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Two divisors of $(n^2+1)/2$ summing up to $\delta * n + \epsilon$, for δ and ϵ even

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TWO DIVISORS OF $(n^2+1)/2$ SUMMING UP TO $\delta n+\varepsilon$, FOR δ AND ε EVEN

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Abstract. In this paper we are dealing with the problem of the existence of two divisors of $(n^2+1)/2$ whose sum is equal to $\delta n+\varepsilon$, in the case when δ and ε are even, or more precisely in the case in which $\delta \equiv \varepsilon+2 \equiv 0$ or $2 \pmod 4$. We will completely solve the cases $\delta = 2, \delta = 4$ and $\varepsilon = 0$.

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1. INTRODUCTION

In [1], Ayad and Luca have proved that there does not exist an odd integer $n > 1$ and two positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = n + 1$. In [2], Dujella and Luca have dealt with a more general issue, where $n+1$ was replaced with an arbitrary linear polynomial $\delta n+\varepsilon$, where $\delta > 0$ and ε are given integers. The reason that d_1 and d_2 are congruent to 1 modulo 4 comes from the fact that $(n^2+1)/2$ is odd and is a sum of two coprime squares $((n+1)/2)^2 + ((n-1)/2)^2$. Such numbers have the property that all their prime factors are congruent to 1 modulo 4. Since $d_1 + d_2 = \delta n+\varepsilon$, then there are two cases: it is either $\delta \equiv \varepsilon \equiv 1 \pmod 2$, or $\delta \equiv \varepsilon+2 \equiv 0$ or $2 \pmod 4$. In [2] authors have focused on the first case.

In this paper, we deal with the second case, the case where $\delta \equiv \varepsilon+2 \equiv 0$ or $2 \pmod 4$. We completely solve cases when $\delta = 2, \delta = 4$ and $\varepsilon = 0$. We prove that there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n+\varepsilon$ for $\varepsilon \equiv 0 \pmod 4$ and we prove an analoguos result for $\varepsilon \equiv 2 \pmod 4$ and divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n+\varepsilon$. In case when $\delta \geq 6$ is a positive integer of the form $\delta = 4k+2,\ k \in \mathbb{N}$ we prove that there does not exist an odd integer n such that there exists a pair of divisors d_1, d_2 of $\frac{n^2+1}{2}$ with the property $d_1 + d_2 = \delta n$. We also prove that there exist infinitely many odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$.

2. THE CASE $\delta = 2$

Theorem 1. *If $\varepsilon \equiv 0 \pmod 4$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$.*

Proof. Let $\varepsilon \equiv 0 \pmod 4$. We want to find a positive odd integer n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$. Let $g = \gcd(d_1, d_2)$. We can write $d_1 = gd_1', d_2 = gd_2'$. Since $gd_1'd_2' = \operatorname{lcm}(d_1, d_2)$ divides $\frac{n^2+1}{2}$, we conclude that there exists a positive integer d such that

$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we can easily obtain

$$(d_2 - d_1)^2 = (2n+\varepsilon)^2 - 4\frac{g(n^2+1)}{2d},$$

$$(d_2 - d_1)^2 = 4n^2 + 4\varepsilon n + \varepsilon^2 - 2\frac{g(n^2+1)}{d},$$

$$d(d_2 - d_1)^2 = 4n^2 d + 4d\varepsilon n + \varepsilon^2 d - 2n^2 g - 2g,$$

$$d(d_2 - d_1)^2 = (4d - 2g)n^2 + 4d\varepsilon n + \varepsilon^2 d - 2g,$$

$$d(4d-2g)(d_2-d_1)^2 \\ = (4d-2g)^2 n^2 + 4(4d-2g)d\varepsilon n + 4d^2\varepsilon^2 - 8dg - 2\varepsilon^2 dg + 4g^2. \tag{2.1}$$

For $X = (4d-2g)n + 2d\varepsilon, Y = d_2 - d_1$, the equation (2.1) becomes

$$X^2 - d(4d-2g)Y^2 = 8dg + 2\varepsilon^2 dg - 4g^2.$$

For $g = 1$ the previous equation becomes

$$X^2 - 2d(2d-1)Y^2 = 8d + 2\varepsilon^2 d - 4,$$

$$X^2 - 2d(2d-1)Y^2 = 2d(4+\varepsilon^2) - 4. \tag{2.2}$$

The equation (2.2) is a Pellian equation. The right-hand side of (2.2) is nonzero.

