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TORSION UNITS OF INTEGRAL GROUP RING OF THE SIMPLE GROUP $S_4(4)$

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Abstract. We consider the Zassenhaus conjecture for the normalized unit group of the integral group ring of the sympletic simple group $S_4(4)$. As a consequence, we confirm for this group the prime graph conjecture.

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1. Introduction, conjectures and main results

In the integral group ring $\mathbb{Z}G$, where G is a finite group, we consider the group of normalized units

$$V(\mathbb{Z}G) = \left\{u = \sum_{g\in G} \alpha_g g \in U(\mathbb{Z}G) \mid \varepsilon(u) = \sum_{g\in G} \alpha_g = 1\right\}.$$

A conjecture of Zassenhaus **(ZC)** states that:

> **(ZC)** Every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate in the rational group algebra $\mathbb{Q}G$ to an element $g \in G$. That is, for every $u \in V(\mathbb{Z}G)$, there exists $v \in \mathbb{Q}G$ such that $vuv^{-1} = g \in G$.

The main tool in the investigation of the Zassenhaus conjecture **(ZC)** for finite groups with small numbers of normal subgroups is the Luthar-Passi method, which appeared in [24] to show **(ZC)** in the case where $G = A_5$. M. Hertweck (see [19,20]) applied the theory of Brauer characters to the Luthar-Passi method and used it to study the conjecture of Zassenhaus **(ZC)** for $G = PSL(2, p^n)$.

Now the Luthar-Passi method is very useful to study of **(ZC)** for arbitrary groups. As recent results, we cite [4, 18, 21–23, 25–27]. Now, we need to introduce some notation. For $P_d(H)$ denote the set of all prime divisors of the orders of elements of torsion part $t(H)$ of a group H. The prime graph (or Gruenberg-Kegel graph) of H, denoted by $\pi(H)$, is the graph with vertices as the primes in $P_d(H)$ and we connect of p to q if there is an element of order pq in the torsion part $t(H)$. In [23] was proposed the following conjecture,

(PGC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

Clearly, this conjecture is a weaker version of the conjecture of Zassenhaus **(ZC)**. In [23], W. Kimerle showed that **(PGC)** holds for finite Frobenius groups and solvable groups. V. Bovdi and M. Hertweck in [3] completed the p-version of the Zassenhaus conjecture for Frobenius groups. In [2,5–15] the conjecture **(PGC)** was studied for several simple sporadic groups.

Here, we are investigating the conjecture **(ZC)** for the symplectic simple group $S_4(4)$. Using the Luthar-Passi method, we can not prove the rational conjugacy **(ZC)** for all torsion units in $V(\mathbb{Z}S_4(4))$, but for units of order $\{3,5,17\}$ we are able to prove **(ZC)**. Also our main result gives a lot of information about the partial augmentations of some units. Finally, as a consequence we obtain that **(PGC)** is valid for this group.

Let $G = S_4(4)$. It is well known (see [17]) that $|G| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ and $\exp(G) = 2^2 \cdot 3 \cdot 5 \cdot 17$. Let

$$\begin{aligned}\mathcal{C} = \{&C_1, C_{2a}, C_{2b}, C_{2c}, C_{3a}, C_{3b}, C_{4a}, C_{4b}, C_{5a}, C_{5b}, C_{5c}, C_{5d}, C_{5e}, C_{6a}, C_{6b},\\ &C_{10a}, C_{10b}, C_{10c}, C_{10d}, C_{15a}, C_{15b}, C_{15c}, C_{15d}, C_{17a}, C_{17b}, C_{17c}, C_{17d}\}\end{aligned}$$

be the collection of all conjugacy classes of $S_4(4)$, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order k. Denote by

$$\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$$

the partial augmentation of u with respect to C_{nt}. From the Berman–Higman Theorem (see [1]) one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \tag{1.1}$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt}.

