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New integral inequalities of Hermite-Hadamard type for n-times differentiable s-logarithimcally convex functions with applications

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NEW INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR n-TIMES DIFFERENTIABLE s-LOGARITHMICALLY CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, some new integral inequalities of Hermite-Hadamard type are presented for functions whose nth derivatives in absolute value are s-logarithmically convex. From our results, several inequalities of Hermite-Hadamard type can be derived in terms of functions whose first and second derivatives in absolute value are s-logarithmically convex functions as special cases. Our results may provide refinements of some results for s-logarithmically convex functions already exist in literature. Finally, applications to special means of the established results are given.

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1. Introduction

A function $f : I \to \mathbb{R}$, $\varnothing \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) is known as the Hermite-Hadamard inequality (see [8]). The inequalities (1.1) hold in reversed direction if f is concave.

For recent results on Hermite-Hadamard type integral inequalities for convex functions see [5, 7, 10–13, 15, 18, 19] and closely related references therein.

The classical convexity has been generalized in diverse ways such as s-convexity, m-convexity, (α, m)-convexity, h-convexity, logarithmically-convexity, s- logarithmically convexity, (α, m)- logarithmically convexity and h-log-convexity. Many papers have been written by a number of mathematicians concerning Hermite-Hadamard type inequalities for these classes of convex functions see for instance the recent papers [1–4, 6, 8, 9, 14, 17, 20–25, 27] and the references therein.

The notion of logarithmically convex functions is defined as follows.

Definition 1 ([1, 25, 26])**.** If a function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies

$$f(\lambda x + (1-\lambda) y) \leq [f(x)]^{\lambda} [f(y)]^{1-\lambda}, \tag{1.2}$$

for all x, $y \in I$, $\lambda \in [0,1]$, the function f is called logarithmically convex on I. If the inequality (1.2) reverses, the function f is called logarithmically concave on I.

The concept of logarithmically convex functions was further generalized as in the definition below.

Definition 2 ([1, 25, 26])**.** For some $s \in (0,1]$, a positive function $f : I \subseteq \mathbb{R} \to (0, \infty)$ is said to be s-logarthmically convex on I if and only if

$$f(\lambda x + (1-\lambda) y) \leq [f(x)]^{\lambda^s} [f(y)]^{(1-\lambda)^s}$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$.

It is obvious that when $s = 1$ in Definition 2, the s-logarithmically convexity becomes the usual logarithmically convexity.

Xi et al. [25], obtained the following Hermite-Hadamard type inequalities for s-logarithmically convex functions.

Theorem 1 ([25])**.** *Let $f : I \subseteq [0, \infty) \to (0, \infty)$ be a differentiable function on I°, a, $b \in I^\circ$ with $a < b$ and $f' \in L([a,b])$. If $\left|f'\right|^q$ for $q \geq 1$ is s-logarithmically convex on $[a,b]$ for some given $s \in (0,1]$, then*

$$\left| f(a) - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4} \left(\frac{1}{2}\right)^{1-1/q} \left\{ 3^{(q-1)/q} [L_1(\mu, q)]^{1/q} + [L_2(\mu, q, b)]^{1/q} \right\}, \tag{1.3}$$

where

$$L_1(\mu, q) \leq \begin{cases} \left|f'(a) f'(b)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f'(a)\right|, \left|f'(b)\right| \leq 1, \\ \left|f'(a) f'(b)\right|^{q/(2s)} F_1(\mu_2), & 1 \leq \left|f'(a)\right|, \left|f'(b)\right|, \\ \left|f'(a) f'(b)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f'(a)\right| \leq 1 < \left|f'(b)\right|, \\ \left|f'(a) f'(b)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f'(b)\right| \leq 1 < \left|f'(a)\right|, \end{cases}$$

$$L_2(\mu,q,u) \leq \begin{cases} \left|f'(u)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f'(a)\right|, \left|f'(b)\right| \leq 1, \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_2), & 1 \leq \left|f'(a)\right|, \left|f'(b)\right|, \\ \left|f'(u)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f'(a)\right| \leq 1 < \left|f'(b)\right|, \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f'(b)\right| \leq 1 < \left|f'(a)\right|, \end{cases}$$

