Miskolc Mathematical Notes
Vol. 16 (2015), No 1, pp. 361-368

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1260

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APPROXIMATELY ALGEBRAIC TENSOR PRODUCTS

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Received 23 May, 2014

Abstract. Let X and Y be normed spaces over a complete field $\mathbb{F}$ with dual spaces X' and Y' respectively. Under certain hypotheses, for given $x \in X$, $y \in Y$ and a mapping u from $X' \times Y'$ to $\mathbb{F}$, we apply Hyers–Ulam approach to find a unique bounded bilinear mapping v near to u such that $||v|| = ||x \otimes y||$.

2010 *Mathematics Subject Classification:* 46B10; 47L50; 39B72

Keywords: algebraic tensor product, dual space, stability

1. Introduction

Let X, Y, and Z be normed linear spaces over the same field $\mathbb{F}$. A mapping $\phi : X \times Y \longrightarrow Z$ is said to be bilinear if the mappings $x \longmapsto \phi(x, y)$ and $y \longmapsto \phi(x, y)$ are linear. A bilinear mapping $\phi : X \times Y \longrightarrow Z$ is said to be bounded if there exists $M > 0$ such that $||\phi(x, y)|| \leq M||x||||y||$ for all $x \in X$ and $y \in Y$. The norm of ϕ is then defined by

$$||\phi|| := \sup\{||\phi(x, y)|| : (x, y) \in \mathcal{B}_X \times \mathcal{B}_Y\},$$

where $\mathcal{B}_X := \{x \in X : ||x|| \leq 1\}$. The set of all bounded bilinear mappings from $X \times Y$ to Z is denoted by $\mathcal{BL}(X \times Y, Z)$. Let X' and Y' be dual spaces of X and Y respectively. For given $x \in X$ and $y \in Y$, $x \otimes y$ is an element of $\mathcal{BL}(X' \times Y', \mathbb{F})$ defined by $x \otimes y(f, g) := f(x)g(y)$ for all $f \in X'$ and $g \in Y'$. The algebraic tensor product of X and Y, $X \otimes Y$, is defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in $\mathcal{BL}(X' \times Y', \mathbb{F})$ (see [3]).

A classical question in the theory of functional equations is the following (see [4], [6], [7], [9], [10], [8], [12], [14], [15], [20], [19], [17], [18], [21], [13], [22]): *When is it true that a function which approximately satisfies a functional equation ζ must be close to an exact solution of ζ?*
If the problem accepts a solution, we say that the equation ζ is stable. There are cases in which each approximate solution is actually a true solution. In such cases, we call the equation ζ superstable.

The first stability problem concerning group homomorphisms was raised by Ulam [22] during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940. Ulam's problem was partially solved by Hyers [7] for mappings between Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [16] has provided a lot of influence in the development of what is called the generalized Hyers-Ulam stability or the Hyers-Ulam-Rassias stability of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Gavruta [5] in 1994 by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. Badora [2] proved the generalized Hyers-Ulam stability of ring homomorphisms, which generalizes the result of D. G. Bourgin. Miura [11] proved the generalized Hyers-Ulam stability of Jordan homomorphisms.

In this paper, under certain hypotheses and using Hyers–Ulam approach, we find a unique bounded bilinear mapping v near to a given mapping $u : X' \times Y' \longrightarrow \mathbb{F}$ such that $||v|| = ||x \otimes y||$ for $x \in X$, $y \in Y$. Throughout this paper, it is assumed that X and Y are normed spaces over a complete field $\mathbb{F}$ with dual spaces X' and Y' respectively.

2. Results

Theorem 1. *Let $u : X' \times Y' \to \mathbb{F}$ be a mapping for which there exist positive real valued functions φ_1, φ_2, and φ on $X' \times X' \times Y'$, $X' \times Y' \times Y'$, and $X' \times Y'$, respectively such that*

$$\tilde{\varphi}(f,g) := \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi_1(2^i f, 2^i f, g) < \infty, \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi_1(2^n f_1, 2^n f_2, g) = \lim_{n\to\infty} \frac{1}{2^n} \varphi_2(2^n f, g_1, g_2) = \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n f, g) = 0, \tag{2.2}$$

$$|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \le \varphi_1(f_1, f_2, g), \tag{2.3}$$

$$|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \le \varphi_2(f, g_1, g_2) \tag{2.4}$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$, and $c \in \mathbb{F}$. Then, there exists a unique bilinear mapping v from $X' \times Y'$ to $\mathbb{F}$ such that

$$|u(f,g) - v(f,g)| \le \tilde{\varphi}(f,g) \quad (f \in X', g \in Y'). \tag{2.5}$$

Moreover, if the mapping u satisfies

$$||u(f,g)| - |f(x)g(y)|| \le \varphi(f,g) \tag{2.6}$$

for some fixed $x \in X$ and $y \in Y$, then $||v|| = ||x \otimes y||$ and so in particular v is bounded.