Our main result is the following

Theorem 1. *Let G denote the symplectic group $S_4(4)$. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. Denote by $\mathfrak{P}(u)$ the tuple*

$$\begin{aligned}(&\nu_{2a}, \nu_{2b}, \nu_{2c}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{5d}, \nu_{5e}, \nu_{6a}, \nu_{6b}, \nu_{10a},\\ &\nu_{10b}, \nu_{10c}, \nu_{10d}, \nu_{15a}, \nu_{15b}, \nu_{15c}, \nu_{15d}, \nu_{17a}, \nu_{17b}, \nu_{17c}, \nu_{17d}) \in \mathbb{Z}^{26}\end{aligned}$$

of partial augmentations of u in $V(\mathbb{Z}G)$. The following properties hold.

(i) *There is no elements of orders 34, 51, 85 in $V(\mathbb{Z}G)$.*

(ii) *If $|u| \in \{3,5,17\}$, then u is rationally conjugate to some $g \in G$.*

(iii) *If $|u| = 2$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{2a}, ν_{2b} and ν_{2c}, and the tuple $(\nu_{2a}, \nu_{2b}, \nu_{2c})$ is one of*

$$\{(1,-3,3), (2,-2,1), (1,-2,2), (0,-2,3), (3,-1,-1), (2,-1,0),$$

$$(1,-1,1),(0,-1,2),(-1,-1,3),(2,0,-1),(1,0,0),(0,0,1),$$
$$(-1,0,2),(-2,0,3),(1,1,-1),(0,1,0),(-1,1,1),(-2,1,2),$$
$$(-3,1,3),(0,2,-1),(-1,2,0),(-2,2,1),(-1,3,-1)\}.$$

As an immediate consequence of part (i) of the Theorem we obtain

Corollary 1. *If $G = S_4(4)$ then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

2. Preliminaries

The following result is a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

Proposition 1 (see [24]). *Let $u \in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

This fact shows that several partial augmentations are zero.

Proposition 2 (see [19], Proposition 3.1; [20], Proposition 2.2). *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar–Passi method.

Proposition 3 (see [20,24]). *Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G, then for every integer l the number*

$$\mu_l(u,\chi,p) = \frac{1}{k}\sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d)z^{-dl}\} \tag{2.1}$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u,\chi,*)$ for $\mu_l(u,\chi,0)$.
Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4 (see [16]). *The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

3. Proof of the Theorem

Throughout this section we denote $S_4(4)$ by G. The character table of G, as well as the p-Brauer character tables, which will be denoted by $\mathfrak{BCT}(p)$ where $p \in \{2,3,5,17\}$, can be found using the computational algebra system GAP [17]. For the

characters and conjugacy classes we will use throughout the paper the same notation, indexation inclusive, as used in the GAP Character Table Library.

First of all we start to investigate units of orders 2, 3, 5 and 17, since the group G possesses elements of these orders. After this, by Proposition 4, the order of each torsion unit divides the exponent of G, so to prove the Kimmerle's conjecture, it remains to consider units of orders 34, 51 and 85. We prove that no units of all these orders do appear in $V(\mathbb{Z}G)$.

Now we consider each case of possible orders separately.

Let u be an involution. By (1.1) and Proposition 2 we get

$$\nu_{2a} + \nu_{2b} + \nu_{2c} = 1.$$

Put $t_1 = 3\nu_{2a} + 3\nu_{2b} - \nu_{2c}$ and $t_2 = 5\nu_{2a} - 3\nu_{2b} + \nu_{2c}$. By (2.1) we obtain the system of inequalities

$$\mu_1(u,\chi_2,*) = \tfrac{1}{2}(2t_1+18) \geq 0; \qquad \mu_0(u,\chi_2,*) = \tfrac{1}{2}(-2t_1+18) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{2}(2t_2+34) \geq 0; \qquad \mu_1(u,\chi_3,*) = \tfrac{1}{2}(-2t_2+34) \geq 0,$$

from which $-9 \leq t_1 \leq 9$ and $-17 \leq t_2 \leq 17$. Furthermore, from the system of linear equations

$$\nu_{2a} + \nu_{2b} + \nu_{2c} = 1, \quad 3\nu_{2a} + 3\nu_{2b} - \nu_{2c} = t_1, \quad 5\nu_{2a} - 3\nu_{2b} + \nu_{2c} = t_2,$$

we select only integer solutions, and using the condition that all $\mu_i(u,\chi_j,*)$ are non-negative integers, we obtain twenty three tuples $(\nu_{2a},\nu_{2b},\nu_{2c})$.

