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A ONE-SIDED THEOREM FOR THE PRODUCT OF ABEL AND CESÀRO SUMMABILITY METHODS

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Abstract. In this paper, a one-sided condition is given to recover (C,α) summability of a sequence from its $(A)(C,\alpha+1)$ summability. Our result extends and generalizes the well known classical Tauberian theorems given for Abel and Cesàro summability methods.

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1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series of real numbers with partial sums $s_n = \sum_{k=0}^{n} a_k$. For all nonnegative integers m, we define

$$(n\Delta)_m s_n = n\Delta((n\Delta)_{m-1} s_n),$$

where $(n\Delta)_0 s_n = s_n$ and $(n\Delta)_1 s_n = n\Delta s_n$.

The backward difference Δs_n of s_n is defined to be $\Delta s_n = s_n - s_{n-1}$, $n \geq 1$, with $\Delta s_0 = s_0$.

Let A_n^{α} be defined by generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^{\alpha} x^n$ $(|x| < 1)$, where

$$A_0^{\alpha} = 0, \quad A_n^{\alpha} = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$$

for $\alpha > -1$.

A sequence (s_n) is said to be summable by the Cesàro mean of order α, or (C,α) summable to s, where $\alpha > -1$, and we write $s_n \to s$ (C,α) if

$$s_n^{\alpha} = \frac{S_n^{\alpha}}{A_n^{\alpha}} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k \to s$$

as $n \to \infty$.

We write $\tau_n = n a_n$ and denote the (C,α) mean of (τ_n) by τ_n^{α}. Borwein [4] showed that if a sequence is (C,α) summable to s for any $\alpha > -1$, it is (C,β) summable to

s for any $\beta > \alpha$. It is also well known that the (C,α) summability method is regular (see [3]). Note that $(C,0)$ summability reduces to the ordinary convergence.

A sequence (s_n) is said to be Abel summable to s, and we write $s_n \to s$ (A) if the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $0 \leq x < 1$ and tends to s as $x \to 1^-$. It is well known that if a sequence is (C,α) summable to s for any $\alpha > -1$, then it is Abel summable to s (see [2]).

A sequence (s_n) is said to be $(A)(C,\alpha)$ summable to s, and we write $s_n \to s$ $(A)(C,\alpha)$ if the series $\sum_{n=0}^{\infty}(s_n^{\alpha} - s_{n-1}^{\alpha})x^n$, with $s_{-1}^{\alpha} = 0$, is convergent for $0 \leq x < 1$ and tends to s as $x \to 1^-$. Note that $(A)(C,\alpha)$ summability reduces to the Abel summability when $\alpha = 0$.

The identity $s_n - s_n^1 = \tau_n^1$ is known as the Kronecker identity and it will be used in the proof of the main result. Throughout this paper we use the symbols $s_n = o(1)$ and $s_n = O(1)$ to mean that $s_n \to 0$ as $n \to \infty$ and (s_n) is bounded for large enough n.

2. Preliminary results

A theorem due to Abel [1] states that if (s_n) converges to s, then it is Abel summable to s. The converse Abel's theorem is not necessarily true. For example the series $\sum_{n=0}^{\infty}(-1)^n$ is not convergent, but it is Abel summable to $1/2$. However, the converse of Abel's theorem may be valid under some condition which we call Tauberian condition. Any theorem stating that convergence follows from a summability method and a Tauberian condition is called a Tauberian theorem.

By imposing some restriction on a_n, Tauber [17] obtained the first partial converses of Abel's theorem.

Theorem 1. *If (s_n) is Abel summable to s and $\tau_n = o(1)$, then (s_n) converges to s.*

Theorem 2. *If (s_n) is Abel summable to s and $\tau_n^1 = o(1)$, then (s_n) converges to s.*

Littlewood [12] replaced the condition $\tau_n = o(1)$ by $\tau_n = O(1)$ and later Hardy and Littlewood [9] obtained the following one-sided Tauberian theorem.

Theorem 3. *If (s_n) is Abel summable to s and $\tau_n \geq -H$ for some nonnegative constant H, then (s_n) converges to s.*

A generalization of Theorem 3 was given by Szász [16].

