepsilon_s(H)$.

Let H be a weak Hopf algebra. The following results hold from Böhm et al. ([1]), for all $h,g\in H$,

(W1) H_t and H_s are two sub-algebras of H,

(W2) $\Delta(1)=1_{(1)}\otimes 1_{(2)}\in H_s\otimes H_t,\ \ \varepsilon_t(h)\varepsilon_s(g)=\varepsilon_s(g)\varepsilon_t(h)$,

(W3) $\Delta(\varepsilon_t(h))=1_{(1)}\varepsilon_t(h)\otimes 1_{(2)},\ \ \Delta(\varepsilon_s(g))=1_{(1)}\otimes\varepsilon_s(g)1_{(2)}$,

(W4) $h_{(1)}\otimes\varepsilon_s(h_{(2)})=h1_{(1)}\otimes S(1_{(2)}),\ \ \varepsilon_t(h_{(1)})\otimes h_{(2)}=S(1_{(1)})\otimes 1_{(2)}h$,

(W5) $h_{(1)} \otimes \varepsilon_t(h_{(2)}) = 1_{(1)}h \otimes 1_{(2)},\ \ \varepsilon_s(h_{(1)}) \otimes h_{(2)} = 1_{(1)} \otimes h1_{(2)}$,
(W6) $\varepsilon_t \circ \varepsilon_t = \varepsilon_t,\ \ \varepsilon_s \circ \varepsilon_s = \varepsilon_s$,
(W7) $\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s,\ \ \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t$,
(W8) $S(hg) = S(g)S(h),\ S(h_{(2)}) \otimes S(h_{(1)}) = S(h)_{(1)} \otimes S(h)_{(2)}$ and $S(1) = 1,\ \varepsilon \circ S = \varepsilon$,
(W9) $h_1\varepsilon_s(g) \otimes h_2 = h_{(1)} \otimes h_{(2)}S(\varepsilon_s(g)),\ \ h_1 \otimes \varepsilon_t(g)h_2 = S(\varepsilon_t(g))h_{(1)} \otimes h_{(2)}$.

2.2. *Quasitriangular weak Hopf algebras*

Recall from Nikshych et al. in [8] that a *quasitriangular weak Hopf algebra* is a pair (H, R), where H is a weak Hopf algebra and $R = R^1 \otimes R^2 \in \Delta^{op}(1)(H \otimes H)\Delta(1)$ such that

(Q1) There exists $\bar{R} \in \Delta(1)(H \otimes_k H)\Delta^{op}(1)$ with $R\bar{R} = \Delta^{op}(1)$ and $\bar{R}R = \Delta(1)$.
(Q2) For all $h \in H$, we have

$$\begin{cases} \Delta^{op}(h)R = R\Delta(h), \\ (id \otimes \Delta)R = R_{13}R_{12}, \\ (\Delta \otimes id)R = R_{13}R_{23} \end{cases}$$

where Δ^{op} denotes the comultiplication opposite to Δ, $R_{12} = R \otimes 1,\ \ R_{23} = 1 \otimes R$, etc., as usual.

Let (H, R) be a quasitriangular weak Hopf algebra. Then the following six identities hold:

$$\begin{cases} (\varepsilon_s \otimes id)(R) = \Delta(1),\ \ (id \otimes \varepsilon_s)(R) = (S \otimes id)\Delta^{op}(1), \\ (\varepsilon_t \otimes id)(R) = \Delta^{op}(1),\ \ (id \otimes \varepsilon_t)(R) = (S \otimes id)\Delta(1), \\ (S \otimes id)(R) = (id \otimes S^{-1})(R) = \bar{R},\ \ (S \otimes S)(R) = R. \end{cases}$$

2.3. *Weak module algebras*

Let H be a weak Hopf algebra. An algebra A is called a left *weak H-module algebra*, if A is a left H-module via $h \otimes a \mapsto h \cdot a$ such that, for any $a, b \in A$ and $h \in H$,

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \qquad h \cdot 1_A = \varepsilon_t(h) \cdot 1_A.$$

Let H be a weak Hopf algebra and A a weak left H-module algebra. Recall from Nikshych in [8] that a *weak smash product* $A\sharp H$ of A and H is defined on a k-vector space $A \otimes_{H_t} H$, where H is a left H_t -module via its multiplication and A is a right H_t- module via

$$a \cdot x = S^{-1}(x) \cdot a = a(x \cdot 1_A),\ a \in A,\ x \in H_t.$$

Its multiplication is given by the following formula:

$$(a\sharp h)(b\sharp g) = a(h_{(1)} \cdot b)\sharp h_{(2)}g,$$

for any $a, b \in A$ and $g, h \in H$. Then $A\sharp H$ is an associative algebra with unit $1_A\sharp 1_H$.