Proof. Putting $c = 1$ and replacing f_1 and f_2 in (2.3) by f and dividing both sides by 2, we get

$$|\frac{1}{2}u(2f,g) - u(f,g)| \leq \frac{1}{2}\varphi_1(f,f,g) \tag{2.7}$$

for all $f \in X'$ and $g \in Y'$. Replacing f by $2f$ in (2.7) and dividing both sides by 2, we find that

$$|\frac{1}{2^2}u(2^2f,g) - \frac{1}{2}u(2f,g)| \leq \frac{1}{2^2}\varphi_1(2f,2f,g) \tag{2.8}$$

for all $f \in X'$ and $g \in Y'$. Combining (2.7) with (2.8), we obtain

$$|\frac{1}{2^2}u(2^2f,g) - u(f,g)| \leq \frac{1}{2}\varphi_1(f,f,g) + \frac{1}{2^2}\varphi_1(2f,2f,g)$$

for all $f \in X'$ and $g \in Y'$. By induction on n, we conclude that

$$|\frac{1}{2^n}u(2^nf,g) - u(f,g)| \leq \sum_{i=0}^{n-1}\frac{1}{2^{i+1}}\varphi_1(2^if,2^if,g) \tag{2.9}$$

for all $f \in X'$ and $g \in Y'$. We now turn to use the Cauchy convergence criterion. Replace f by 2^kf in (2.9) and divide both sides by 2^k, where k is an arbitrary positive integer, to get

$$|\frac{1}{2^{n+k}}u(2^{n+k}f,g) - \frac{1}{2^k}u(2^kf,g)| \leq \sum_{i=k}^{n+k-1}\frac{1}{2^{i+1}}\varphi_1(2^if,2^if,g)$$

for all $f \in X'$, $g \in Y'$, and all positive integers $n \geq k$. It follows from the last inequality and (2.1) that the sequence $\{\frac{1}{2^n}u(2^nf,g)\}$ is a Cauchy sequence for all $f \in X'$ and $g \in Y'$. Since $\mathbb{F}$ is a complete field, this sequence converges. Define $v(f,g) := \lim_{n\to\infty}\frac{1}{2^n}u(2^nf,g)$. Taking the limit as $n \to \infty$ in (2.9), we find that the inequality (2.5) holds for all $f \in X'$ and $g \in Y'$. Replace f_1 and f_2 in (2.3) by 2^nf_1 and 2^nf_2 respectively and divide both sides by 2^n and take the limit as $n \to \infty$ and apply then (2.2) to get the mapping $f \longmapsto v(f,g)$ is linear. By a similar way one can replace f in (2.4) by 2^nf and divide both sides by 2^n to deduce that the mapping $g \longmapsto v(f,g)$ is linear. Consequently, the mapping v is bilinear. Our next claim is to prove that v is unique. Let v' be another mapping satisfying (2.5). Hence,

$$\begin{aligned}
|v(f,g) - v'(f,g)| &= \frac{1}{2^k}|v(2^kf,g) - v'(2^kf,g)| \\
&\leq \frac{2}{2^k}\tilde{\varphi}(2^kf,g) \\
&= 2\sum_{i=k}^{\infty}\frac{1}{2^{i+1}}\varphi_1(2^if,2^if,g)
\end{aligned}$$

for all $f \in X'$ and $g \in Y'$. Passing to the limit as $k \to \infty$, we conclude that v is unique. Replace f by $2^n f$ in (2.6) and divide both sides by 2^n, to arrive at

$$\left| \frac{1}{2^n}|u(2^n f, g)| - |f(x)g(y)| \right| \leq \frac{1}{2^n}\varphi(2^n f, g) \tag{2.10}$$

for all $f \in X'$ and $g \in Y'$. Taking the limit as $n \to \infty$ in (2.10) and applying the definition of the norm, we conclude that $||v|| = ||x \otimes y||$ and so v is bounded. □

Remark 1. Under the same hypotheses of Theorem 1, with (2.1) and (2.2) replaced by

$$\tilde{\varphi}(f, g) := \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi_2(f, 2^i g, 2^i g) < \infty, \tag{2.11}$$

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi_1(f_1, f_2, 2^n g) = \lim_{n\to\infty} \frac{1}{2^n}\varphi_2(f, 2^n g_1, 2^n g_2) = \lim_{n\to\infty} \frac{1}{2^n}\varphi(f, 2^n g) = 0, \tag{2.12}$$

there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ satisfying (2.5). Note that by using (2.4) and the same method as in the proof of Theorem 1, we can define $v(f, g) := \lim_{n\to\infty} \frac{1}{2^n} u(f, 2^n g)$.