Theorem 4. *If (s_n) is Abel summable to s and $\tau_n^1 \geq -H$ for some nonnegative constant H, then (s_n) is $(C,1)$ summable to s.*

Pati [14] have recently obtained more general Tauberian theorems generalizing the classical results for the product of the Abel and (C,α) summability methods.

Theorem 5. *If (s_n) is $(A)(C,\alpha)$ summable to s, where $\alpha > 0$, and $\tau_n^{\alpha} \geq -H$ for some nonnegative constant H, then (s_n) is (C,α) summable to s.*

Theorem 6. *The necessary and sufficient condition that the $(A)(C,\alpha+1)$ summability of (s_n) to s, where $\alpha > -1$, implies the (C,α) summability of (s_n) to s, is that $\tau_n^{\alpha+1} = o(1)$.*

Tauberian theorems in the sense of Pati were generalized by Çanak et al. [6], Çanak and Erdem [5] and Erdem and Çanak [7]. Çanak et al. [6] proved that if (s_n) is $(A)(C,\alpha)$ summable to s and $(n\Delta)_m \tau_n^{\alpha+m} = o(1)$ for $m = 1,2$, then (s_n) is convergent to s. Later, Erdem and Çanak [7] proved the main result in Çanak et al. [6] for all integers $m \geq 1$. Recently, Çanak and Erdem [5] have recovered convergence, (C,α) convergence, and (C,α) slow oscillation of (s_n) depending on the conditions given in terms of $(n\Delta)_m \tau_n^{\alpha+m}$ for some special cases of m.

In this paper, we recover (C,α) convergence of (s_n) from its $(A)(C,\alpha+1)$ summability under the one-sided boundedness of $((n\Delta)_m \tau_n^{\alpha+m})$, where $m \geq 1$ and $\alpha > -1$.

3. Main result

Our result is based on Theorem 1 and Theorem 3.

Theorem 7. *If (s_n) is $(A)(C,\alpha+1)$, where $\alpha > -1$, summable to s, and for some integer $m \geq 0$,*

$$(n\Delta)_m \tau_n^{\alpha+m} \geq -H \tag{3.1}$$

then (s_n) is (C,α) summable to s.

From Theorem 7, we deduce the following corollary:

Corollary 1. *If (s_n) is $(A)(C,\alpha+1)$, where $\alpha > -1$, summable to s, and for some integer $m \geq 0$,*

$$(n\Delta)_m \tau_n^{m} \geq -H \tag{3.2}$$

then (s_n) converges to s.

4. Auxiliary results

We need the following lemmas for the proof of Theorem 7.

Lemma 1 ([10, 11])**.** *For $\alpha > -1$, $\tau_n^{\alpha} = n\Delta s_n^{\alpha} = n(s_n^{\alpha} - s_{n-1}^{\alpha})$.*

Lemma 2 ([8, 11])**.** *For $\alpha > -1$, $\tau_n^{\alpha+1} = (\alpha+1)(s_n^{\alpha} - s_n^{\alpha+1})$.*

Lemma 3 ([6])**.** *For $\alpha > -1$, $n\Delta\tau_n^{\alpha+1} = (\alpha+1)(\tau_n^{\alpha} - \tau_n^{\alpha+1})$.*

Lemma 4 ([13])**.** *For $-1 < \alpha < \beta$, $(A)(C,\alpha) \subset (A)(C,\beta)$.*

Lemma 5 ([7]). *Let $\alpha > -1$. For any integer $m \geq 2$,*

$$(n\Delta)_m \tau_n^{\alpha+m} = \sum_{j=1}^{m} (-1)^{j+1} A_m^{(j)}(\alpha) n\Delta \tau_n^{(\alpha+j)},$$

where

$$A_m^{(j)}(\alpha) = a_m^{(j-1)}(\alpha) + a_m^{(j)}(\alpha), \qquad a_m^{(0)}(\alpha) = 0,$$

and

$$a_m^{(j)}(\alpha) = \prod_{k=j+1}^{m} (\alpha+k) \left[\sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq m \\ r<s \Rightarrow t_r \leq t_s}} (\alpha+t_1)(\alpha+t_2)\ldots(\alpha+t_{j-1}) \right]$$

where $j = 1, 2, 3, \ldots, m$.