Our goal is to make the right-hand side of (2.2) a perfect square. That condition can be satisfied by taking $d = \frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1$. With this choice of d, we get

$$2d(4+\varepsilon^2) - 4 = 2\left(\frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1\right)(4+\varepsilon^2) - 4 = \left(\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)\right)^2.$$

Pellian equation (2.2) becomes

$$X^2-2d(2d-1)Y^2=\left(\frac{1}{2}(\varepsilon^2-2\varepsilon+4)\right)^2. \tag{2.3}$$

Now, like in [2], we are trying to solve (2.3). We let

$$X=\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U,\ \ Y=\frac{1}{2}(\varepsilon^2-2\varepsilon+4)V.$$

The equation (2.3) becomes

$$U^2-2d(2d-1)V^2=1. \tag{2.4}$$

Equation (2.4) is a Pell equation which has infinitely many positive integer solutions (U,V), and consequently, there exist infinitely many positive integer solutions (X,Y) of (2.3). The least positive integer solution of (2.4) can be found using the continued fraction expansion of number $\sqrt{2d(2d-1)}$.

We can easily get $\sqrt{2d(2d-1)}=[2d-1;\overline{2,4d-2}]$. All positive solutions of (2.4) are given by (U_m,V_m) for some $m\geq 0$. The first few solutions are
$(U_0,V_0)=(1,0)$,
$(U_1,V_1)=(4d-1,2)$,
$(U_2,V_2)=(32d^2-16d+1,16d-4)$,
$(U_3,V_3)=(256d^3-192d^2+36d-1,128d^2-64d+6),\ldots.$

Generally, solutions of (2.4) are generated by recursive expressions

$$U_0=1,\ \ U_1=4d-1,\ \ U_{m+2}=2(4d-1)U_{m+1}-U_m,$$
$$V_0=0,\ \ V_1=2,\ \ V_{m+2}=2(4d-1)V_{m+1}-V_m,\ \ m\in\mathbb{N}_0. \tag{2.5}$$

By induction on m, one gets that $U_m\equiv 1 \pmod{(4d-2)}, m\geq 0$. Indeed, $U_0=1\equiv 1 \pmod{(4d-2)}$, $U_1=4d-1\equiv 1 \pmod{(4d-2)}$. Assume that $U_m\equiv U_{m-1}\equiv 1 \pmod{(4d-2)}$. For U_{m+1} we get

$$U_{m+1}=2(4d-1)U_m-U_{m-1}\equiv 2-1\equiv 1 \pmod{(4d-2)}.$$

Now, it remains to compute the corresponding values of n which arise from $X=(4d-2)n+2d\varepsilon$ and $X=\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U$. We obtain

$$n=\frac{\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U-2d\varepsilon}{4d-2}.$$

We want the above number n to be a positive integer.

From $d=\frac{1}{8}\varepsilon^2-\frac{1}{2}\varepsilon+1$, it follows $4d-2=\frac{1}{2}\varepsilon^2-2\varepsilon+2$. Note that ε is even. So, congruences

$$\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U-2d\varepsilon\equiv 4d+\varepsilon-2-2d\varepsilon\equiv-(2d-1)\varepsilon\equiv 0 \pmod{(4d-2)},$$

show us that all numbers n generated in the specified way are integers.

The first few values of number n, which we get from U_1, U_2, U_3, are

$$\begin{cases} n = \frac{1}{2}(\varepsilon^2 - 3\varepsilon + 6), \\ d_1 = 1, \\ d_2 = \varepsilon^2 - 2\varepsilon + 5. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^4 - 6\varepsilon^3 + 20\varepsilon^2 - 33\varepsilon + 34), \\ d_1 = \varepsilon^2 - 2\varepsilon + 5, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^6 - 10\varepsilon^5 + 50\varepsilon^4 - 148\varepsilon^3 + 281\varepsilon^2 - 323\varepsilon + 198), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 49\varepsilon^4 - 142\varepsilon^3 + 262\varepsilon^2 - 292\varepsilon + 169. \end{cases}$$

□

3. THE CASE $\delta = 4$

Theorem 2. *If $\varepsilon \equiv 2 \pmod 4$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.*

Proof. Proof of this theorem will be slightly different from the proof of Theorem 1. Instead of assuming that $\varepsilon \equiv 2 \pmod 4$, we will distiguish two cases: in one case we will be dealing with $\varepsilon \equiv 6 \pmod 8$ and we will apply strategies from [2] and in the other case we will be dealing with $\varepsilon \equiv 2 \pmod 8$ and we will use different methods in obtaining results.

