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BI-STARLIKE FUNCTION OF COMPLEX ORDER ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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Abstract. In the present paper, two new subclasses of bi-starlike and bi-convex function of complex order associated with hypergeometric functions are introduced and coefficient estimates for functions in these classes are obtained. Several new (or known) consequences of the results are also pointed out.

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1. INTRODUCTION

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let \mathcal{S} denote the class of all functions in \mathcal{A} which are univalent in Δ . Some of the important and well-investigated subclasses of the univalent function class \mathcal{S} include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in Δ and the class $\mathcal{K}(\alpha)$ of convex functions of order α ($0 \leq \alpha < 1$) in Δ . It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and $f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4)$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ . Let Σ denote the class of bi-univalent functions in Δ given by (1.1). Earlier, Brannan and Taha [3] introduced certain subclasses of bi-univalent function

class Σ , namely bi-starlike functions of order α denoted by $\mathcal{S}_{\Sigma}^*(\alpha)$ and bi-convex function of order α denoted by $\mathcal{K}_{\Sigma}(\alpha)$ corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively.

A function $f(z) \in \mathcal{A}$ is in the class of strongly bi-starlike functions $\mathcal{S}_{\Sigma}^*[\alpha]$ ([3, 19]) of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2}, \text{ and } \left| \arg \left(\frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2}$$

and strongly bi-convex functions $\mathcal{K}_{\Sigma}^*[\alpha]$ ([3, 19]) of order α ($0 < \alpha \leq 1$)

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\alpha\pi}{2}, \text{ and } \left| \arg \left(1 + \frac{wg''(w)}{g'(w)} \right) \right| < \frac{\alpha\pi}{2}$$

where g is given by (1.2). For each of the function classes $\mathcal{S}_{\Sigma}^*[\alpha]$ and $\mathcal{K}_{\Sigma}[\alpha]$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found [3, 19]. But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} := \{1, 2, 3, \dots\})$$

is still an open problem (see [2, 3, 8, 12, 19]). Many researchers (see [16, 17, 20, 21]) have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

An analytic function f is subordinate to an analytic function g , written $f(z) \prec g(z)$, provided there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk Δ , $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}_{\Sigma}^*(\phi)$ and $\mathcal{K}_{\Sigma}(\phi)$.

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better. The convolution or Hadamard product of two

functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \Delta \tag{1.3}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

For complex parameters $\alpha_1, \dots, \alpha_l$ and β_1, \dots, β_m ($\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$${}_lF_m(z) \equiv {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \tag{1.4}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \Delta)$$

where \mathbb{N} denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\dots(a+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.5}$$

For positive real values of $\alpha_1, \dots, \alpha_l$ and β_1, \dots, β_m ($\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m$), let

$$\mathcal{H}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) : \mathfrak{S} \rightarrow \mathfrak{S}$$

be a linear operator defined by

$$(\mathcal{H}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m))(f)(z) = z {}_lF_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) * f(z)$$

$$(\mathcal{H}_m^l)(f)(z) = z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n \tag{1.6}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{1}{(n-1)!} \tag{1.7}$$

$\alpha_i > 0, (i = 1, 2, \dots, l), \beta_j > 0, (j = 1, 2, \dots, m), l \leq m + 1; l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For notational simplicity, we use a shorter notation

$$\mathcal{H}_m^l[\alpha_1, \beta_1]f(z) = \mathcal{H}_m^l f(z)$$

for

$$\mathcal{H}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) = \mathcal{H}_m^l[\alpha_1, \beta_1]$$

in the sequel. It follows from (1.6) that

$$\mathcal{H}_1^2[1, 1]f(z) = f(z), \mathcal{H}_1^2[2, 1]f(z) = z f'(z).$$

The linear operator $\mathcal{H}_m^l[\alpha_1, \beta_1]$ is called Dziok-Srivastava operator (see [6]). Further by using the Gaussian hypergeometric function given by (1.6), Hohlov [7] introduced a generalized convolution operator $H_{a,b,c}$ as $H_{a,b,c}f(z) = z {}_2F_1(a, b, c; z) *$

$f(z)$, contains as special cases most of the known linear integral or differential operators. For the suitable choices of l, m in turn the operator $\mathcal{H}_m^l[\alpha_1, \beta_1]$ includes various operators as remarked below:

Remark 1. For $f \in \mathcal{A}$,

$$\begin{aligned} \mathcal{H}_1^2(a, 1; c)f(z) &= \mathcal{L}(a, c)f(z) \\ &= \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \right) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n \end{aligned}$$

was considered by Carlson and Shaffer [4].