Lemma 6 ([7]). *Let $\alpha > -1$. For any integer $m \geq 2$,*

$$(\alpha+j) A_m^{(j-1)}(\alpha+1) + (\alpha+j+1) A_m^{(j)}(\alpha+1) = A_{m+1}^{(j)}(\alpha),$$

where $A_m^{(j)}(\alpha)$ is as in Lemma 5.

Lemma 7 ([15]). *For $\alpha > -1$, $\sigma_n(s^\alpha) = \dfrac{1}{\alpha+1} s_n^{\alpha+1} + \left(1 - \dfrac{1}{\alpha+1}\right) \sigma_n(s^{\alpha+1})$.*

Lemma 8. *i) For all $\lambda > 1$ and large enough n, that is, when $[\lambda n] > n$,*

$$s_n^\alpha - s_n^{\alpha+1} = \frac{\alpha}{\alpha+1} \left[\frac{[\lambda n]+1}{[\lambda n]-n} \left(\sigma_{[\lambda n]}(s^{\alpha+1}) - \sigma_n(s^{\alpha+1}) \right) + (\sigma_n(s^{\alpha+1}) - s_n^{\alpha+1}) \right]$$
$$+ \frac{1}{\alpha} \frac{[\lambda n]+1}{[\lambda n]-n} \left(s_{[\lambda n]}^{\alpha+1} - s_n^{\alpha+1} \right) - \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \left(s_k^\alpha - s_n^\alpha \right) \tag{4.1}$$

ii) For all $0 < \lambda < 1$ and large enough n, that is, when $n > [\lambda n]$,

$$s_n^\alpha - s_n^{\alpha+1} = \frac{\alpha}{\alpha+1} \left[\frac{[\lambda n]+1}{n-[\lambda n]} \left(\sigma_n(s^{\alpha+1}) - \sigma_{[\lambda n]}(s^{\alpha+1}) \right) + (\sigma_n(s^{\alpha+1}) - s_n^{\alpha+1}) \right]$$
$$+ \frac{1}{\alpha} \frac{[\lambda n]+1}{n-[\lambda n]} \left(s_n^{\alpha+1} - s_{[\lambda n]}^{\alpha+1} \right) - \frac{1}{n-[\lambda n]} \sum_{k=[\lambda n]+1}^{n} \left(s_n^\alpha - s_k^\alpha \right), \tag{4.2}$$

where $[\lambda n]$ denotes the integer part of the product λn.

Proof. Let $\tau_{n,[\lambda n]}^\alpha = \dfrac{1}{[\lambda n]-n} \displaystyle\sum_{k=n+1}^{[\lambda n]} s_k^\alpha$. Then we have,

$$\tau_{n,[\lambda n]}^\alpha - s_n^{\alpha+1} = \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} s_k^\alpha - s_n^{\alpha+1}$$

$$
\begin{aligned}
&= \frac{1}{[\lambda n]-n}\left(\sum_{k=0}^{[\lambda n]} s_k^\alpha - \sum_{k=0}^{n} s_k^\alpha\right) - s_n^{\alpha+1} \\
&= \frac{[\lambda n]+1}{([\lambda n]-n)([\lambda n]+1)}\sum_{k=0}^{n} s_k^\alpha - \frac{n+1}{([\lambda n]-n)(n+1)}\sum_{k=0}^{n} s_k^\alpha - s_n^{\alpha+1} \\
&= \frac{1}{[\lambda n]-n}\big(([\lambda n]+1)\sigma_{[\lambda n]}(s^\alpha) - (n+1)\sigma_n(s^\alpha)\big) - s_n^{\alpha+1}
\end{aligned}
$$