We start with the case when $\varepsilon \equiv 6 \pmod 8$. We want to find odd positive integers n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.

Let $g = \gcd(d_1, d_2)$, $d_1 = gd_1', d_2 = gd_2'$ and d is a positive integer which satisfies the equation

$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we obtain

$$(d_2 - d_1)^2 = (4n + \varepsilon)^2 - 4\frac{g(n^2+1)}{2d},$$

$$d(d_2-d_1)^2=(16d-2g)n^2+8d\varepsilon n+\varepsilon^2 d-2g,$$

$$d(16d-2g)(d_2-d_1)^2 \tag{3.1}$$

$$=(16d-2g)^2n^2+8(16d-2g)d\varepsilon n+16d^2\varepsilon^2-32dg-2\varepsilon^2 dg+4g^2.$$

Let $X=(16d-2g)n+4d\varepsilon,\ \ Y=d_2-d_1$. Equation (3.1) becomes

$$X^2-2d(8d-g)Y^2=32dg+2\varepsilon^2 dg-4g^2. \tag{3.2}$$

For $g=1$ the previous expression becomes

$$X^2-2d(8d-1)Y^2=2d(16+\varepsilon^2)-4. \tag{3.3}$$

It is obvious that (3.3) is a Pellian equation. The right-hand side of (3.3) is nonzero.

Our goal is to make the right-hand side of (3.3) a perfect square. That condition can be satisfied by taking $d=\frac{1}{32}\varepsilon^2-\frac{1}{8}\varepsilon+\frac{5}{8}$. With this choice for d, we get

$$2d(16+\varepsilon^2)-4=2\left(\frac{1}{32}\varepsilon^2-\frac{1}{8}\varepsilon+\frac{5}{8}\right)(16+\varepsilon^2)-4=\left(\frac{1}{4}(\varepsilon^2-2\varepsilon+16)\right)^2.$$

So, Pellian equation (3.3) becomes

$$X^2-2d(8d-1)Y^2=\left(\frac{1}{4}(\varepsilon^2-2\varepsilon+16)\right)^2. \tag{3.4}$$

Let

$$X=\frac{1}{4}(\varepsilon^2-2\varepsilon+16)W,\ \ Y=\frac{1}{4}(\varepsilon^2-2\varepsilon+16)Z.$$

The equation (3.4) becomes

$$W^2-2d(8d-1)Z^2=1. \tag{3.5}$$

The equation (3.5) is a Pell equation which has infinitely many positive integer solutions (W,Z), and consequently, there exist infinitely many positive integer solutions (X,Y) of (3.4). The least positive integer solution of (3.5) can be found using the continued fraction expansion of number $\sqrt{2d(8d-1)}$.

We can easily get

$$\sqrt{2d(8d-1)}=[4d-1;\overline{1,2,1,8d-2}].$$

All positive solutions of (3.5) are given by (W_m,Z_m) for some $m\geq 0$. The first few solutions are

$(W_0,Z_0)=(1,0)$,
$(W_1,Z_1)=(16d-1,4)$,
$(W_2,Z_2)=(512d^2-64d+1,128d-8),\ldots$.

Generally, solutions of (3.5) are generated by recursive expressions

$$W_0=1,\ \ W_1=16d-1,\ \ W_{m+2}=2(16d-1)W_{m+1}-W_m,$$

$$Z_0=0,\ \ Z_1=4,\ \ Z_{m+2}=2(16d-1)Z_{m+1}-Z_m,\ \ m\in\mathbb{N}_0.$$

By induction on m, one gets that $W_m \equiv 1 \pmod{(16d-2)}, m \geq 0$. Indeed, $W_0 = 1 \equiv 1 \pmod{(16d-2)}$, $W_1 = 16d-1 \equiv 1 \pmod{(16d-2)}$. Assume that $W_m \equiv W_{m-1} \equiv 1 \pmod{(16d-2)}$. For W_{m+1} we get

$$W_{m+1} = 2(16d-1)W_m - W_{m-1} \equiv 2-1 \equiv 1 \pmod{(16d-2)}.$$

Now, it remains to compute the corresponding values of n which arise from $X = (16d-2)n + 4d\varepsilon$ and $X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W$. We obtain

$$n = \frac{\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon}{16d-2}.$$

We want to prove that number n is a positive integer.