Remark 2. For $f \in \mathcal{A}$, and

$$\mathcal{H}_1^2(\delta + 1, 1; 1)f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = \mathcal{D}^\delta f(z), (\delta > -1)$$

given by $\mathcal{D}^\delta f(z) = z + \sum_{n=2}^{\infty} \binom{\delta + n - 1}{n - 1} a_n z^n$, was introduced by Ruscheweyh [15].

Remark 3. For $f \in \mathcal{A}$, and $\mathcal{H}_1^2(c + 1, 1; c + 2)f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt = \mathcal{J}_c f(z)$ where $c > -1$. The operator \mathcal{J}_c was introduced by Bernardi [1]. In particular, the operator \mathcal{J}_1 was studied earlier by Libera [9] and Livingston [10].

Remark 4. For $f \in \mathcal{A}$, $\mathcal{H}_1^2(2, 1; 2 - \lambda)f(z) = \Gamma(2 - \lambda) z^\lambda \mathcal{D}_z^\lambda f(z) = \Omega^\lambda f(z)$, $\lambda \notin \mathbb{N} \setminus \{1\}$ called Owa-Srivastava operator [18] and Ω^λ is also called Srivastava-Owa fractional derivative operator, where $\mathcal{D}_z^\lambda f(z)$ denotes the fractional derivative of $f(z)$ of order λ , studied by Owa [13].

Motivated by the earlier work of Deniz [5], in the present paper we introduce two new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C} \setminus \{0\}$, involving Dziok-Srivastava operator \mathcal{H}_m^l and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of the function class Σ . Several related classes are also considered, and connection to earlier known results are made.

1.1. The function class of bi-univalent functions

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ , satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad (B_1 > 0). \quad (1.8)$$

Further we assume $\alpha_i > 0, (i = 1, 2, \dots, l), \beta_j > 0, (j = 1, 2, \dots, m), l \leq m + 1; l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in our present study unless otherwise stated.

Definition 1. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma, \phi}^{l,m}(\gamma, \lambda)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left(\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} + \frac{\lambda z^2(\mathcal{H}_m^l f(z))''}{\mathcal{H}_m^l f(z)} - 1 \right) \prec \phi(z) \tag{1.9}$$

and

$$1 + \frac{1}{\gamma} \left(\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \frac{\lambda w^2(\mathcal{H}_m^l g(w))''}{\mathcal{H}_m^l g(w)} - 1 \right) \prec \phi(w) \tag{1.10}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0, z, w \in \Delta$ and the function g is given by (1.2).

Definition 2. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{K}_{\Sigma, \phi}^{l,m}(\gamma, \lambda)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left(\frac{[z(\mathcal{H}_m^l f(z))' + \lambda z^2(\mathcal{H}_m^l f(z))'']'}{(\mathcal{H}_m^l f(z))'} - 1 \right) \prec \phi(z) \tag{1.11}$$

and

$$1 + \frac{1}{\gamma} \left(\frac{[w(\mathcal{H}_m^l g(w))' + \lambda w^2(\mathcal{H}_m^l g(w))'']'}{(\mathcal{H}_m^l g(w))'} - 1 \right) \prec \phi(w) \tag{1.12}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0, z, w \in \Delta$ and the function g is given by (1.2).