$$F_1(\nu) = \begin{cases} \frac{1}{\ln \nu}\left(2\nu - 1 - \frac{\nu-1}{\ln \nu}\right) & \nu \neq 1, \\ \frac{3}{2} & \nu = 1, \end{cases}$$

$$F_2(\nu) = \begin{cases} \frac{1}{\ln \nu}\left(\nu - \frac{\nu-1}{\ln \nu}\right) & \nu \neq 1, \\ \frac{1}{2} & \nu = 1, \end{cases}$$

and

$$\mu_1 = \left|\frac{f'(a)}{f'(b)}\right|^{sq/2}, \mu_2 = \left|\frac{f'(a)}{f'(b)}\right|^{q/(2s)}, \mu_3 = \frac{\left|f'(a)\right|^{sq/2}}{\left|f'(b)\right|^{q/(2s)}}, \mu_4 = \frac{\left|f'(a)\right|^{q/(2s)}}{\left|f'(b)\right|^{qs/2}}.$$

Theorem 2 ([25]). *Under the conditions of Theorem 1, we have*

$$\left|f(b) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[L_2(\mu,q,a)\right]^{1/q} + 3^{(q-1)/q}\left[L_1\left(\mu^{-1},q\right)\right]^{1/q}\right\}, \quad (1.4)$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ *and* μ_i *for* $i = 1, 2, 3, 4$ *are defined as in Theorem 1.*

Theorem 3 ([25]). *Under the conditions of Theorem 1, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[L_2(\mu,q,b)\right]^{1/q} + \left[L_1\left(\mu^{-1},q,a\right)\right]^{1/q}\right\}, \quad (1.5)$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ *and* μ_i *for* $i = 1, 2, 3, 4$ *are defined as in Theorem 1.*

Applications to special means of positive numbers of the above results can also be seen in [25].

For further results on Hermite-Hadamard type inequalities for s-logarithmically convex we refer the reader to [1, 9, 26, 27]. The main purpose of the present paper is to establish a new Hermite-Hadamard type integral inequalities in Section 2 by using the notion of s-logarithmically convexity and new identity for n-times differentiable functions from [15]. The applications of our results to special means of positive real numbers are also given in Section 3.

2. Main results

The following Lemmas are essential in establishing our main results in this section.

Lemma 1 ([15]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for some $n \in \mathbb{N}$, where $a, b \in I^\circ$ with $a < b$, we have the identity*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)$$
$$= \frac{(b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t)\, f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt$$
$$+ \frac{(-1)^n (b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t)\, f^{(n)}\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right)dt, \quad (2.1)$$

where an empty sum is understood to be nil.

Lemma 2 ([16]). *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^1 t^n \mu^t\,dt = \frac{(-1)^{n+1} n!}{(\ln\mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\ln\mu)^{k+1}} \quad (2.2)$$

for $n \in \mathbb{N}$.

Lemma 3. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^1 (1-t)^n \mu^t\,dt = \frac{n!\mu}{(\ln\mu)^{n+1}} - n!\sum_{k=0}^{n} \frac{1}{(n-k)!\,(\ln\mu)^{k+1}} \quad (2.3)$$

for $n \in \mathbb{N}$.

Proof. By making the substitution $t = 1-u$ in (2.2), in which μ is replaced by $\frac{1}{\mu}$, we get (2.3). □

Lemma 4 ([3]). *For $\alpha > 0$ and $\mu > 0$, we have*

$$J(\alpha,\mu) := \int_0^1 (1-t)^{\alpha-1}\mu^t dt = \sum_{k=1}^{\infty} \frac{(\ln\mu)^{k-1}}{(\alpha)_k} < \infty, \tag{2.4}$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+k-1).$$