2.4. *Weak quantum Yetter-Drinfeld H-module algebras*

A k-algebra A is called a *weak quantum Yetter-Drinfeld H-module algebra*, if A satisfies the following conditions:

(WQ1) A is a weak left $H-$module algebra,

(WQ2) A is a right H^{op}-comodule algebra, i.e., the comodule structure map $\rho: A \to A \otimes H$ satisfies

$$a_{[0]} \otimes a_{[1]} = 1_{(1)} \cdot a_{[0]} \otimes 1_{(2)} a_{[1]},\ \rho(ab) = a_{[0]} b_{[0]} \otimes b_{[1]} a_{[1]},\ \rho(1) = (id \otimes \varepsilon_t)\rho(1),$$

where $\rho(a) = a_{[0]} \otimes a_{[1]}$ denotes the coaction.

(WQ3) (WQ1) and (WQ2) satisfy the Yetter-Drinfeld condition

$$(h_{(2)} \cdot a)_{[0]} \otimes (h_{(2)} \cdot a)_{[1]} h_{(1)} = h_{(1)} \cdot a_{[0]} \otimes h_{(2)} a_{[1]},$$

for all $h \in H, a \in A$.

Remark 1. The Yetter-Drinfeld condition is equivanlent to

$$\rho(h \cdot a) = h_{(2)} \cdot a_{[0]} \otimes h_{(3)} a_{[1]} S^{-1}(h_{(1)}). \tag{2.1}$$

Example 1. Let H be a weak Hopf algebra. $\{1_{(1)} h S(1_{(2)}) | h \in H\}$ is a weak quantum Yetter-Drinfeld H-module algebra with the action and coaction given as follows:

$$g \cdot (1_{(1)} h S(1_{(2)})) = g_{(1)} h S(g_{(2)}),\ \rho(1_{(1)} h S(1_{(2)})) = 1_{(1)} h_{(2)} S(1_{(2)}) \otimes S^{-1}(h_{(1)}).$$

Example 2. Let (H, R) be a quasitriangular weak Hopf algebra. Given any left weak H-module algebra A, one can define a right H^{op}-coaction on A as follows:

$$\rho(a) = R^2 \cdot a \otimes R^1.$$

With the above coaction, it is easily checked that A is a weak quantum Yetter-Drinfeld H-module algebra.

3. THE MAIN RESULTS

In this section, we assume that H is a weak Hopf algebra with bijective antipode S, and A a weak left H -module algebra, and $A \sharp H$ the weak smash product algebra.

Definition 1. Let H be a weak Hopf algebra and A a weak quantum Yetter-Drinfeld H-module algebra. A is called a *generalized quantum commutative algebras*, if A satisfies, for all $a, b \in H$

$$ab = b_{[0]}(b_{[1]} \cdot a).$$

Lemma 1. *For any weak quantum Yetter-Drinfeld H-module algebra A. Then, for all $a \in A$,*

$$1_{[0]} \otimes 1_{[1](1)} \otimes 1_{[1](2)} = 1_{[0]} \otimes 1_{(1)} 1_{[1]} \otimes 1_{(2)} \tag{3.1}$$

$$a_{[0]} \otimes \varepsilon_t(a_{[1]}) = a 1_{[0]} \otimes 1_{[1]}, \tag{3.2}$$

$$a_{[0]}\otimes\varepsilon_s(a_{[1]}) = 1_{[0]}a\otimes S(1_{[1]}). \tag{3.3}$$

Lemma 2. *For any generalized quantum commutative algebra A. Then, for all $a,b\in A$,*

$$ab = (S(a_{[1]})\cdot b)a_{[0]} \Longleftrightarrow ab = b_{[0]}(b_{[1]}\cdot a). \tag{3.4}$$

Proof. For all $a,b\in A$, we have

$$\begin{aligned}(S(a_{[1]})\cdot b)a_{[0]} &= a_{[0][0]}(a_{[0][1]}S(a_{[1]})\cdot b)\\ &= a_{[0]}(\varepsilon_t(a_{[1]})\cdot b)\\ &= a1_{[0]}(1_{[1]}\cdot b) = ab.\end{aligned}$$

Conversely, For all $a,b\in A$, we have

$$\begin{aligned}b_{[0]}(b_{[1]}\cdot a) &= (S(b_{[0][1]})b_{[1]}\cdot a)b_{[0][0]}\\ &= (S(b_{[1](1)})b_{[1](2)}\cdot a)b_{[0]}\\ &= (S(1_{[1]})\cdot a)1_{[0]}b = ab.\end{aligned}$$

So we finish the proof. □

Lemma 3. *For any left weak H-module algebra A. Then M is a left $A\sharp H$-module if and only if M is both a left H-module and a left A-module and satisfies the following compatible condition*

$$h\cdot(a\cdot m) = (h_{(1)}\cdot a)\cdot(h_{(2)}\cdot m),$$

for all $h\in H$, $a\in A$ and $m\in M$.