Let u be a unit of order 3. By (1.1) and Proposition 2 we get

$$\nu_{3a} + \nu_{3b} = 1.$$

Put $t_1 = \nu_{3a} + 4\nu_{3b}$. Using $\mathfrak{BCT}(2)$ and by (2.1) we obtain the system of inequalities

$$\mu_0(u,\chi_7,2) = \tfrac{1}{3}(2t_1+16) \geq 0, \qquad \mu_1(u,\chi_7,2) = \tfrac{1}{3}(-t_1+16) \geq 0,$$

from which $t_1 \in \{1+3k \mid -3 \leq k \leq 5\}$. Using inequalities

$$\mu_0(u,\chi_2,2) = \tfrac{1}{3}(-4\nu_{3a}+2\nu_{3b}+4) \geq 0, \qquad \mu_0(u,\chi_4,2) = \tfrac{1}{3}(2\nu_{3a}-4\nu_{3b}+4) \geq 0,$$

we obtain only two integral solutions $(\nu_{3a},\nu_{3b}) \in \{(0,1),(1,0)\}$.

Let u be a unit of order 5. By (1.1) and Proposition 2 we get

$$\nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d} + \nu_{5e} = 1.$$

Put

$$\alpha = 3\nu_{5a} + 3\nu_{5b} + 3\nu_{5c} + 3\nu_{5d} - 2\nu_{5e} \text{ and } \beta = 4\nu_{5a} + 4\nu_{5b} - \nu_{5c} - \nu_{5d} - \nu_{5e}.$$

Using $\mathfrak{BCT}(2)$ and by (2.1) we obtain the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{5}(4\alpha+18) \geq 0; \qquad \mu_1(u,\chi_2,*) = \tfrac{1}{5}(-\alpha+18) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{5}(4\beta+34) \geq 0; \qquad \mu_0(u,\chi_6,2) = \tfrac{1}{5}(-4\beta+16) \geq 0,$$

so $\alpha \in \{-2,3,8,13,18\}$ and $\beta \in \{-6,-1,4\}$ and we obtain only five integral solutions

$$(v_{5a}, v_{5b}) \in \{(1,0,0,0,0),(0,0,0,0,1),(0,0,0,1,0),(0,0,1,0,0),(0,1,0,0,0)\}.$$

Let u be a unit of order 17. By (1.1) and Proposition 2 we get

$$v_{17a} + v_{17b} + v_{17c} + v_{17d} = 1.$$

Put

$$t_1 = 13v_{17a} - 4v_{17b} - 4v_{17c} - 4v_{17d}, \quad t_2 = 4v_{17a} - 13v_{17b} + 4v_{17c} + 4v_{17d},$$
$$\text{and} \quad t_3 = 4v_{17a} + 4v_{17b} - 13v_{17c} + 4v_{17d}.$$

By (2.1) we have

$$\mu_1(u,\chi_2,2) = \tfrac{1}{17}(t_1+4) \geq 0; \qquad \mu_3(u,\chi_{12},2) = \tfrac{1}{17}(-t_1+64) \geq 0;$$
$$\mu_6(u,\chi_{12},2) = \tfrac{1}{17}(t_2+64) \geq 0; \qquad \mu_2(u,\chi_2,2) = \tfrac{1}{17}(-t_2+4) \geq 0;$$
$$\mu_2(u,\chi_{12},2) = \tfrac{1}{17}(t_3+64) \geq 0; \qquad \mu_3(u,\chi_2,2) = \tfrac{1}{17}(-t_3+4) \geq 0.$$

This yields

$$t_1 \in \{-4,13,30,47,64\}, \quad t_2 \in \{-64,-47,-30,-13,4\},$$
$$\text{and} \quad t_3 \in \{-64,-47,-30,-13,4\}.$$

Using inequality

$$\mu_6(u,\chi_2,2) = \tfrac{1}{17}(-4v_{17a} - 4v_{17b} - 4v_{17c} + 13v_{17d} + 4) \geq 0$$

we have the four trivial solutions.