Theorem 4. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for some $n \in \mathbb{N}$, where a, $b \in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for some $s \in (0,1]$ and $q \in [1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^n}{2^{n+1}n!}\left(\frac{n}{n+1}\right)^{1-\frac{1}{q}} \left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{\left[F_1(\mu,n)\right]^{\frac{1}{q}} + \left[F_1\left(\mu^{-1},n\right)\right]^{\frac{1}{q}}\right\}, \tag{2.5}$$

where $\mu = \left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2}$,

$$(\delta,\theta) = \begin{cases} (s/2, s/2), & \text{if } 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \le 1, \\ (1-s/2, 1-s/2), & \text{if } 1 \le \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ (s/2, 1-s/2), & \text{if } 0 < \left|f^{(n)}(a)\right| \le 1 \le \left|f^{(n)}(b)\right|, \\ (1-s/2, s/2) & \text{if } 0 < \left|f^{(n)}(b)\right| \le 1 \le \left|f^{(n)}(a)\right|, \end{cases}$$

and

$$F_1(\nu,n) = \begin{cases} \frac{n!\nu(\ln\nu-1)}{(\ln\nu)^{n+1}} + \frac{1}{\ln\nu} - n!\sum_{k=1}^{n}\frac{\ln\nu-1}{(n-k)!(\ln\nu)^{k+1}}, & \nu \ne 1, \\ \frac{n}{n+1}, & \nu = 1. \end{cases}$$

Proof. From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2} - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right.$$

$$-\frac{1}{b-a}\int_a^b f(x)\,dx\Bigg| \le \frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right.$$

$$\left.+\left(\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right\}. \quad (2.6)$$

Since for $0<\xi\le 1\le\eta$, $0\le\lambda\le 1$ and $0<s\le 1$. Then

$$\xi^{\lambda^s}\le\xi^{s\lambda} \quad\text{and}\quad \eta^{\lambda^s}\le\eta^{\lambda s+1-s}. \quad (2.7)$$

When $0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\le 1$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\le\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{sq\left(\frac{1-t}{2}\right)}\left|f^{(n)}(b)\right|^{sq\left(\frac{1+t}{2}\right)}dt$$
$$=\left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^t\,dt$$
$$=\left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2}F_1(\mu,n). \quad (2.8)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\le\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{sq\left(\frac{1-t}{2}\right)}\left|f^{(n)}(a)\right|^{sq\left(\frac{1+t}{2}\right)}dt$$
$$=\left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{-t}\,dt$$
$$=\left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2}F_1\left(\mu^{-1},n\right). \quad (2.9)$$

When $\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\ge 1$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$

$$\leq \left|f^{(n)}(a)\,f^{(n)}(b)\right|^{q(1-s/2)}\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^t\,dt$$
$$= \left|f^{(n)}(a)\,f^{(n)}(b)\right|^{q(1-s/2)} F_1(\mu,n). \quad (2.10)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\,f^{(n)}(b)\right|^{q(1-s/2)}\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-t}\,dt$$
$$= \left|f^{(n)}(a)\,f^{(n)}(b)\right|^{q(1-s/2)} F_1\left(\mu^{-1},n\right). \quad (2.11)$$

When $0<\left|f^{(n)}(a)\right|\leq 1\leq\left|f^{(n)}(b)\right|$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-1/2s)}\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^t\,dt$$
$$= \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-s/2)} F_1(\mu,n). \quad (2.12)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-1/2s)}\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-t}\,dt$$
$$= \left|f^{(n)}(a)\right|^{q/2s}\left|f^{(n)}(b)\right|^{q(1-s/2)} F_1\left(\mu^{-1},n\right). \quad (2.13)$$

When $0<\left|f^{(n)}(b)\right|\leq 1\leq\left|f^{(n)}(a)\right|$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^t\,dt$$
$$= \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2} F_1(\mu,n). \quad (2.14)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\le \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}\int_0^1(1-t)^{n-1}(n-1+t)\mu^{-t}dt$$
$$=\left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}F_1\left(\mu^{-1},n\right), \quad (2.15)$$

where $\mu=\left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2}$. A combination of (2.8)-(2.15) into (2.6) gives the desired result. This completes the proof of the Theorem. □

Corollary 1. *Under the assumptions of Theorem 4, if $q=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{F_1(\mu,n)+F_1\left(\mu^{-1},n\right)\right\}, \quad (2.16)$$

where $F_1(\nu,n)$, μ and (δ,θ) are defined as in Theorem 4.