By lemma 7, we have

$$
\begin{aligned}
\tau_{n,[\lambda n]}^\alpha - s_n^{\alpha+1} &= \frac{1}{[\lambda n]-n}\left(([\lambda n]+1)\left(\frac{\alpha}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right)\right. \\
&\quad \left. -(n+1)\left(\frac{\alpha}{\alpha+1}\sigma_n(s^{\alpha+1}) + \frac{1}{\alpha+1}s_n^{\alpha+1}\right)\right) - s_n^{\alpha+1} \\
&= \frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right. \\
&\quad \left. -\frac{(n+1)\alpha}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{n+1}{\alpha+1}s_n^{\alpha+1} - ([\lambda n]-n)s_n^{\alpha+1}\right).
\end{aligned}
$$

or

$$
\begin{aligned}
s_n^\alpha - s_n^{\alpha+1} &= s_n^\alpha - \tau_{n,[\lambda n]}^\alpha + \tau_{n,[\lambda n]}^\alpha - s_n^{\alpha+1} \\
&= s_n^\alpha - \tau_{n,[\lambda n]}^\alpha + \frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right. \\
&\quad \left. -\frac{(n+1)\alpha}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right) \\
&= s_n^\alpha - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} s_k^\alpha + \frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1})\right. \\
&\quad \left. +\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1} - \frac{\alpha(n+1)}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right)
\end{aligned}
$$

We finally have

$$
\begin{aligned}
s_n^\alpha - s_n^{\alpha+1} = \frac{1}{[\lambda n]-n}&\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right. \\
&\left. -\frac{\alpha(n+1)}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{[\lambda n]-n}\left(\sum_{k=n+1}^{[\lambda n]} s_k^{\alpha}-([\lambda n]-n)s_n^{\alpha}\right)\\
&=\frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1})-\frac{\alpha(n+1)}{\alpha+1}\sigma_n(s^{\alpha+1})\right.\\
&\quad+\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_n(s^{\alpha+1})-\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_n(s^{\alpha+1})\\
&\quad\left.+\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}-\frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right)\\
&\quad-\frac{1}{[\lambda n]-n}\left(\sum_{k=n+1}^{[\lambda n]} s_k^{\alpha}-\sum_{k=n+1}^{[\lambda n]} s_n^{\alpha}\right)\\
&=\frac{1}{[\lambda n]-n}\left[\frac{\alpha([\lambda n]+1)}{\alpha+1}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\right.\\
&\quad-\frac{\alpha}{\alpha+1}\left((n+1)\sigma_n(s^{\alpha+1})-([\lambda n]+1)\sigma_n(s^{\alpha+1})\right)\\
&\quad\left.+\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}-\frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right]\\
&\quad-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(s_k^{\alpha}-s_n^{\alpha}\right)\\
&=\frac{1}{[\lambda n]-n}\left[\frac{\alpha([\lambda n]+1)}{\alpha+1}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\right.\\
&\quad-\frac{\alpha(n-[\lambda n])}{\alpha+1}\sigma_n(s^{\alpha+1})+\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}+\frac{\alpha(n-[\lambda n])}{\alpha+1}s_n^{\alpha+1}\\
&\quad\left.-\frac{[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right]-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(s_k^{\alpha}-s_n^{\alpha}\right)\\
&=\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\right.\\
&\quad\left.+(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1})\right]+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{[\lambda n]}^{\alpha+1}-s_n^{\alpha+1}\right)\\
&\quad-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(s_k^{\alpha}-s_n^{\alpha}\right).
\end{aligned}
$$

This completes the proof. □

The proof for ii) is similar to that of i).

5. Proof of Theorem 7

By hypothesis, $s_n^{\alpha+1} \to s\ (A)$. By Lemma 4, we have $s_n^{\alpha+1} \to s\ (A)$, $s_n^{\alpha+2} \to s\ (A), \ldots, s_n^{\alpha+m} \to s\ (A)$ where m is any positive integer. Hence, by Lemma 2, we get

$$\begin{aligned}(\alpha+2)(s_n^{\alpha+1} - s_n^{\alpha+2}) &= \tau_n^{\alpha+2}\\ (\alpha+3)(s_n^{\alpha+2} - s_n^{\alpha+3}) &= \tau_n^{\alpha+3}\\ &\vdots\\ (\alpha+m+1)(s_n^{\alpha+m} - s_n^{\alpha+m+1}) &= \tau_n^{\alpha+m+1}.\end{aligned}$$