Let u be a unit of order 34. By (1.1) and Proposition 2 we have

$$v_{2a} + v_{2b} + v_{2c} + v_{17a} + v_{17b} + v_{17c} + v_{17d} = 1.$$

Put

$$t_1 = 6v_{2a} + 6v_{2b} - 2v_{2c} - v_{17a} - v_{17b} - v_{17c} - v_{17d},$$
$$t_2 = 5v_{2a} - 3v_{2b} + v_{2c}, \quad t_3 = 3v_{2a} - 5v_{2b} - v_{2c},$$
$$t_4 = 15v_{2a} + 15v_{2b} - v_{2c} - 4v_{17a} + 13v_{17b} - 4v_{17c} - 4v_{17d}$$
$$t_5 = 15v_{2a} + 15v_{2b} - v_{2c} + 13v_{17a} - 4v_{17b} - 4v_{17c} - 4v_{17d}$$
$$t_6 = 15v_{2a} + 15v_{2b} - v_{2c} - 4v_{17a} - 4v_{17b} - 4v_{17c} + 13v_{17d}$$

Since $|u^{17}| = 2$, for any character χ of G we need to consider twenty three cases, defined by part (ii) of the Theorem:

(1) Let

$$\chi(u^{17}) \in \{\chi(2a), \chi(2a) - 2\chi(2b) + 2\chi(2c), 2\chi(2a) - \chi(2b), -\chi(2b) + 2\chi(2c),$$
$$\chi(2b), -\chi(2a) + 2\chi(2c), -2\chi(2a) + \chi(2b) + 2\chi(2c), -\chi(2a) + 2\chi(2b)\}.$$

Then by (2.1) we obtain two incompatible inequalities

$$\mu_{17}(u,\chi_2,*) = \tfrac{1}{34}(16t_1+\alpha) \geq 0, \qquad \mu_0(u,\chi_2,*) = \tfrac{1}{34}(-16t_1+\beta) \geq 0,$$

where $(\alpha,\beta)\in\{(40,28),(24,44)\}$.

(2) Let

$$\chi(u^{17})\in\{\chi(2c),2\chi(2a)-2\chi(2b)+\chi(2c),\chi(2a)-\chi(2b)+\chi(2c),\\ -\chi(2a)+\chi(2b)+\chi(2c),-2\chi(2a)+2\chi(2b)+\chi(2c)\}.$$

Using (2.1), we obtain the incompatible system of inequalities

$$\mu_{17}(u,\chi_2,*)=\tfrac{1}{34}(16t_1+32)\geq 0;\\ \mu_0(u,\chi_2,*)=\tfrac{1}{34}(-16t_1+36)\geq 0;\\ \mu_1(u,\chi_2,*)=\tfrac{1}{34}(-t_1+15)\geq 0.$$

(3) Let

$$\chi(u^{17})\in\{-2\chi(2b)+3\chi(2c),-2\chi(2a)+3\chi(2c),\\ 2\chi(2a)-\chi(2c),2\chi(2b)-\chi(2c)\}.$$

Using (2.1), we obtain the system of inequalities

$$\mu_0(u,\chi_3,*)=\tfrac{1}{34}(32t_2+\alpha)\geq 0;\qquad \mu_{17}(u,\chi_3,*)=\tfrac{1}{34}(-32t_2+\beta)\geq 0,$$

where $(\alpha,\beta)\in\{(52,16),(20,48)\}$, which has no integral solution.