Remark 5. A function $f(z) \in \Sigma$ given by (1.1) and for $\lambda = 0$, we note that

$$\mathcal{S}_{\Sigma, \phi}^{l,m}(\gamma, \lambda) \equiv \mathcal{S}_{\Sigma, \phi}^{l,m}(\gamma)$$

satisfies the following conditions :

$$\left[1 + \frac{1}{\gamma} \left(\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} - 1 \right) \right] \prec \phi(z) \text{ and } \left[1 + \frac{1}{\gamma} \left(\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} - 1 \right) \right] \prec \phi(w)$$

and

$$\mathcal{K}_{\Sigma, \phi}^{l,m}(\gamma, \lambda) \equiv \mathcal{K}_{\Sigma, \phi}^{l,m}(\gamma)$$

satisfies the following conditions :

$$\left[1 + \frac{1}{\gamma} \left(\frac{z(\mathcal{H}_m^l f(z))''}{(\mathcal{H}_m^l f(z))'} \right) \right] \prec \phi(z) \text{ and } \left[1 + \frac{1}{\gamma} \left(\frac{w(\mathcal{H}_m^l g(w))''}{(\mathcal{H}_m^l g(w))'} \right) \right] \prec \phi(w)$$

where $\gamma \in \mathbb{C} \setminus \{0\}, z, w \in \Delta$ and the function g is given by (1.2).

Remark 6. A function $f(z) \in \Sigma$ given by (1.1) and for $\gamma = 1$, we note that $\mathcal{S}_{\Sigma, \phi}^{l,m}(\gamma, \lambda) \equiv \mathcal{S}_{\Sigma, \phi}^{l,m}(\lambda)$ and $\mathcal{K}_{\Sigma, \phi}^{l,m}(\gamma, \lambda) \equiv \mathcal{K}_{\Sigma, \phi}^{l,m}(\lambda)$ satisfies the following conditions respectively:

$$\left(\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} + \frac{\lambda z^2(\mathcal{H}_m^l f(z))''}{\mathcal{H}_m^l f(z)} \right) \prec \phi(z),$$

$$\left(\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \frac{\lambda w^2(\mathcal{H}_m^l g(w))''}{\mathcal{H}_m^l g(w)} \right) \prec \phi(w),$$

$$\left(\frac{[z(\mathcal{H}_m^l f(z))' + \lambda z^2(\mathcal{H}_m^l f(z))'']'}{(\mathcal{H}_m^l f(z))'} \right) \prec \phi(z),$$

and

$$\left(\frac{[w(\mathcal{H}_m^l g(w))' + \lambda w^2(\mathcal{H}_m^l g(w))'']'}{(\mathcal{H}_m^l g(w))'} \right) \prec \phi(w)$$

where $\lambda \geq 0, z, w \in \Delta$ and the function g is given by (1.2).

2. COEFFICIENT ESTIMATES FOR THE FUNCTION CLASS $\mathcal{S}_{\Sigma, \phi}^{l, m}(\gamma, \lambda)$ AND

$$\mathcal{K}_{\Sigma, \phi}^{l, m}(\gamma, \lambda)$$

For deriving our main results, we need the following lemma.

Lemma 1 ([14]). *If $h \in \mathcal{P}$, then*

$$|c_k| \leq 2$$

for each k , where \mathcal{P} is the family of all functions h analytic in Δ for which $\Re(h(z)) > 0$ and

$$h(z) = 1 + c_1 z + c_2 z^2 + \dots \text{ for } z \in \Delta.$$

Lemma 2 (see [14]). *Let the function $\varphi(z)$ given by*

$$\varphi(z) = \sum_{n=1}^{\infty} C_n z^n \quad (z \in \Delta)$$

be convex in \mathbb{U} . Suppose also that the function $\mathfrak{h}(z)$ given by

$$\mathfrak{h}(z) = \sum_{n=1}^{\infty} \mathfrak{h}_n z^n$$

is holomorphic in Δ . If

$$\mathfrak{h}(z) \prec \varphi(z) \quad (z \in \Delta),$$

then

$$|\mathfrak{h}_n| \leq |C_n| \quad (n \in \mathbb{N}).$$

Theorem 1. *Let $f(z)$ is given by (1.1) be in the class $\mathcal{S}_{\Sigma, \phi}^{l, m}(\gamma, \lambda)$, $\gamma \in \mathbb{C} \setminus \{0\}$ and $\lambda \geq 0$. Then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2|}} \quad (2.1)$$

and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1+2\lambda)^2 \Gamma_2^2} + \frac{|\gamma| B_1}{2(1+3\lambda) \Gamma_3}. \quad (2.2)$$