Since $s_n^{\alpha+k} \to s\ (A)$ for $k = 1, 2, \ldots, m+1$, we have

$$\begin{aligned}\tau_n^{\alpha+2} &\to 0\ (A)\\ \tau_n^{\alpha+3} &\to 0\ (A)\\ &\vdots\\ \tau_n^{\alpha+m+1} &\to 0\ (A)\end{aligned} \tag{5.1}$$

and by Lemma 3,

$$\begin{aligned}(\alpha+3)(\tau_n^{\alpha+2} - \tau_n^{\alpha+3}) &= n\Delta\tau_n^{\alpha+3} \to 0\ (A)\\ (\alpha+4)(\tau_n^{\alpha+3} - \tau_n^{\alpha+4}) &= n\Delta\tau_n^{\alpha+4} \to 0\ (A)\\ &\vdots\\ (\alpha+m+1)(\tau_n^{\alpha+m} - \tau_n^{\alpha+m+1}) &= n\Delta\tau_n^{\alpha+m+1} \to 0\ (A).\end{aligned}$$

Since

$$(n\Delta)_m \tau_n^{\alpha+m} \geq -H \tag{5.2}$$

then

$$(n\Delta)_m \tau_n^{\alpha+m+j} \geq -H_1 \tag{5.3}$$

for $j = 1, \ldots, m-1$, by Lemma 5, we have

$$\begin{aligned}(n\Delta)_{m-1}\tau_n^{\alpha+m+1} &= \sum_{j=1}^{m-1} (-1)^{j+1} A_{m-1}^{j} (\alpha+2) n\Delta\tau_n^{\alpha+2+j}\\ &= A_{m-1}^{1}(\alpha+2)n\Delta\tau_n^{\alpha+3} - A_{m-1}^{2}(\alpha+2)n\Delta\tau_n^{\alpha+4} + \ldots\\ &\quad + (-1)^m A_{m-1}^{m-1}(\alpha+2)n\Delta\tau_n^{\alpha+m+1}.\end{aligned}$$

For $j = 1, \ldots, m-1$ we have $n\Delta\tau_n^{\alpha+2+j} \to 0\ (A)$. Hence, we get

$$(n\Delta)_{m-1}\tau_n^{\alpha+m+1} \to 0\ (A) \tag{5.4}$$

It follows from (5.3) that

$$(n\Delta)_{m-1}\tau_n^{\alpha+m+1} = o(1) \tag{5.5}$$

by Theorem 3. By Lemma 3, we obtain

$$(\alpha+m+1)((n\Delta)_{m-1}\tau_n^{\alpha+m} - (n\Delta)_{m-1}\tau_n^{\alpha+m+1}) = (n\Delta)_m\tau_n^{\alpha+m+1}. \tag{5.6}$$

Substituting (5.3) and (5.5) into (5.6), we have

$$(n\Delta)_{m-1}\tau_n^{\alpha+m} \geq -H_2. \tag{5.7}$$

Since

$$\begin{aligned}(n\Delta)_{m-2}\tau_n^{\alpha+m} &= \sum_{j=1}^{m-2}(-1)^{j+1}A_{m-2}^{j}(\alpha+2)n\Delta\tau_n^{\alpha+2+j}\\ &= A_{m-2}^{1}(\alpha+2)n\Delta\tau_n^{\alpha+3} - A_{m-2}^{2}(\alpha+2)n\Delta\tau_n^{\alpha+4}+\dots\\ &\quad+(-1)^{m-2}A_{m-2}^{m-2}(\alpha+2)n\Delta\tau_n^{\alpha+m}\end{aligned}$$

by Lemma 5, we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m} \to 0\ (A). \tag{5.8}$$

From (5.7) and (5.8), we obtain, by Theorem 3,

$$(n\Delta)_{m-2}\tau_n^{\alpha+m} = o(1). \tag{5.9}$$

By Lemma 3, we obtain

$$(\alpha+m)((n\Delta)_{m-2}\tau_n^{\alpha+m-1} - (n\Delta)_{m-2}\tau_n^{\alpha+m}) = (n\Delta)_{m-1}\tau_n^{\alpha+m}. \tag{5.10}$$

Substituting (5.7) and (5.9) into (5.10), we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-1} \geq -H_3. \tag{5.11}$$