(4) Let $\chi(u^{17})=\chi(2a)-3\chi(2b)+3\chi(2c)$. Using (2.1), we calculate the following system of inequalities

$$\mu_{17}(u,\chi_2,*)=\tfrac{1}{34}(16t_1+16)\geq 0;\qquad \mu_1(u,\chi_2,*)=\tfrac{1}{34}(-t_1-1)\geq 0;\\ \mu_1(u,\chi_3,*)=\tfrac{1}{34}(2t_2)\geq 0;\qquad \mu_{17}(u,\chi_3,*)=\tfrac{1}{34}(-32t_2)\geq 0;\\ \mu_2(u,\chi_4,*)=\tfrac{1}{34}(2t_3+4)\geq 0;\qquad \mu_0(u,\chi_4,*)=\tfrac{1}{34}(-32t_3+4)\geq 0;\\ \mu_4(u,\chi_{17},*)=\tfrac{1}{34}(t_4+271)\geq 0;\qquad \mu_1(u,\chi_{17},*)=\tfrac{1}{34}(-t_4+205)\geq 0;\\ \mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+254)\geq 0;\qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+188)\geq 0;\\ \mu_{12}(u,\chi_{17},*)=\tfrac{1}{34}(t_6+254)\geq 0;\qquad \mu_3(u,\chi_{17},*)=\tfrac{1}{34}(-t_6+188)\geq 0.$$

It follows that $t_1=-1$, $t_2=0$, $t_3=-2$,

$$t_4\in\{1+34k\mid -8\leq k\leq 6\},\ t_5,t_6\in\{18+34k\mid -8\leq k\leq 5\},$$

and we have no solutions.

(5) Let $\chi(u^{17})=3\chi(2a)-\chi(2b)-\chi(2c)$. Using (2.1), we obtain the system of inequalities

$$\mu_2(u,\chi_2,*)=\tfrac{1}{34}(t_1+3)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{34}(-16t_1+20)\geq 0;\\ \mu_1(u,\chi_3,*)=\tfrac{1}{34}(2t_2)\geq 0;\qquad \mu_{17}(u,\chi_3,*)=\tfrac{1}{34}(-32t_2)\geq 0;\\ \mu_2(u,\chi_4,*)=\tfrac{1}{34}(2t_3+4)\geq 0;\qquad \mu_0(u,\chi_4,*)=\tfrac{1}{34}(-32t_3+4)\geq 0;\\ \mu_4(u,\chi_{17},*)=\tfrac{1}{34}(t_4+190)\geq 0;\qquad \mu_1(u,\chi_{17},*)=\tfrac{1}{34}(-t_4+252)\geq 0;\\ \mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+190)\geq 0;\qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+252)\geq 0;$$

$\mu_{12}(u,\chi_{17},*) = \frac{1}{34}(t_6+207) \geq 0; \quad \mu_3(u,\chi_{17},*) = \frac{1}{34}(-t_6+269) \geq 0.$

It follows that $t_1 = -3$, $t_2 = 0$, $t_3 = -2$,

$$t_4, t_5 \in \{14+34k \mid -6 \leq k \leq 7\},\ t_6 \in \{31+34k \mid -7 \leq k \leq 7\},$$

and we have no solutions again.

(6) Let $\chi(u^{17}) = -\chi(2a) - \chi(2b) + 3\chi(2c)$. Again, using (2.1), we obtain the system of inequalities

$\mu_{17}(u,\chi_2,*) = \frac{1}{34}(16t_1+16) \geq 0; \quad \mu_1(u,\chi_2,*) = \frac{1}{34}(-t_1-1) \geq 0;$

$\mu_0(u,\chi_3,*) = \frac{1}{34}(32t_2+36) \geq 0; \quad \mu_{17}(u,\chi_3,*) = \frac{1}{34}(-32t_2+32) \geq 0;$

$\mu_{17}(u,\chi_4,*) = \frac{1}{34}(32t_3+32) \geq 0; \quad \mu_0(u,\chi_4,*) = \frac{1}{34}(-32t_3+36) \geq 0;$