Corollary 2. *Under the assumptions of Theorem 4, if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le(b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{\left[F_1(\mu,1)\right]^{\frac{1}{q}}+\left[F_1\left(\mu^{-1},1\right)\right]^{\frac{1}{q}}\right\}, \quad (2.17)$$

where

$$F_1(\nu,1)=\begin{cases}\frac{\nu(\ln\nu-1)+1}{(\ln\nu)^2}, & \nu\ne 1\\ \frac{1}{2}, & \nu=1\end{cases}, \quad \mu=\left|\frac{f'(b)}{f'(a)}\right|^{sq/2}$$

and

$$(\delta,\theta)=\begin{cases}(s/2,s/2), & \text{if } 0<\left|f'(a)\right|,\left|f'(b)\right|\le 1,\\ (1-s/2,1-s/2), & \text{if } 1\le\left|f'(a)\right|,\left|f'(b)\right|,\\ (s/2,1-s/2), & \text{if } 0<\left|f'(a)\right|\le 1\le\left|f'(b)\right|,\\ (1-s/2,s/2) & \text{if } 0<\left|f'(b)\right|\le 1\le\left|f'(a)\right|.\end{cases}$$

Corollary 3. *If we take $q = 1$ in Corollary 2, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{b-a}{4}\right)\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{\left[F_1(\mu,1)\right]+\left[F_1\left(\mu^{-1},1\right)\right]\right\}, \quad (2.18)$$

where $F_1(\nu,1)$, μ and (δ,θ) are defined as in Corollary 2.

Corollary 4. *Suppose the assumptions of Theorem 4 are fulfilled and if $n = 2$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{\left[F_1(\mu,2)\right]^{\frac{1}{q}}+\left[F_1\left(\mu^{-1},2\right)\right]^{\frac{1}{q}}\right\}, \quad (2.19)$$

where

$$F_1(\nu,2)=\begin{cases}\frac{2\nu(\ln\nu-1)-(\ln\nu)^2+2}{(\ln\nu)^3}, & \nu\neq 1,\\ \frac{2}{3}, & \nu=1,\end{cases} ;\ \mu=\left|\frac{f''(b)}{f''(a)}\right|^{sq/2}$$

and

$$(\delta,\theta)=\begin{cases}(s/2,s/2), & \text{if } 0<\left|f''(a)\right|,\left|f''(b)\right|\leq 1,\\ (1-s/2,1-s/2), & \text{if } 1\leq\left|f''(a)\right|,\left|f''(b)\right|,\\ (s/2,1-s/2), & \text{if } 0<\left|f''(a)\right|\leq 1\leq\left|f''(b)\right|,\\ (1-s/2,s/2) & \text{if } 0<\left|f''(b)\right|\leq 1\leq\left|f''(a)\right|.\end{cases}$$

Corollary 5. *If $q = 1$ in Corollary 4, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{16}\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{\left[F_1(\mu,2)\right]+\left[F_1\left(\mu^{-1},2\right)\right]\right\}, \quad (2.20)$$

where $F_1(\nu,2)$, μ and (δ,θ) are defined as in Corollary 4.

Theorem 5. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for some $n \in \mathbb{N}$, where a, $b \in I^{\circ}$ with $a < b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for some $s \in (0,1]$ and $q \in (1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{n+1}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{\left[F_2(\mu,n)\right]^{\frac{1}{q}}+\left[F_2\left(\mu^{-1},n\right)\right]^{\frac{1}{q}}\right\}, \quad (2.21)$$

where

$$F_2(\nu,n)=\begin{cases}\sum_{k=1}^{\infty}\frac{(\ln\nu)^{k-1}}{(nq-q+1)_k}<\infty, & \nu\neq 1,\\ \frac{1}{nq-q+1}, & \nu=1,\end{cases}$$
$$(nq-q+1)_k=(nq-q+1)(nq-q+2)\dots(nq-q+k),$$

μ and (δ,θ) are defined as in Theorem 4.