Lemma 4. *Let H be a weak Hopf algebra and A a generalized quantum commutative algebra. Then, for all $a,b\in A$,*

$$a_{[0]}b_{[0]}\otimes b_{[1]}a_{[1]} = b_{[0]}(b_{[1](2)}\cdot a)_{[0]}\otimes(b_{[1](2)}\cdot a)_{[1]}b_{[1](1)}. \tag{3.5}$$

Proof. For all $a,b\in A$, apply ρ to the identity $ab = b_{[0]}(b_{[1]}\cdot a)$, we have

$$\begin{aligned}\rho(b_{[0]}(b_{[1]}\cdot a)) &= b_{[0][0]}(b_{[1]}\cdot a)_{[0]}\otimes(b_{[1]}\cdot a)_{[1]}b_{[0][1]}\\ &= b_{[0]}(b_{[1](2)}\cdot a)_{[0]}\otimes(b_{[1](2)}\cdot a)_{[1]}b_{[1](1)}\\ &= \rho(ab).\end{aligned}$$

The proof is completed. □

Lemma 5. *Let H be a weak Hopf algebra and A a generalized quantum commutative algebra. If M is a left $A\sharp H$-module, then M is an A-bimodule with the right module action of A on M as follows*

$$\leftharpoonup: M\otimes A\to M, m\otimes a\mapsto m\leftharpoonup a = a_{[0]}\cdot(a_{[1]}\cdot m),$$

and for all $h\in H$,

$$h\cdot(m\leftharpoonup a) = (h_{(1)}\cdot m)\leftharpoonup(h_{(2)}\cdot a).$$

Proof. First, we shall check that M is a right A-module. In fact, for all $m \in M$ and $a, b \in A$, we have

$$\begin{aligned}
(m \leftharpoonup a) \leftharpoonup b &= (a_{[0]} \cdot (a_{[1]} \cdot m)) \leftharpoonup b \\
&= b_{[0]} \cdot (b_{[1]} \cdot (a_{[0]} \cdot (a_{[1]} \cdot m))) \\
&= b_{[0]} \cdot ((b_{[1](1)} \cdot a_{[0]}) \cdot (b_{[1](2)} a_{[1]} \cdot m)) \\
&= (b_{[0]} (b_{[1](2)} \cdot a)_{[0]}) \cdot ((b_{[1](2)} \cdot a)_{[1]} b_{[1](1)} \cdot m) \\
&= a_{[0]} b_{[0]} \cdot (b_{[1]} a_{[1]} \cdot m) \\
&= (ab)_{[0]} ((ab)_{[1]} \cdot m) \\
&= m \leftharpoonup ab
\end{aligned}$$

$$\begin{aligned}
m \leftharpoonup 1 &= 1_{[0]} \cdot (1_{[1]} \cdot m) \\
&= 1_{[0]} (1_{(1)} \cdot 1_A) \cdot (1_{(2)} 1_{[1]} \cdot m) \\
&= 1_{[0]} (S(1_{(1)}) \cdot 1_A) \cdot (1_{(2)} \varepsilon_t (1_{[1]}) \cdot m) \\
&= (1_{[0]} (\varepsilon_t (1_{[1]}) S(1_{(1)}) \cdot 1_A)) \cdot (1_{(2)} \cdot m) \\
&= (1_{[0]} (1_{[1]} S(1_{(1)}) \cdot 1_A)) \cdot (1_{(2)} \cdot m) \\
&= (S(1_{(1)}) \cdot 1_A) \cdot (1_{(2)} \cdot m) \\
&= (1_{(1)} \cdot 1_A) \cdot (1_{(2)} \cdot m) = m.
\end{aligned}$$