Since

$$\begin{aligned}(n\Delta)_{m-3}\tau_n^{\alpha+m-1} &= \sum_{j=1}^{m-3}(-1)^{j+1}A_{m-3}^{j}(\alpha+2)n\Delta\tau_n^{\alpha+2+j}\\ &= A_{m-3}^{1}(\alpha+2)n\Delta\tau_n^{\alpha+3} - A_{m-3}^{2}(\alpha+2)n\Delta\tau_n^{\alpha+4}+\dots\\ &\quad+(-1)^{m-2}A_{m-3}^{m-3}(\alpha+2)n\Delta\tau_n^{\alpha+m-2}\end{aligned}$$

by Lemma 3, we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-1} \to 0\ (A). \tag{5.12}$$

From (5.11) and (5.12), we have, by Theorem 3,

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-1} = o(1). \tag{5.13}$$

By Lemma 3, we obtain

$$(\alpha+m-1)((n\Delta)_{m-3}\tau_n^{\alpha+m-2} - (n\Delta)_{m-3}\tau_n^{\alpha+m-1}) = (n\Delta)_{m-2}\tau_n^{\alpha+m-1}. \tag{5.14}$$

Substituting (5.11) and (5.13) into (5.14), we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-2} \geq -H_4. \tag{5.15}$$

Continuing in this way, we obtain

$$(n\Delta)_2\tau_n^{\alpha+4} \to 0\ (A). \tag{5.16}$$

and

$$(n\Delta)_3\tau_n^{\alpha+4} \geq -H_5. \tag{5.17}$$

From (5.16) and (5.17), we have, by Theorem 3,

$$(n\Delta)_2\tau_n^{\alpha+4} = o(1). \tag{5.18}$$

By Lemma 3, we obtain

$$(\alpha+4)((n\Delta)_2\tau_n^{\alpha+3} - (n\Delta)_2\tau_n^{\alpha+4}) = (n\Delta)_3\tau_n^{\alpha+4}. \tag{5.19}$$

Substituting (5.17) and (5.18) into (5.19), we have

$$(n\Delta)_2\tau_n^{\alpha+3} \geq -H_6. \tag{5.20}$$

From $n\Delta\tau_n^{\alpha+3} \to 0\ (A)$ and (5.20), we have, by Theorem 3,

$$n\Delta\tau_n^{\alpha+3} = o(1). \tag{5.21}$$

By Lemma 3, we obtain

$$(\alpha+3)(n\Delta\tau_n^{\alpha+2} - n\Delta\tau_n^{\alpha+3}) = (n\Delta)_2\tau_n^{\alpha+3}. \tag{5.22}$$

Substituting (5.20) and (5.21) into (5.22), we have

$$n\Delta\tau_n^{\alpha+2} \geq -H_7. \tag{5.23}$$

It follows from (5.1) and (5.23) by Theorem 1, we get

$$\tau_n^{\alpha+2} = o(1). \tag{5.24}$$

Substituting (5.23) and (5.24) into

$$(\alpha+2)(\tau_n^{\alpha+1} - \tau_n^{\alpha+2}) = n\Delta\tau_n^{\alpha+2}$$

we have

$$\tau_n^{\alpha+1} \geq -H_8.$$

Since $s_n^{\alpha+1} \to s\ (A)$ and

$$\tau_n^{\alpha+1} = n\Delta s_n^{\alpha+1} \geq -H_8, \tag{5.25}$$

we obtain, by Theorem 3,

$$s_n^{\alpha+1} \to s. \tag{5.26}$$

Now we need to show that

$$\tau_n^{\alpha} = n\Delta s_n^{\alpha} \geq -C \tag{5.27}$$

for some constant C.