From $d = \frac{1}{32}\varepsilon^2 - \frac{1}{8}\varepsilon + \frac{5}{8}$,it follows $8d - 1 = \frac{1}{4}\varepsilon^2 - \varepsilon + 4$. Number $\frac{\varepsilon}{2}$ is an odd integer. Thus, the congruences

$$\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon \equiv 8d - 1 + \frac{\varepsilon}{2} - 4d\varepsilon \equiv (8d-1)(1 - \frac{\varepsilon}{2}) \equiv 0 \pmod{(16d-2)}$$

show us that all numbers n generated in the specified way are integers.

The first few values of number n, which we get from W_1, W_2, W_3, are

$$\begin{cases} n = \frac{1}{4}(\varepsilon^2 - 3\varepsilon + 18), \\ d_1 = 1 \\ d_2 = \varepsilon^2 - 2\varepsilon + 17. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^4 - 6\varepsilon^3 + 44\varepsilon^2 - 105\varepsilon + 322), \\ d_1 = \varepsilon^2 - 2\varepsilon + 17, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^6 - 10\varepsilon^5 + 86\varepsilon^4 - 388\varepsilon^3 + 1529\varepsilon^2 - 3155\varepsilon + 5778), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 85\varepsilon^4 - 382\varepsilon^3 + 1486\varepsilon^2 - 3052\varepsilon + 5473. \end{cases}$$

Now, we deal with the case when $\varepsilon \equiv 2 \pmod{8}$. Let $\varepsilon = 8k + 2,\ k \in \mathbb{N}_0$. For $g = \frac{1}{4}\varepsilon^2 + 4$ and $g = d_1$, the equation (3.2) becomes

$$X^2 - 2d(8d-g)Y^2 = \frac{2d-1}{4}\varepsilon^4 + 8\varepsilon^2(2d-1) + 64(2d-1).$$

The right-hand side of the equation will be a perfect square if $2d-1$ is a perfect square. Motivated by the experimental data, we take

$$d = \frac{1}{512}\varepsilon^4 - \frac{1}{64}\varepsilon^3 + \frac{7}{64}\varepsilon^2 - \frac{5}{16}\varepsilon + \frac{41}{32}.$$

We get

$$2d-1 = 16k^4+8k^2+1 = (4k^2+1)^2.$$

So, the equation (3.2) becomes

$$X^2 - 2d(8d-g)Y^2 = \left(\frac{1}{32}(\varepsilon^2+16)(\varepsilon^2-4\varepsilon+20)\right)^2. \qquad (3.6)$$

We consider the corresponding Pell equation

$$U^2 - 2d(8d-g)V^2 = 1. \qquad (3.7)$$

Let (U_0, V_0) be the least positive integer solution of (3.7). That equation has infinitely many solutions. From (3.7) we get that

$$U^2 \equiv 1 \pmod{(16d-2g)}.$$

We deal with the case where $g = d_1 = \frac{1}{4}\varepsilon^2+4$ and from the experimental data we can set

$$d_2 = d_1^2 - 16kd_1, \;\; k \in \mathbb{N}_0.$$

For $Y = d_2 - d_1$ we get

$$Y = \left(\frac{1}{4}\varepsilon^2+4\right)^2 - (2\varepsilon-3)\left(\frac{1}{4}\varepsilon^2+4\right) = \frac{\varepsilon^4}{16} - \frac{\varepsilon^3}{2} + \frac{11\varepsilon^2}{4} - 8\varepsilon + 28.$$

From (3.6), we obtain:

$$X = \frac{(\varepsilon^2+16)(\varepsilon^6-16\varepsilon^5+140\varepsilon^4-768\varepsilon^3+3120\varepsilon^2-8704\varepsilon+14400)}{2048}.$$

We claim that X satisfies the congruence

$$X \equiv 4d\varepsilon \pmod{(16d-2g)}. \qquad (3.8)$$

Indeed,

$$16d-2g = \frac{\varepsilon^4}{32} - \frac{\varepsilon^3}{4} + \frac{5\varepsilon^2}{4} - 5\varepsilon + \frac{25}{2},$$

$$X - 4d\varepsilon = \left(\frac{\varepsilon^4}{32} - \frac{\varepsilon^3}{4} + \frac{5\varepsilon^2}{4} - 5\varepsilon + \frac{25}{2}\right)\left(\frac{\varepsilon^4}{64} - \frac{\varepsilon^3}{8} + \frac{13\varepsilon^2}{16} - \frac{9\varepsilon}{4} + 9\right).$$