Now, we shall check M is an A-bimodule, i.e., $(a \cdot m) \leftharpoonup b = a \cdot (m \leftharpoonup b)$. As a matter of fact,

$$\begin{aligned}
(a \cdot m) \leftharpoonup b &= b_{[0]} \cdot (b_{[1]} \cdot (a \cdot m)) \\
&= b_{[0]} \cdot ((b_{[1](1)} \cdot a) \cdot (b_{[1](2)} \cdot m)) \\
&= (b_{[0]} (b_{[1](1)} \cdot a)) \cdot (b_{[1](2)} \cdot m) \\
&= (b_{[0][0]} (b_{[0][1]} \cdot a)) \cdot (b_{[1]} \cdot m) \\
&= a b_{[0]} \cdot (b_{[1]} \cdot m) \\
&= a \cdot (m \leftharpoonup b).
\end{aligned}$$

Finally, for all $a \in A, m \in M$ and $h \in H$,

$$\begin{aligned}
(h_{(1)} \cdot m) \leftharpoonup (h_{(2)} \cdot a) &= (h_{(2)} \cdot a)_{[0]} ((h_{(2)} \cdot a)_{[1]} h_{(1)} \cdot m) \\
&= (h_{(1)} \cdot a_{[0]}) (h_{(2)} a_{[1]} \cdot m) \\
&= h \cdot (a_{[0]} \cdot (a_{[1]} \cdot m)) = h \cdot (m \leftharpoonup a).
\end{aligned}$$

The proof is completed. □

Lemma 6. *For all left $A\sharp H$-module M, we have*

$$_{A\sharp H}Hom(A,M)\cong M^{H}, F: f\mapsto f(1_A),$$

where $M^{H}=\{m\in M|h\cdot m=\varepsilon_t(h)\cdot m,\quad \forall h\in H\}$ *and A is left $A\sharp H$-module via*

$$(a\sharp h)\cdot b=a(h\cdot b).$$

Proof. For given $0\neq m\in M^{H}$, we define a map via

$$f: a\mapsto (a\sharp 1_H)\cdot m.$$

Throughout standard computation, we can show that $f\in_{A\sharp H}\mathrm{Hom}(A,M)$. Based on this, we can check that F is bijective in a straightforward way. □

Lemma 7. *Let H be a weak Hopf algebra and A a generalized quantum commutative algebra. Then, for all left $A\sharp H$-modules M and N.*

(1) *$Hom_A(M_A,N_A)\in_{A\sharp H} M$,*

(2) *$Hom_A(M_A,N_A)^{H}={}_{A\sharp H}Hom(M,N)$,*

where $Hom_A(M_A,N_A)$ denotes the space of the right A-module homomorphisms.

Proof. (1) Let $M,N\in_{A\sharp H} M$. Then M,N are both A-bimodules from Lemma 5. For all $a\sharp h\in A\sharp H$, $f\in\mathrm{Hom}_A(M_A,N_A)\in_{A\sharp H} M$, define action of $A\sharp H$ on $\mathrm{Hom}_A(M_A,N_A)\in_{A\sharp H} M$ by

$$((a\sharp h)\cdot f)(m)=a\cdot(h_{(1)}\cdot f(S(h_{(2)})\cdot m)),$$

for all $m\in M$. We shall check $(a\sharp h)\cdot f\in\mathrm{Hom}_A(M_A,N_A)$. For all $c\in A$, we have

$$\begin{aligned}((a\sharp h)\cdot f)(m\leftharpoonup c)&=a\cdot(h_{(1)}\cdot f(S(h_{(2)})\cdot(m\leftharpoonup c)))\\&=a\cdot((h_{(1)}\cdot f(S(h_{(4)})\cdot m))\leftharpoonup h_{(2)}S(h_{(3)})\cdot c)\\&=a\cdot((1_{(1)}h_{(1)}\cdot f(S(h_{(2)})\cdot m))\leftharpoonup 1_{(2)}\cdot c)\\&=a\cdot((h_{(1)}\cdot f(S(h_{(2)})\cdot m))\leftharpoonup c)\\&=(a\cdot(h_{(1)}\cdot f(S(h_{(2)})\cdot m)))\leftharpoonup c\\&=((a\sharp h)\cdot f)(m)\leftharpoonup c.\end{aligned}$$

It is checked directly that $\mathrm{Hom}_A(M_A,N_A)\in_{A\sharp H} M$ is a left $A\sharp H$-module.

(2) For all $f\in_{A\sharp H}\mathrm{Hom}(M,N)$, then f is both a left H-module morphism and a left A-module morphism between M and N. We shall check that $f\in\mathrm{Hom}_A(M_A,N_A)^{H}$. For all $m\in M$ and $a\in A$, since

$$f(m\leftharpoonup a)=f(a_{[0]}\cdot(a_{[1]}\cdot m))=a_{[0]}\cdot(a_{[1]}\cdot f(m))=f(m)\leftharpoonup m,$$

we conclude that f is a right A-linear. Also, for any $h\in H$ and $m\in M$, we have

$$(h\cdot f)(m)=h_{(1)}\cdot f(S(h_{(2)})\cdot m)=h_{(1)}S(h_{(2)})\cdot f(m)=\varepsilon_t(h)\cdot f(m),$$

i.e., $f\in\mathrm{Hom}_A(M_A,N_A)^{H}$. Conversely, First, we can define the left H_s-action on M by restricting the $A\sharp H$-action on $M: x\cdot m=m\leftharpoonup(h\cdot 1_A)$, for all $m\in M,h\in H_s$.