From (5.2), by Lemma 5, we have, for $j=1,\dots,m-1$,

$$\begin{aligned}(n\Delta)_{m-1}\tau_n^{\alpha+m} &= \sum_{j=1}^{m-1}(-1)^{j+1}A_{m-1}^{j}(\alpha+1)n\Delta\tau_n^{\alpha+1+j}\\ &= A_{m-1}^{1}(\alpha+1)n\Delta\tau_n^{\alpha+2}-A_{m-1}^{2}(\alpha+1)n\Delta\tau_n^{\alpha+3}+\dots\\ &\quad+(-1)^{m}A_{m-1}^{m-1}(\alpha+1)n\Delta\tau_n^{\alpha+m}.\end{aligned}$$

We have $n\Delta\tau_n^{\alpha+1+j}\to 0\ (A)$ for $j=1,\dots,m-1$. Hence, we get

$$(n\Delta)_{m-1}\tau_n^{\alpha+m}\to 0\ (A). \tag{5.28}$$

It follows from (5.2) that

$$(n\Delta)_{m-1}\tau_n^{\alpha+m}=o(1) \tag{5.29}$$

by Theorem 3. By Lemma 3, we obtain

$$(\alpha+m)((n\Delta)_{m-1}\tau_n^{\alpha+m-1}-(n\Delta)_{m-1}\tau_n^{\alpha+m})=(n\Delta)_m\tau_n^{\alpha+m}. \tag{5.30}$$

Substituting (5.2) and (5.29) into (5.30), we have

$$(n\Delta)_{m-1}\tau_n^{\alpha+m-1}\geq -H_{10}. \tag{5.31}$$

Since

$$\begin{aligned}(n\Delta)_{m-2}\tau_n^{\alpha+m-1} &= \sum_{j=1}^{m-2}(-1)^{j+1}A_{m-2}^{j}(\alpha+1)n\Delta\tau_n^{\alpha+1+j}\\ &= A_{m-2}^{1}(\alpha+1)n\Delta\tau_n^{\alpha+2}-A_{m-2}^{2}(\alpha+1)n\Delta\tau_n^{\alpha+3}+\dots\\ &\quad+(-1)^{m-2}A_{m-2}^{m-2}(\alpha+1)n\Delta\tau_n^{\alpha+m-1}\end{aligned}$$

by Lemma 5, we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-1}\to 0\ (A). \tag{5.32}$$

From (5.31) and (5.32), we obtain, by Theorem 3,

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-1}=o(1). \tag{5.33}$$

By Lemma 3, we obtain

$$(\alpha+m-1)((n\Delta)_{m-2}\tau_n^{\alpha+m-2}-(n\Delta)_{m-2}\tau_n^{\alpha+m-1})=(n\Delta)_{m-1}\tau_n^{\alpha+m-1}. \tag{5.34}$$

Substituting (5.31) and (5.33) into (5.34), we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-2}\geq -H_{11}. \tag{5.35}$$

Since

$$\begin{aligned}(n\Delta)_{m-3}\tau_n^{\alpha+m-2} &= \sum_{j=1}^{m-3}(-1)^{j+1}A_{m-3}^{j}(\alpha+1)n\Delta\tau_n^{\alpha+1+j}\\ &= A_{m-3}^{1}(\alpha+1)n\Delta\tau_n^{\alpha+2}-A_{m-3}^{2}(\alpha+1)n\Delta\tau_n^{\alpha+3}+\dots\end{aligned}$$

$$+(-1)^{m-2}A_{m-3}^{m-3}(\alpha+1)n\Delta\tau_n^{\alpha+m-3}$$

by Lemma 3, we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-2} \to 0\ (A). \tag{5.36}$$

From (5.35) and (5.36), we have, by Theorem 3,

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-2} = o(1). \tag{5.37}$$

By Lemma 3, we obtain

$$(\alpha+m-2)((n\Delta)_{m-3}\tau_n^{\alpha+m-3} - (n\Delta)_{m-3}\tau_n^{\alpha+m-2}) = (n\Delta)_{m-2}\tau_n^{\alpha+m-2}. \tag{5.38}$$

Substituting (5.35) and (5.37) into (5.38), we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-3} \geq -H_{12}. \tag{5.39}$$

Continuing in this way, we obtain

$$(n\Delta)_2\tau_n^{\alpha+3} \to 0\ (A). \tag{5.40}$$

and

$$(n\Delta)_3\tau_n^{\alpha+3} \geq -H_{13}. \tag{5.41}$$

From (5.40) and (5.41), we have, by Theorem 3,

$$(n\Delta)_2\tau_n^{\alpha+3} = o(1). \tag{5.42}$$

By Lemma 3, we obtain

$$(\alpha+3)((n\Delta)_2\tau_n^{\alpha+2} - (n\Delta)_2\tau_n^{\alpha+3}) = (n\Delta)_3\tau_n^{\alpha+3}. \tag{5.43}$$