From $n = \frac{X-4d\varepsilon}{16d-2g}$, we get

$$n = \frac{\varepsilon^4}{64} - \frac{\varepsilon^3}{8} + \frac{13\varepsilon^2}{16} - \frac{9\varepsilon}{4} + 9 = 64k^4 + 28k^2 + 7,$$

and we see that n is an odd integer. Thus, if we define

$$(X_0,Y_0)=\left(\frac{(\varepsilon^2+16)(\varepsilon^6-16\varepsilon^5+140\varepsilon^4-768\varepsilon^3+3120\varepsilon^2-8704\varepsilon+14400)}{2048},\right.$$

$$\left.\frac{1}{16}(\varepsilon^2+16)(\varepsilon^2-8\varepsilon+28)\right),$$

we see that (X_0,Y_0) is a solution of (3.6) which satisfies the congruence (3.8). We have proved that for every $\varepsilon\equiv 2 \pmod 8$ there exists at least one odd integer n which satisfies the conditions of Theorem 2. Our goal is to prove that there exist infinitely many such integers n that satisfy the properties of Theorem 2.

If (X_0,Y_0) is a solution of (3.6), solutions of (3.6) are also

$$(X_i,Y_i)=\left(X_0+\sqrt{2d(8d-g)}Y_0\right)\left(U_0+\sqrt{2d(8d-g)}V_0\right)^{2i},\ \ i=0,1,2,\dots \tag{3.9}$$

From the equation (3.9), we get

$$X_i\equiv U_0^{2i}X_0\equiv X_0\equiv 4d\varepsilon \pmod{(16d-2g)}.$$

So, there are infinitely many solutions (X_i,Y_i) of (3.6) that satisfy the congruence (3.8). Therefore, by

$$n=\frac{X_i-4d\varepsilon}{16d-2g},$$

we get infinitely many integers n with the required properties. It is easy to see that number n defined in this way is odd. Indeed, we have $16d-2g\equiv 2 \pmod 4$, $X_0\equiv 2 \pmod 4$, and since (3.7) implies that U_0 is odd and V_0 is even, we get from (3.8) that

$$X_i-4d\varepsilon\equiv X_i\equiv U_0^{2i}X_0\equiv X_0\equiv 2 \pmod 4,$$

so n is odd. □

4. The case $\varepsilon=0$

Proposition 1. *There exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors* d_1,d_2 *of* $\frac{n^2+1}{2}$ *such that* $d_1+d_2=2n$. *These solutions satisfy* $\gcd(d_1,d_2)=1$ *and* $d_1d_2=\frac{n^2+1}{2}$.

Proof. We want to find a positive odd integer n and positive divisors d_1,d_2 of $\frac{n^2+1}{2}$ such that $d_1+d_2=2n$. Let $g=\gcd(d_1,d_2)$. Then $g|(2n)$ and $g|(n^2+1)$ which implies that $g|((2n)^2+4)$ so we can conclude that $g|4$. Because g is the greatest common divisor of d_1,d_2 and d_1,d_2 are odd numbers, we can also conclude that g is an odd number. So, $g=1$. Like we did in the proofs of the previous theorems, we define a positive integer d which satisfies the equation $d_1d_2 = \frac{n^2+1}{2d}$. From the identity

$$(d_2-d_1)^2 = (d_1+d_2)^2 - 4d_1d_2,$$

we can easily obtain

$$(d_2-d_1)^2 = (2n)^2 - 2\frac{(n^2+1)}{d},$$
$$d(d_2-d_1)^2 = 4n^2d - 2n^2 - 2.$$

Let $d_2-d_1 = 2y$, so we get

$$(4d-2)n^2 - 4dy^2 = 2,$$
$$(2d-1)n^2 - 2dy^2 = 1. \tag{4.1}$$

We will use the next lemma, which is Criterion 1 from [3] to check if there exists a solution for (4.1).