$\mu_4(u,\chi_{17},*) = \frac{1}{34}(t_4+254) \geq 0; \quad \mu_1(u,\chi_{17},*) = \frac{1}{34}(-t_4+188) \geq 0;$

$\mu_2(u,\chi_{17},*) = \frac{1}{34}(t_5+254) \geq 0; \quad \mu_9(u,\chi_{17},*) = \frac{1}{34}(-t_5+188) \geq 0;$

$\mu_{12}(u,\chi_{17},*) = \frac{1}{34}(t_6+271) \geq 0; \quad \mu_3(u,\chi_{17},*) = \frac{1}{34}(-t_6+205) \geq 0.$

It follows that $t_1 = -1$, $t_2 = 1$, $t_3 = -1$,

$$t_4, t_5 \in \{18+34k \mid -8 \leq k \leq 5\},\ t_6 \in \{1+34k \mid -8 \leq k \leq 6\},$$

and we have no solutions again.

(7) Let $\chi(u^{17}) = \chi(2a) + \chi(2b) - \chi(2c)$. Then, by (2.1), we obtain the system of inequalities

$\mu_2(u,\chi_2,*) = \frac{1}{34}(t_1+3) \geq 0; \quad \mu_0(u,\chi_2,*) = \frac{1}{34}(-16t_1+20) \geq 0;$

$\mu_0(u,\chi_3,*) = \frac{1}{34}(32t_2+36) \geq 0; \quad \mu_{17}(u,\chi_3,*) = \frac{1}{34}(-32t_2+32) \geq 0;$

$\mu_{17}(u,\chi_4,*) = \frac{1}{34}(32t_3+32) \geq 0; \quad \mu_0(u,\chi_4,*) = \frac{1}{34}(-32t_3+36) \geq 0;$

$\mu_4(u,\chi_{17},*) = \frac{1}{34}(t_4+190) \geq 0; \quad \mu_1(u,\chi_{17},*) = \frac{1}{34}(-t_4+252) \geq 0;$

$\mu_2(u,\chi_{17},*) = \frac{1}{34}(t_5+190) \geq 0; \quad \mu_9(u,\chi_{17},*) = \frac{1}{34}(-t_5+252) \geq 0;$

$\mu_{12}(u,\chi_{17},*) = \frac{1}{34}(t_6+207) \geq 0; \quad \mu_3(u,\chi_{17},*) = \frac{1}{34}(-t_6+269) \geq 0.$

It follows that $t_1 = -3$, $t_2 = 1$, $t_3 = -1$,

$$t_4, t_5 \in \{14+34k \mid -6 \leq k \leq 7\},\ t_6 \in \{31+34k \mid -7 \leq k \leq 7\},$$

and we have no solutions.

(8) Let $\chi(u^{17}) = -3\chi(2a) + \chi(2b) + 3\chi(2c)$. Then, by (2.1), we obtain the system of inequalities

$\mu_{17}(u,\chi_2,*) = \frac{1}{34}(16t_1+16) \geq 0; \quad \mu_1(u,\chi_2,*) = \frac{1}{34}(-t_1-1) \geq 0;$

$\mu_0(u,\chi_3,*) = \frac{1}{34}(32t_2+4) \geq 0; \quad \mu_2(u,\chi_3,*) = \frac{1}{34}(-2t_2+4) \geq 0;$

$\mu_{17}(u,\chi_4,*) = \frac{1}{34}(32t_3) \geq 0; \quad \mu_1(u,\chi_4,*) = \frac{1}{34}(-2t_3) \geq 0;$

$\mu_4(u,\chi_{17},*) = \frac{1}{34}(t_4+254) \geq 0; \quad \mu_1(u,\chi_{17},*) = \frac{1}{34}(-t_4+188) \geq 0;$

$$\mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+271)\geq 0; \qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+205)\geq 0;$$
$$\mu_{12}(u,\chi_{17},*)=\tfrac{1}{34}(t_6+254)\geq 0; \qquad \mu_3(u,\chi_{17},*)=\tfrac{1}{34}(-t_6+188)\geq 0.$$

It follows that $t_1=-1$, $t_2=2$, $t_3=0$,

$$t_4,t_6\in\{18+34k\mid -8\leq k\leq 5\},\ t_5\in\{1+34k\mid -8\leq k\leq 6\},$$

and we have no solutions.