Proof. Using Lemma 1, the Hölder inequality and the s-logarithmically convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1(n-1+t)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1(1-t)^{q(n-1)}\left[\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}\right]dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1(1-t)^{q(n-1)}\left[\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}\right]dt\right)^{1/q}\right\}. \quad (2.22)$$

The proof follows by using similar arguments as in proving Theorem 4 and using Lemma 4. □

Corollary 6. *Under the assumptions of Theorem 5, if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{\left[F_2(\mu,1)\right]^{\frac{1}{q}}+\left[F_2\left(\mu^{-1},1\right)\right]^{\frac{1}{q}}\right\}, \tag{2.23}$$

where

$$F_2(\nu,1)=\begin{cases}\sum_{k=1}^{\infty}\frac{(\ln\nu)^{k-1}}{k!}<\infty, & \nu\neq 1,\\ 1, & \nu=1,\end{cases} \;;\; \mu=\left|\frac{f'(b)}{f'(a)}\right|^{sq/2}$$

and (δ,θ) is defined as in Corollary 2.

Corollary 7. *Under the assumptions of Theorem 5, if $n=2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{16}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{\left[F_2(\mu,2)\right]^{\frac{1}{q}}+\left[F_2\left(\mu^{-1},2\right)\right]^{\frac{1}{q}}\right\}, \tag{2.24}$$

where

$$F_2(\nu,2)=\begin{cases}\sum_{k=1}^{\infty}\frac{(\ln\nu)^{k-1}}{(q+1)_k}<\infty, & \nu\neq 1,\\ \frac{1}{q+1}, & \nu=1,\end{cases}$$
$$(q+1)_k=(q+1)(q+2)\ldots(q+k),\; \mu=\left|\frac{f''(b)}{f''(a)}\right|^{sq/2}$$

and (δ,θ) are defined as in Corollary 4.

Theorem 6. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)}\in L([a,b])$ for some $n\in\mathbb{N}$, where $a,\ b\in I^{\circ}$ with $a<b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for $q\in(1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{n^{n+1-\frac{1}{q}}(b-a)^n}{2^{n+1}n!}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}$$

$$\times \left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{[F_3(\mu)]^{\frac{1}{q}}+\left[F_3\left(\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (2.25)$$

where

$$F_3(\nu)=\begin{cases}\frac{\nu-1}{\ln\nu}, & \nu\neq 1,\\ 1, & \nu=1,\end{cases}, \mu=\left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2},$$

$$B(z;\alpha,\beta)=\int_0^z t^{\alpha-1}(1-t)^{1-\beta}\,dt, 0\leq z\leq 1, \alpha>0, \beta>0$$

is the incomplete Beta function and (δ,θ) *are defined as in Theorem 4.*

Proof. Using Lemma 1, the Hölder inequality and the s-logarithmically convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right\}. \quad (2.26)$$

By using (2.7) and the fact that

$$\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt$$
$$=n^{\frac{nq+q-1}{q-1}}\int_0^{\frac{1}{n}}t^{\frac{(n-1)q}{q-1}}(1-t)^{\frac{q}{q-1}}\,dt=n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right),$$

we get the required inequality (2.25) from (2.26). □

Corollary 8. *Suppose the assumptions of Theorem 6 are satisfied and if $n = 1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{\left[F_3(\mu)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (2.27)$$

where

$$F_3(\nu)=\begin{cases}\frac{\nu-1}{\ln\nu}, & \nu\neq 1,\\ 1, & \nu=1,\end{cases}\; ;\mu=\left|\frac{f'(b)}{f'(a)}\right|^{sq/2}$$

and (δ,θ) are defined as in Corollary 2.

Corollary 9. *Suppose the assumptions of Theorem 6 are satisfied and if $n = 2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2^{1+1/q}}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{\left[F_3(\mu)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (2.28)$$

where

$$F_3(\nu)=\begin{cases}\frac{\nu-1}{\ln\nu}, & \nu\neq 1,\\ 1, & \nu=1,\end{cases}\; ;\mu=\left|\frac{f''(b)}{f''(a)}\right|^{sq/2},$$

$B(z;\alpha,\beta)$ is the incomplete Beta function as defined in Theorem 6 and (δ,θ) are defined as in Corollary 4.