Lemma 1. *Let $a > 1$, b be positive integers such that $\gcd(a,b) = 1$ and $D = ab$ is not a perfect square. Moreover, let (u_0, v_0) denote the least positive integer solution of the Pell equation*

$$u^2 - Dv^2 = 1.$$

Then equation $ax^2 - by^2 = 1$ has a solution in positive integers x, y if and only if

$$2a|(u_0+1) \text{ and } 2b|(u_0-1).$$

□

We want to solve the Pell equation

$$U^2 - 2d(2d-1)V^2 = 1, \tag{4.2}$$

where $n = U$, $y = V$. The continued fraction expansion of the number $\sqrt{2d(2d-1)}$ is already known from Theorem 1 where we have obtained

$$\sqrt{2d(2d-1)} = [2d-1; \overline{2, 4d-2}].$$

The least positive integer solution of the Pell equation (4.2) is $(4d-1, 2)$. In our case, we want to find solutions of (4.1), so we apply Lemma 1 which gives us conditions that have to be fulfilled. It has to be that

$$2(2d-1)|4d \text{ and } 4d|(4d-2),$$

which is not true for $d \in \mathbb{N}$. So, for Pellian equation (4.1) there are no integer solutions (n, y) when $a = 2d-1 > 1$. Finally, we have to check the remaining case for $a = 1$, which is the case that is not included in Lemma 1.

If $a = 2d-1 = 1$, then $d = 1$. From (4.1) and $d = 1$, we get the Pell equation

$$n^2 - 2y^2 = 1, \tag{4.3}$$

which has infinitely many solutions $n = U_m,\ \ y = V_m,\ \ m \in \mathbb{N}_0$ where

$$U_0 = 1,\ \ U_1 = 3,\ \ U_{m+2} = 6U_{m+1} - U_m,$$

$$V_0 = 0,\ \ V_1 = 2,\ \ V_{m+2} = 6V_{m+1} - V_m,\ \ m \in \mathbb{N}_0.$$

The first few values (U_i, V_i) are

$$(U_0, V_0) = (1,0), (U_1, V_1) = (3,2), (U_2, V_2) = (17,12), (U_3, V_3) = (99,70), \ldots.$$

From those solutions we can easily generate (n, d_1, d_2)

$$(n, d_1, d_2) = (3,1,5),\ \ (17,5,29),\ \ (99,29,169), \ldots.$$

We have proved that there exist infinitely many odd positive integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. We have also proved that $g = 1$ and $d = 1$, so we conclude that numbers d_1 and d_2 are coprime and that $d_1 d_2 = \frac{n^2+1}{2}$. □

Theorem 3. *Let $\delta \geq 6$ be a positive integer such that $\delta = 4k+2, k \in \mathbb{N}$. Then there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.*

Proof. Suppose on the contrary that this is not so and let the number δ be the smallest positive integer $\delta = 4k+2,\ \ k \in \mathbb{N}$ for which there exists an odd integer n and a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$. Let $g = \gcd(d_1, d_2) > 1$. Since $d_1 = g d_1',\ \ d_2 = g d_2'$, it follows that $g|(n^2+1)$ and $g|(\delta n)$ and we conclude that $g|((\delta n)^2 + \delta^2)$, which implies that $g|\delta^2$. This means that g and δ have a common prime factor p. Let $d_1 = p d_1'', d_2 = p d_2'', \delta = p\delta''$. Then, we have $p d_1'' + p d_2'' = p\delta'' n$, so we can conclude $d_1'' + d_2'' = \delta'' n$ where d_1'', d_2'' are divisors of $\frac{n^2+1}{2}$. It is clear that $\delta'' < \delta$ and if it also satisfies $\delta'' \neq 2$, the existence of the number δ'' contradicts the minimality of δ. So, if $\delta'' \neq 2$, then we must have $g = 1$.

If $\delta'' = 2$, it follows from Proposition 1 that $\gcd(d_1'', d_2'') = 1$ and $d_1'' d_2'' = \frac{n^2+1}{2}$. But, $\gcd(d_1, d_2) = p d_1'' d_2''$ should be a divisor of $\frac{n^2+1}{2}$ which is not possible because $p > 1$. So, in this case we also conclude that $g = 1$.