Substituting (5.41) and (5.42) into (5.43), we have

$$(n\Delta)_2\tau_n^{\alpha+2} \geq -H_{14}. \tag{5.44}$$

From $n\Delta\tau_n^{\alpha+2} \to 0\ (A)$ and (5.44), we have, by Theorem 3,

$$n\Delta\tau_n^{\alpha+2} = o(1). \tag{5.45}$$

By Lemma 3, we obtain

$$(\alpha+2)(n\Delta\tau_n^{\alpha+1} - n\Delta\tau_n^{\alpha+2}) = (n\Delta)_2\tau_n^{\alpha+2}. \tag{5.46}$$

Substituting (5.44) and (5.45) into (5.46), we have

$$n\Delta\tau_n^{\alpha+1} \geq -H_{15}. \tag{5.47}$$

Substituting (5.47), (5.25) and (5.27) into

$$(\alpha+1)(\tau_n^{\alpha} - \tau_n^{\alpha+1}) = n\Delta\tau_n^{\alpha+1}$$

we have

$$\tau_n^{\alpha} = n\Delta s_n^{\alpha} \geq -H_{16}. \tag{5.48}$$

By Lemma 8 i) and (5.48), we have

$$s_n^{\alpha} - s_n^{\alpha+1} = \frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1}) - \sigma_n(s^{\alpha+1})\right) + \left(\sigma_n(s^{\alpha+1})\ - s_n^{\alpha+1}\right)\right]$$

$$+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta s_j^{\alpha}$$

$$\leq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]$$

$$+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)+\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\frac{H}{j}$$

$$\leq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]$$

$$+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)+H\log\frac{[\lambda n]}{n}$$

Taking the lim sup of both sides, we get

$$\limsup_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\leq\limsup_{n\to\infty}\left\{\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\right.\right.$$
$$\left.\left.+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)\right\}+H\log\lambda,$$

where $H>0$. Since

$$\frac{[\lambda n]+1}{[\lambda n]-n}\leq\frac{2\lambda}{\lambda-1}\tag{5.49}$$

for $\lambda>1$ and sufficiently large n and $s_n^{\alpha+1}\to s$, we have

$$\lim_{\lambda\to1^+}\limsup_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\leq 0\tag{5.50}$$

By Lemma 8 ii) and (5.48), we have

$$s_n^{\alpha}-s_n^{\alpha+1}=\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]$$

$$+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{[\lambda n]}^{\alpha+1}\right)+\frac{1}{n-[\lambda n]}\sum_{k=[\lambda n]+1}^{n}\sum_{j=n+1}^{k}\Delta s_j^{\alpha}$$

$$\geq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]$$

$$+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{\lambda n}^{\alpha+1}\right)+\frac{1}{n-[\lambda n]}\sum_{k=[\lambda n]+1}^{n}\sum_{j=n+1}^{k}\frac{H}{j}$$

$$\geq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]$$

$$+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{[\lambda n]}^{\alpha+1}\right)-H\log\frac{[\lambda n]}{n}$$

Taking the lim inf of both sides, we get

$$\liminf_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\geq\liminf_{n\to\infty}\left\{\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)\right.\right.$$
$$\left.\left.+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{[\lambda n]}^{\alpha+1}\right)\right\}+H\log\lambda$$

where $H>0$. Since

$$\frac{[\lambda n]+1}{n-[\lambda n]}\leq\frac{2\lambda}{1-\lambda} \tag{5.51}$$

for $0<\lambda<1$ and sufficiently large n and $s_n^{\alpha+1}\to s$, we have

$$\lim_{\lambda\to 1^+}\liminf_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\geq 0. \tag{5.52}$$

Combining (5.50) and (5.52) provides

$$\lim s_n^{\alpha}=\lim s_n^{\alpha+1}.$$

This completes the proof.

We like to note that we used H to denote a constant, possibly different at each occurrence above.

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