Proof. It follows from (1.9) and (1.10) that

$$1 + \frac{1}{\gamma} \left(\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} + \frac{\lambda z^2 (\mathcal{H}_m^l f(z))''}{\mathcal{H}_m^l f(z)} - 1 \right) = \phi(u(z)) \quad (2.3)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \frac{\lambda w^2 (\mathcal{H}_m^l g(w))''}{\mathcal{H}_m^l g(w)} - 1 \right) = \phi(v(w)). \quad (2.4)$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \dots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \dots,$$

or equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \dots \right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2} \left[q_1 z + \left(q_2 - \frac{q_1^2}{2} \right) z^2 + \dots \right].$$

Then $p(z)$ and $q(z)$ are analytic in Δ with $p(0) = 1 = q(0)$. Since $u, v : \Delta \rightarrow \Delta$, the functions $p(z)$ and $q(z)$ have a positive real part in Δ , and for each i ,

$$|p_i| \leq 2 \quad \text{and} \quad |q_i| \leq 2.$$

Since $p(z)$ and $q(w)$ in \mathcal{P} , we have the following forms:

$$\begin{aligned} \phi(u(z)) &= \phi \left(\frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \dots \right] \right) \\ &= \frac{1}{2} B_1 p_1 z + \left[\frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 \right] z^2 \end{aligned} \quad (2.5)$$

and

$$\phi(v(w)) = \phi \left(\frac{1}{2} \left[q_1 w + \left(q_2 - \frac{q_1^2}{2} \right) w^2 + \dots \right] \right)$$

$$= \frac{1}{2}B_1q_1w + \left[\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 \right] w^2 \quad (2.6)$$

It follows from (2.3) -(2) that

$$\frac{1}{\gamma}(1+2\lambda)\Gamma_2a_2 = \frac{1}{2}B_1p_1, \quad (2.7)$$

$$\frac{1}{\gamma} [2(1+3\lambda)\Gamma_3a_3 - (1+2\lambda)\Gamma_2^2a_2^2] = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2p_1^2, \quad (2.8)$$

$$-\frac{1}{\gamma}(1+2\lambda)\Gamma_2a_2 = \frac{1}{2}B_1q_1, \quad (2.9)$$

and

$$\frac{1}{\gamma} ([4(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]a_2^2 - 2(1+3\lambda)\Gamma_3a_3) = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2q_1^2. \quad (2.10)$$

From (2.7) and (2.9), we get

$$p_1 = -q_1 \quad (2.11)$$

and

$$8(1+2\lambda)^2\Gamma_2^2a_2^2 = \gamma^2B_1^2(p_1^2 + q_1^2). \quad (2.12)$$

Upon adding (2.8) and (2.10), if we make use of (2.11) and (2.12), we obtain

$$\begin{aligned} & (4\{\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2\})a_2^2 \\ & = \gamma^2B_1^3(p_2 + q_2). \end{aligned}$$

which yields

$$a_2^2 = \frac{\gamma^2B_1^3(p_2 + q_2)}{4\{\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2\}}$$

Since, by definition, $p(z), q(w) \in h(\Delta)$, by applying Lemma (1) for the coefficients p_2, q_2 and $B_1 > 0$, we have

$$|a_2| \leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.8) from (2.10) and using (2.11), we get

$$\frac{4}{\gamma}(1+3\lambda)\Gamma_3(a_3 - a_2^2) = \frac{B_1}{2}(p_2 - q_2).$$

Upon substituting the value of a_2^2 from (2.12), we get

$$a_3 = \frac{\gamma^2B_1^2(p_1^2 + q_1^2)}{8(1+2\lambda)^2\Gamma_2^2} + \frac{\gamma B_1(p_2 - q_2)}{8(1+3\lambda)\Gamma_3}.$$

Applying Lemma (1) once again for the coefficients p_1, p_2, q_1 and q_2 , we get