In the following corollaries, as a consequence of Theorem 1, we show the Rassias stability of algebraic tensor products.

Corollary 1. *Let $x \in X$, $y \in Y$, and $u : X' \times Y' \to \mathbb{F}$ be a mapping such that*

$$||u(f, g)| - |f(x)g(y)|| \leq \alpha + \beta(||f||^p + ||g||^p) + \gamma ||f||^p ||g||^p, \tag{2.13}$$

$$|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \leq \alpha + \beta(||f_1||^q + ||f_2||^q + ||g||^q) + \gamma ||f_1||^{\frac{q}{2}} ||f_2||^{\frac{q}{2}} ||g||^q,$$

$$|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \leq \alpha + \beta(||f||^r + ||g_1||^r + ||g_2||^r) + \gamma ||f||^r ||g_1||^{\frac{r}{2}} ||g_2||^{\frac{r}{2}}$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$, and $c \in \mathbb{F}$, where p, q, r, α, β, and γ are constants with $0 \leq p, q, r < 1$, $\alpha > 0$, and $\beta, \gamma \geq 0$. Then, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f, g) - v(f, g)| \leq \alpha + \beta(2k||f||^q + ||g||^q) + \gamma k |f||^q ||g||^q \tag{2.14}$$

for all $f \in X'$ and $g \in Y'$, where $k = \frac{1}{2-2^q}$.

Remark 2. Under the hypotheses of Corollary 1 and using Remark 1, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f, g) - v(f, g)| \leq \alpha + \beta(||f||^r + 2k||g||^r) + \gamma k |f||^r ||g||^r$$

for all $f \in X'$ and $g \in Y'$, where $k = \frac{1}{2-2^r}$.

Theorem 2. *Let $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ be linearly independent sets in X and Y respectively and u be a mapping from $X'\times Y'$ to $\mathbb{F}$ for which there exist mappings $\varphi_1 : X'\times X'\times Y' \longrightarrow \mathbb{R}^+$, $\varphi_2 : X'\times Y'\times Y' \longrightarrow \mathbb{R}^+$, and $\varphi : X'\times Y' \longrightarrow \mathbb{R}^+$ satisfying (2.1), (2.2), (2.3), (2.4) and*

$$\left| |u(f,g)| - \sum_{i=1}^{m} |f(x_i)g(y_i)| \right| \leq \varphi(f,g) \tag{2.15}$$

for all $f \in X'$, $g \in Y'$. Then, there exists a unique mapping $v \in \mathcal{BL}(X'\times Y', \mathbb{F})$ such that

$$|u(f,g) - v(f,g)| \leq \tilde{\varphi}(f,g) \ \ (f \in X', g \in Y'), \ \ ||v|| \leq \sum_{i=1}^{m} ||x_i \otimes y_i||. \tag{2.16}$$

In the following our interest is to provide a dual for Theorem 1.

Theorem 3. *Let $x \in X$, $y \in Y$, and let $u : X'\times Y' \to \mathbb{F}$ be a mapping for which there exist mappings $\varphi_1 : X'\times X'\times Y' \longrightarrow \mathbb{R}^+$, $\varphi_2 : X'\times Y'\times Y' \longrightarrow \mathbb{R}^+$, and $\varphi : X'\times Y' \longrightarrow \mathbb{R}^+$ satisfying (2.3), (2.4), (2.6), and*

$$\tilde{\varphi}(f,g) := \sum_{i=0}^{\infty} 2^i \varphi_1(\frac{f}{2^{i+1}}, \frac{f}{2^{i+1}}, g) < \infty, \tag{2.17}$$

$$\lim_{n\to\infty} 2^n \varphi_1(\frac{f_1}{2^n}, \frac{f_2}{2^n}, g) = \lim_{n\to\infty} 2^n \varphi_2(\frac{f}{2^n}, g_1, g_2) = \lim_{n\to\infty} 2^n \varphi(\frac{f}{2^n}, g) = 0 \tag{2.18}$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$. Then, there exists a unique mapping $v \in \mathcal{BL}(X'\times Y', \mathbb{F})$ satisfying (2.5).