Remark 1. We can get several interesting inequalities for log-convex functions by setting $s = 1$ in the above results. However, the details are left to the interested reader.

3. Applications to special means

For positive numbers $a > 0$, $b > 0$, define

$$A(a,b)=\frac{a+b}{2},\; G(a,b)=\sqrt{ab},\; H(a,b)=\frac{2ab}{a+b},$$

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\\ a, & a=b\end{cases}$$

and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq 0,-1 \text{ and } a \neq b, \\ \frac{b-a}{\ln b - \ln a}, & p = -1 \text{ and } a \neq b, \\ I(a,b), & p = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

It is well known that A, G, H, $L = L_{-1}$, $I = L_0$ and L_p are called the arithmetic, geometric, harmonic, logarithmic, exponential and generalized logarithmic means of positive numbers a and b.

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

Theorem 7. *Let* $0 < a < b \leq 1$, $r < 0$, $r \neq -1$, $s \in (0,1]$ *and* $q \geq 1$.

(1) *If* $r \neq -2$, *then*

$$\begin{aligned} &\left|A\left(a^{r+1},b^{r+1}\right) - \left[L_{r+1}(a,b)\right]^{r+1}\right| \\ &\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} |r+1| \left[G\left(a^r,b^r\right)\right]^2 \\ &\times \left\{ a^{-rs}\left[\frac{2\left(1-b^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)\right)}{rqs(\ln b - \ln a)}\right]^{1/q} \right. \\ &\left. + b^{-rs}\left[\frac{2\left(a^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)-1\right)}{rqs(\ln b - \ln a)}\right]^{1/q} \right\}. \end{aligned}$$

(2) *If* $r = -2$, *then*

$$\begin{aligned} &\left|\frac{1}{H(a,b)} - \frac{1}{L(a,b)}\right| \\ &\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} \left[G\left(a^{-2},b^{-2}\right)\right]^2 \\ &\times \left\{ a^{2s}\left[\frac{1-b^{qs}L\left(a^{-qs},b^{-qs}\right)}{qs(\ln a - \ln b)}\right]^{1/q} + b^{2s}\left[\frac{a^{qs}L\left(a^{-qs},b^{-qs}\right)-1}{qs(\ln a - \ln b)}\right]^{1/q} \right\}. \end{aligned}$$

Proof. Let $f(x) = \frac{x^{r+1}}{r+1}$ for $0 < x \leq 1$. Then $|f'(x)| = x^r$ and

$$\ln\left|f'(\lambda x + (1-\lambda)y)\right|^q \leq \lambda^s \ln\left|f'(x)\right|^q + (1-\lambda)^s \left|f'(y)\right|^q$$

for $x,\ y \in (0,1]$, $\lambda \in [0,1]$, $s \in (0,1]$ and $q \geq 1$. This shows that $|f'(x)|^q = x^{rq}$ is s-logarithmically convex function on $(0,1]$. Since$|f'(a)| > |f'(b)| = b^r \geq 1$, hence

$$\mu = \left|\frac{f'(a)}{f'(b)}\right|^{sq/2} = \left(\frac{b}{a}\right)^{rqs/2}, \mu^{-1} = \left(\frac{a}{b}\right)^{rqs/2}$$

and

$$\left|f'(a) f'(b)\right|^{(1-s/2)} \left\{[F_1(\mu,1)]^{\frac{1}{q}} + \left[F_1\left(\mu^{-1},1\right)\right]^{\frac{1}{q}}\right\}$$

$$= \left[G\left(a^r,b^r\right)\right]^2 \left\{a^{-rs}\left[\frac{2\left(1-b^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)\right)}{rqs(\ln b - \ln a)}\right]^{1/q}\right.$$

$$\left. + b^{-rs}\left[\frac{2\left(a^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)-1\right)}{rqs(\ln b - \ln a)}\right]^{1/q}\right\}.$$

Substituting the above quantities in Corollary 2, we get the required inequality. □

Remark 2. Many interesting inequalities of means of positive real numbers can be obtained by using the other results, however, the details are left to the interested reader.

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