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1 + 2\lambda)^2 \Gamma_2^2} + \frac{|\gamma| B_1}{2(1 + 3\lambda) \Gamma_3}.$$

□

Theorem 2. Let $f(z)$ given by (1.1) be in the class $K_{\Sigma, \phi}^{l,m}(\gamma, \lambda)$, $\gamma \in \mathbb{C} \setminus \{0\}$ and $\lambda \geq 0$. Then

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{2\{|\gamma[3(1 + 3\lambda)\Gamma_3 - 2(1 + 2\lambda)\Gamma_2^2]B_1^2 + 2(1 + 2\lambda)^2(B_1 - B_2)\Gamma_2^2\}}}$$

(2.13)

and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{4(1 + 2\lambda)^2 \Gamma_2^2} + \frac{|\gamma| B_1}{6(1 + 3\lambda) \Gamma_3}.$$

(2.14)

Proof. We can write the argument inequalities in (1.11) and (1.12) equivalently as follows:

$$1 + \frac{1}{\gamma} \left(\frac{[z(\mathcal{H}_m^l f(z))' + \lambda z^2(\mathcal{H}_m^l f(z))''']}{(\mathcal{H}_m^l f(z))'} - 1 \right) = \phi(u(z))$$

(2.15)

and

$$1 + \frac{1}{\gamma} \left(\frac{[w(\mathcal{H}_m^l g(w))' + \lambda w^2(\mathcal{H}_m^l g(w))''']}{(\mathcal{H}_m^l g(w))'} - 1 \right) = \phi(v(w)),$$

(2.16)

and proceeding as in the proof of Theorem 1, we can arrive the following relations from (2.15) and (2.16)

$$\frac{2}{\gamma}(1 + 2\lambda)\Gamma_2 a_2 = \frac{1}{2} B_1 p_1,$$

(2.17)

$$\frac{1}{\gamma}[6(1 + 3\lambda)\Gamma_3 a_3 - 4(1 + 2\lambda)\Gamma_2^2 a_2^2] = \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2 p_1^2,$$

(2.18)

and

$$-\frac{2}{\gamma}(1 + 2\lambda)\Gamma_2 a_2 = \frac{1}{2} B_1 q_1,$$

(2.19)

$$\frac{1}{\gamma}[6(1 + 3\lambda)(2a_2^2 - a_3)\Gamma_3 - 4(1 + 2\lambda)\Gamma_2^2 a_2^2] = \frac{1}{2} B_1 (q_2 - \frac{q_1^2}{2}) + \frac{1}{4} B_2 q_1^2.$$

(2.20)

From (2.17) and (2.19), we get

$$p_1 = -q_1$$

(2.21)

and

$$32(1 + 2\lambda)^2 \Gamma_2^2 a_2^2 = \gamma^2 B_1^2 (p_1^2 + q_1^2).$$

(2.22)

Now from (2.18), (2.20) and (2.22), we obtain

$$a_2^2 = \frac{\gamma^2 B_1^3 (p_2 + q_2)}{8[\gamma[3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2]B_1^2 + 2(1+2\lambda)^2(B_1 - B_2)\Gamma_2^2]}. \quad (2.23)$$

Applying Lemma (1) for the coefficients p_2 and q_2 , we have the desired inequality given in (2.13).

Next, in order to find the bound on $|a_3|$, by subtracting (2.18) from (2.20), and using (2.21), we get

$$\frac{12}{\gamma}(1+3\lambda)(a_3 - a_2^2)\Gamma_3 = \frac{B_1}{2}(p_2 - q_2).$$

Upon substituting the value of a_2^2 given (2.22), the above equation lead to

$$a_3 = \frac{\gamma B_1 (p_2 - q_2)}{24(1+3\lambda)\Gamma_3} + \frac{\gamma^2 B_1^2 (p_1^2 + q_1^2)}{32(1+2\lambda)^2 \Gamma_2^2}.$$

Applying the Lemma (1) once again for the coefficients p_1, p_2, q_1 and q_2 , we get the desired coefficient given in (2.14). \square

Putting $\lambda = 0$, in Theorems (1) and (2), we can state the coefficient estimates for the functions in the subclasses $\mathcal{S}_{\Sigma, \phi}^{l,m}(\gamma)$ and $\mathcal{K}_{\Sigma, \phi}^{l,m}(\gamma)$ defined in Remark (5).