(9) Let $\chi(u^{17})=-\chi(2a)+3\chi(2b)-\chi(2c)$. Then, by (2.1), we obtain the system of inequalities

$$\mu_2(u,\chi_2,*)=\tfrac{1}{34}(t_1+3)\geq 0; \qquad \mu_0(u,\chi_2,*)=\tfrac{1}{34}(-16t_1+20)\geq 0;$$
$$\mu_0(u,\chi_3,*)=\tfrac{1}{34}(32t_2+4)\geq 0; \qquad \mu_2(u,\chi_3,*)=\tfrac{1}{34}(-2t_2+4)\geq 0;$$
$$\mu_{17}(u,\chi_4,*)=\tfrac{1}{34}(32t_3)\geq 0; \qquad \mu_1(u,\chi_4,*)=\tfrac{1}{34}(-2t_3)\geq 0;$$
$$\mu_4(u,\chi_{17},*)=\tfrac{1}{34}(t_4+190)\geq 0; \qquad \mu_1(u,\chi_{17},*)=\tfrac{1}{34}(-t_4+252)\geq 0;$$
$$\mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+207)\geq 0; \qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+269)\geq 0;$$
$$\mu_{12}(u,\chi_{17},*)=\tfrac{1}{34}(t_6+190)\geq 0; \qquad \mu_3(u,\chi_{17},*)=\tfrac{1}{34}(-t_6+252)\geq 0.$$

It follows that $t_1=-3$, $t_2=2$, $t_3=0$,

$$t_4,t_6\in\{14+34k\mid -6\leq k\leq 7\},\ t_5\in\{31+34k\mid -7\leq k\leq 7\},$$

and we have no solutions.

Let u be a unit of order 51**.** By (1.1) and Proposition 2 we have

$$\nu_{3a}+\nu_{3b}+\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}=1.$$

Put

$$t_1=\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}.$$

Then using (2.1) we obtain the non-compatible system of inequalities

$$\mu_0(u,\chi_2,*)=\tfrac{1}{51}(32t_1+34)\geq 0, \qquad \mu_{17}(u,\chi_2,*)=\tfrac{1}{51}(-16t_1+34)\geq 0.$$

Let u be a unit of order 85**.** By (1.1) and Proposition 2 we have

$$\nu_{5a}+\nu_{5b}+\nu_{5c}+\nu_{5d}+\nu_{5e}+\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}=1.$$

Put

$$t_1=3\nu_{5a}+3\nu_{5b}+3\nu_{5c}+3\nu_{5d}-2\nu_{5e}+\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}.$$

Since $|u^{17}|=5$, for any character χ of G we need to consider five cases, defined by part (iii) of the Theorem:

(1) Let $\chi(u^{17})\in\{\chi(5a),\chi(5b),\chi(5c),\chi(5d)\}$. Using (2.1), we obtain the system of inequalities

$$\mu_0(u,\chi_2,*)=\tfrac{1}{85}(64t_1+46)\geq 0, \qquad \mu_{17}(u,\chi_2,*)=\tfrac{1}{85}(-16t_1+31)\geq 0,$$

which has no integral solution.

(2) Let $\chi(u^{17}) = \chi(5e)$. Again, using (2.1), we obtain the system of inequalities

$$\mu_0(u, \chi_2, *) = \tfrac{1}{85}(64t_1 + 26) \geq 0, \qquad \mu_5(u, \chi_2, *) = \tfrac{1}{85}(-4t_1 + 9) \geq 0,$$

which has no integral solution.

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