Using Remark 1 and Lemma 3, it is easy to see that right A-module homomorphisms are morphisms of left H_s-modules. For all $f \in \mathrm{Hom}_A(M_A, N_A)^H$, we have

$$\begin{aligned} h\cdot f(m) &= h_{(1)}\varepsilon_s(h_{(2)})\cdot f(m) \\ &= h_{(1)}\cdot f(\varepsilon_s(h_{(2)})\cdot m) \\ &= (h_{(1)}\cdot f)(h_{(2)}\cdot m) \\ &= (\varepsilon_t(h_{(1)})\cdot f)(h_{(2)}\cdot m) \\ &= (\varepsilon_t(1_{(1)})\cdot f)(1_{(2)}h\cdot m) \\ &= (1_{(1)}\cdot f)(1_{(2)}h\cdot m) \\ &= 1_{(1)}\cdot f(S(1_{(2)})1_{(3)}h\cdot m) \\ &= 1_{(1)}\varepsilon_s(1_{(2)})\cdot f(h\cdot m) = f(h\cdot m), \end{aligned}$$

i.e., f is a left H-module map. Now, we shall check that f is a left A-module map. Indeed, for all $a \in A$ and $m \in M$,

$$\begin{aligned} f(a\cdot m) &= f(a1_{[0]}\cdot(1_{[1]}\cdot m)) \\ &= f(a_{[0]}\cdot(\varepsilon_t(a_{[1]})\cdot m)) \\ &= f(a_{[0]}\cdot(a_{[1]1}S(a_{[1]2})\cdot m)) \\ &= f(a_{[0][0]}\cdot(a_{[0][1]}S(a_{[1]})\cdot m)) \\ &= f((S(a_{[1]})\cdot m)\leftharpoonup a_{[0]}) \\ &= (S(a_{[1]})\cdot f(m))\leftharpoonup a_{[0]} = a\cdot f(m). \end{aligned}$$

So we get that $f \in {}_{A\sharp H}\mathrm{Hom}(M,N)$. □

Now, we can present the main result in this section.

Theorem 1. *Let H be a weak Hopf algebra and A a generalized quantum commutative. If A is semisimple, then $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module algebra.*

Proof. Assume A is a projective left $A\sharp H$-module, then the functor ${}_{A\sharp H}\mathrm{Hom}(A,-)$ is exact. For any left $A\sharp H$-module M, it is viewed as a right A-module via " $\leftharpoonup$ " in Lemma 5. Since A is semisimple, M is projective as a right A-module. Hence the functor $\mathrm{Hom}_A(M,-)$ is exact. Further, the composition functor ${}_{A\sharp H}\mathrm{Hom}(A,\mathrm{Hom}_A(M,-))$ is also exact. From Lemma 6 and 7, we get

$${}_{A\sharp H}\mathrm{Hom}(A,\mathrm{Hom}_A(M,N)) \cong {}_{A\sharp H}\mathrm{Hom}(M,N),$$

for any left $A\sharp H$-module M and N. Then M is a projective left $A\sharp H$-module, hence $A\sharp H$ is semisimple. The converse is obvious. □

Next, we shall apply Theorem 1 to Example 1. Given a quasitriangular weak Hopf algebra (H,R) and weak H-module algebra A, A is a weak quantum Yetter- Drinfeld

H-module algebra with the coaction defined in Example 1. Then the generalized quantum commutative condition in Definition 1 takes the following form

$$ab = (R^2 \cdot b)(R^1 \cdot a).$$

With the assumption above and by Theorem 1, we have the main result of Zhai and Zhang in [11].

Corollary 1. *Let (H, R) be a quasitriangular weak Hopf algebra and A a quantum commutative. If A is semisimple, then $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module algebra.*

Remark 2. If $\Delta(1) = 1 \otimes 1$, weak Hopf algebras are just Hopf algebras. Corollary 1 recovers to the results of Yang and Wang in [10].

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Author's address

Q. G. Chen
Quan-guo Chen, Yili Normal University, Department of Mathematics, 835000 Yining, China
E-mail address: cqg211@163.com