Corollary 1. Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma, \phi}^{l,m}(\gamma)$, then

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|\gamma(2\Gamma_3 - \Gamma_2^2)B_1^2 + (B_1 - B_2)\Gamma_2^2|}} \quad \text{and} \quad |a_3| \leq \frac{|\gamma|^2 B_1^2}{\Gamma_2^2} + \frac{|\gamma| B_1}{2\Gamma_3}.$$

Corollary 2. Let $f(z)$ given by (1.1) be in the class $\mathcal{K}_{\Sigma, \phi}^{l,m}(\gamma)$, then

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{2|\gamma(3\Gamma_3 - 2\Gamma_2^2)B_1^2 + 4(B_1 - B_2)\Gamma_2^2|}} \quad \text{and} \quad |a_3| \leq \frac{|\gamma|^2 B_1^2}{4\Gamma_2^2} + \frac{|\gamma| B_1}{6\Gamma_3}.$$

Taking $\gamma = 1$, in Theorems (1) and (2), we can state the coefficient estimates for the functions in the subclasses $\mathcal{S}_{\Sigma, \phi}^{l,m}(\lambda)$ and $\mathcal{K}_{\Sigma, \phi}^{l,m}(\lambda)$ defined in Remark (6).

Corollary 3. Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma, \phi}^{l,m}(\lambda)$, then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{B_1^2}{(1+2\lambda)^2 \Gamma_2^2} + \frac{B_1}{2(1+3\lambda)\Gamma_3}.$$

Corollary 4. Let $f(z)$ given by (1.1) be in the class $\mathcal{K}_{\Sigma, \phi}^{l,m}(\lambda)$, then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{2| [3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2] B_1^2 + 2(1+2\lambda)^2 (B_1 - B_2)\Gamma_2^2 |}}$$

and

$$|a_3| \leq \frac{B_1^2}{4(1+2\lambda)^2 \Gamma_2^2} + \frac{B_1}{6(1+3\lambda)\Gamma_3}.$$

3. COROLLARIES AND ITS CONSEQUENCES

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1), \quad (3.1)$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$.

Corollary 5. By choosing $\phi(z)$ of the form (3.1), we state the following results

(1) for functions $f \in \mathcal{S}_{\Sigma, \phi}^{l,m}(\gamma, \lambda)$, by Theorem 1,

$$|a_2| \leq \frac{2|\gamma|\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2 - 2|\gamma|\alpha(1+2\lambda)]\Gamma_2^2 + 4|\gamma|\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2 \alpha^2}{(1+2\lambda)^2 \Gamma_2^2} + \frac{|\gamma|\alpha}{(1+3\lambda)\Gamma_3}.$$

(2) for functions $f \in \mathcal{K}_{\Sigma, \phi}^{l,m}(\gamma, \lambda)$, by Theorem 2,

$$|a_2| \leq \frac{|\gamma|\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2 - 2|\gamma|\alpha(1+2\lambda)]\Gamma_2^2 + 3|\gamma|\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2 \alpha^2}{(1+2\lambda)^2 \Gamma_2^2} + \frac{|\gamma|\alpha}{3(1+3\lambda)\Gamma_3}$$

On the other hand if we take

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \dots \quad (0 \leq \beta < 1), \quad (3.2)$$

then we have $B_1 = B_2 = 2(1-\beta)$.