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4 d_1 d_2,$$

and using $g = 1$, we obtain

$$(d_2 - d_1)^2 = (\delta n)^2 - 2\frac{(n^2+1)}{d},$$

$$d(d_2 - d_1)^2 = \delta^2 n^2 d - 2n^2 - 2,$$

$$d(d_2 - d_1)^2 = (d\delta^2 - 2)n^2 - 2.$$

In the equation

$$(\delta^2 d-2)n^2-d(d_2-d_1)^2=2,$$

we set $(d_2-d_1)=2y$ (number d_2-d_1 is an even number because d_1,d_2 are odd integers), and we get

$$(\delta^2 d-2)n^2-4dy^2=2.$$

If we divide both sides of the above equation by 2, then it becomes

$$(2d(2k+1)^2-1)n^2-2dy^2=1.$$

Now, if we define $\delta'=\frac{\delta}{2}=2k+1$, we get

$$(2\delta'^2 d-1)n^2-2dy^2=1. \tag{4.4}$$

We will prove by applying Lemma 1 that the above Pell equation (4.4) has no solutions.

To be able to apply Lemma 1, we have to deal with an equation of the form

$$x^2-Dy^2=1.$$

We have $a=2d\delta'^2-1$, $a>1$ (because $\delta'\geq 3$) and $D=ab=2d(2\delta'^2 d-1)$ is not a perfect square because $2d(2\delta'^2 d-1)\equiv 2 \pmod 4$. We need to find the least positive integer solution of the equation

$$u^2-2d(2\delta'^2 d-1)v^2=1. \tag{4.5}$$

For that purpose we find the continued fraction expansion of the number

$$\sqrt{2d(2\delta'^2 d-1)},\ \ \delta'\geq 3.$$

We know that

$$\sqrt{2d(2\delta'^2 d-1)}=[a_0;\overline{a_1,a_2,\ldots,a_{l-1},2a_0}],$$

where we calculate numbers a_i recursively

$$a_i=\left\lfloor\frac{s_i+a_0}{t_i}\right\rfloor,\ \ s_{i+1}=a_it_i-s_i,\ \ t_{i+1}=\frac{d-s_{i+1}^2}{t_i}.$$

In our case, we obtain

$$a_0=\lfloor\sqrt{2d(2\delta'^2 d-1)}\rfloor=2d\delta'-1,\ \ s_0=0,\ \ t_0=1;$$
$$s_1=2d\delta'-1,\ \ t_1=4d\delta'-2d-1,\ \ a_1=1;$$
$$s_2=2d\delta'-2d,\ \ t_2=2d,\ \ a_2=2\delta'-2;$$
$$s_3=2d\delta'-2d,\ \ t_3=4d\delta'-2d-1,\ \ a_3=1;$$
$$s_4=2d\delta'-1,\ \ t_4=1,\ \ a_4=2(2d\delta'-1)=2a_0.$$

We get

$$\sqrt{2d(2\delta'^2 d-1)}=[2d\delta'-1;\overline{1,2\delta'-2,1,2(2d\delta'-1)}].$$

Now, we can find the least positive integer solution of the equation (4.5). Because the length of the period of the expansion is $l = 4$, the least positive integer solution of (4.5) is (p_3, q_3), where numbers $p_i, q_i,\ i = 0,1,2,3$ are calculated recursively

$$p_0 = a_0,\ \ p_1 = a_0 a_1 + 1,\ \ p_k = a_k p_{k-1} + p_{k-2},$$
$$q_0 = 1,\ \ q_1 = a_1,\ \ q_k = a_k q_{k-1} + q_{k-2},\ \ k = 2,3.$$

We obtain

$$(p_0,q_0) = (2d\delta' - 1, 1),\ \ (p_1,q_1) = (2d\delta', 1),\ \ (p_2,q_2) = (4\delta'^2 d - 2d\delta' - 1, 2\delta' - 1),$$
$$(p_3,q_3) = (4\delta'^2 d - 1, 2\delta').$$

So, the least positive integer solution is $(p_3,q_3) = (u_0,v_0) = (4\delta'^2 d - 1, 2\delta')$ and we apply Lemma 1.

In our case we have $a = 2\delta'^2 d - 1,\ \ b = 2d$. From Lemma 1 we get

$$(4\delta'^2 d - 2) | 4\delta'^2 d,\ \ 4d | (4\delta'^2 d - 2).$$

We can easily see that $4d | (4\delta'^2 d - 2)$ if and only if $4d | 2$ which is not possible because $d \in \mathbb{N}$. So, the equation (4.4) has no solutions. We have proved that there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$. □

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