Proof. By induction on n, we conclude that

$$|u(f,g) - 2^n u(\frac{f}{2^n}, g)| \leq \sum_{i=0}^{n-1} 2^i \varphi_1(\frac{f}{2^{i+1}}, \frac{f}{2^{i+1}}, g) \tag{2.19}$$

for all $f \in X'$ and $g \in Y'$. Replace f by $\frac{f}{2^k}$ in (2.19) and multiply both sides by 2^k, where k is an arbitrary positive integer, to get

$$|2^k u(\frac{f}{2^k}, g) - 2^{n+k} u(\frac{f}{2^{n+k}}, g)| \leq \sum_{i=k}^{n+k-1} 2^i \varphi_1(\frac{f}{2^{i+1}}, \frac{f}{2^{i+1}}, g)$$

for all $f \in X'$, $g \in Y'$, and all positive integers $n \geq k$. In order to use the Cauchy convergence criterion, the last inequality and (2.17) imply the sequence $\{2^n u(\frac{f}{2^n}, g)\}$ is a Cauchy sequence for all $f \in X'$ and $g \in Y'$. Due to completeness of $\mathbb{F}$, this sequence converges. Define $v(f,g) := \lim_{n\to\infty} 2^n u(\frac{f}{2^n}, g)$. Taking the limit as $n \to \infty$ in (2.19), we deduce that the inequality (2.5) holds for all $f \in X'$ and $g \in Y'$. The rest of the proof is similar to that of Theorem 1. $\square$

Remark 3. Under the same hypotheses of Theorem 3, with (2.17) and (2.18) replaced by

$$\tilde{\varphi}(f,g) := \sum_{i=0}^{\infty} 2^i \varphi_2(f, \frac{g}{2^{i+1}}, \frac{g}{2^{i+1}}) < \infty, \tag{2.20}$$

$$\lim_{n\to\infty} 2^n \varphi_1(f_1, f_2, \frac{g}{2^n}) = \lim_{n\to\infty} 2^n \varphi_2(f, \frac{g_1}{2^n}, \frac{g_2}{2^n}) = \lim_{n\to\infty} 2^n \varphi(f, \frac{g}{2^n}) = 0, \tag{2.21}$$

there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ satisfying (2.5). We remark that by using (2.4) and the same method as in the proof of Theorem 3, one can define $v(f,g) := \lim_{n\to\infty} 2^n u(f, \frac{g}{2^n})$.

Corollary 2. *Let $x \in X$, $y \in Y$, and $u : X' \times Y' \to \mathbb{F}$ be a mapping such that*

$$||u(f,g)| - |f(x)g(y)|| \le \alpha ||f||^p ||g||^p, \tag{2.22}$$

$$|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \le \beta ||f_1||^{\frac{q}{2}} ||f_2||^{\frac{q}{2}} ||g||^q,$$

$$|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \le \gamma ||f||^r ||g_1||^{\frac{r}{2}} ||g_2||^{\frac{r}{2}}$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$, and $c \in \mathbb{F}$, where $p, q, r > 1$, and $\alpha, \beta, \gamma > 0$. Then, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f,g) - v(f,g)| \le \frac{\beta}{2^q - 2} ||f||^q ||g||^q \quad (f \in X', g \in Y').$$

Proof. It is enough to define $\varphi(f,g) := \alpha ||f||^p ||g||^p$, $\varphi_1(f_1, f_2, g) := \beta ||f_1||^{\frac{q}{2}} ||f_2||^{\frac{q}{2}} ||g||^q$, and $\varphi_2(f, g_1, g_2) := \gamma ||f||^r ||g_1||^{\frac{r}{2}} ||g_2||^{\frac{r}{2}}$ for all $f, f_1, f_2 \in X'$ and $g, g_1, g_2 \in Y'$ and then apply Theorem 3. □

Remark 4. Under the hypotheses of Corollary 2 and using Remark 3, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f,g) - v(f,g)| \le \frac{\gamma}{2^r - 2} ||f||^r ||g||^r \quad (f \in X', g \in Y').$$

Theorem 4. *Let $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ be linearly independent sets in X and Y respectively and u be a mapping from $X' \times Y'$ to $\mathbb{F}$ for which there exist mappings $\varphi_1 : X' \times X' \times Y' \longrightarrow \mathbb{R}^+$, $\varphi_2 : X' \times Y' \times Y' \longrightarrow \mathbb{R}^+$, and $\varphi : X' \times Y' \longrightarrow \mathbb{R}^+$ satisfying* (2.17), (2.18), (2.15), (2.3), (2.4). *Then, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ satisfying* (2.16).

Acknowledgement

The authors would like to thank the anonymous referee for his/her valuable comments concerning the statement of Theorem 1.

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