Corollary 6. By choosing $\phi(z)$ of the form (3.2), we state the following results (1) for functions $f \in \mathcal{S}_{\Sigma, \phi}^{l, m}(\gamma, \lambda)$, by Theorem 1,

$$|a_2| \leq \frac{|\gamma| \sqrt{2(1-\beta)}}{\sqrt{|\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]|}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2(1-\beta)^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{|\gamma|(1-\beta)}{(1+3\lambda)\Gamma_3}.$$

(2) for functions $f \in \mathcal{K}_{\Sigma, \phi}^{l, m}(\gamma, \lambda)$, by Theorem 2,

$$|a_2| \leq \frac{|\gamma| \sqrt{(1-\beta)}}{\sqrt{|\gamma[3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2]|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2(1-\beta)^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{|\gamma|(1-\beta)}{3(1+3\lambda)\Gamma_3}$$

Corollary 7. Let $f(z)$ given by (1.1) be in the $\mathcal{S}_{\Sigma, \phi}^{l, m}(\lambda)$ and $\phi(z)$ is of the form (3.1), then from Theorem (1), we have

$$|a_2| \leq \frac{2\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2 - 2\alpha(1+2\lambda)]\Gamma_2^2 + 4\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{\alpha}{(1+3\lambda)\Gamma_3}.$$

Remark 7. From Corollary (7), taking $\lambda = 0$, we obtain the following results

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1-3\alpha)\Gamma_2^2 + 4\alpha\Gamma_3}} \quad \text{and} \quad |a_3| \leq \frac{4\alpha^2}{\Gamma_2^2} + \frac{\alpha}{\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we obtain the results of Srivastava et al [17].

Corollary 8. Let $f(z)$ given by (1.1) be in the $\mathcal{S}_{\Sigma, \phi}^{l, m}(\lambda)$ and $\phi(z)$ is of the form (3.2), then from Theorem (1), we have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{(1-\beta)}{(1+3\lambda)\Gamma_3}.$$

Remark 8. From Corollary (8), taking $\lambda = 0$, we get

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|2\Gamma_3 - \Gamma_2^2|}} \quad \text{and} \quad |a_3| \leq \frac{4(1-\beta)^2}{\Gamma_2^2} + \frac{(1-\beta)}{\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we have

$$|a_2| \leq \sqrt{2(1-\beta)} \quad \text{and} \quad |a_3| \leq 4(1-\beta)^2 + (1-\beta).$$

Corollary 9. Let $f(z)$ given by (1.1) be in the $\mathcal{K}_{\Sigma, \phi}^{l,m}(\lambda)$ and $\phi(z)$ as defined in (3.1) then from Theorem (2), we have

$$|a_2| \leq \frac{\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2 - 2\alpha(1+2\lambda)]\Gamma_2^2 + 3\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{\alpha^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{\alpha}{3(1+3\lambda)\Gamma_3}.$$

Remark 9. From Corollary (9), taking $\lambda = 0$, we get

$$|a_2| \leq \frac{\alpha}{\sqrt{|3\alpha\Gamma_3 + (1-3\alpha)\Gamma_2^2|}} \quad \text{and} \quad |a_3| \leq \frac{\alpha^2}{\Gamma_2^2} + \frac{\alpha}{3\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we have

$$|a_2| \leq \alpha \quad \text{and} \quad |a_3| \leq \alpha^2 + \frac{\alpha}{3}.$$

Corollary 10. Let $f(z)$ given by (1.1) be in the $\mathcal{K}_{\Sigma, \phi}^{l,m}(\lambda)$ and $\phi(z)$ is of the form (3.2), then from Theorem (2), we have

$$|a_2| \leq \sqrt{\frac{1-\beta}{|3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{(1-\beta)}{3(1+3\lambda)\Gamma_3}.$$

Remark 10. From Corollary (10), by taking $\lambda = 0$, we get

$$|a_2| \leq \sqrt{\frac{1-\beta}{|3\Gamma_3 - 2\Gamma_2^2|}} \quad \text{and} \quad |a_3| \leq \frac{(1-\beta)^2}{\Gamma_2^2} + \frac{(1-\beta)}{3\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we have

$$|a_2| \leq \sqrt{1-\beta} \quad \text{and} \quad |a_3| \leq (1-\beta)^2 + \frac{(1-\beta)}{3}.$$

Concluding remarks: By further specializing the parameters l, m , one can define various other interesting subclasses of Σ based on the differential operators stated in the Remarks 1 to 4, and we can easily derived analogous results (as in Theorems 1 and 2) and the corresponding corollaries as mentioned above. The details involved may be left as an exercise for the interested reader.

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