Miskolc Mathematical Notes
Vol. 16 (2015), No 1, pp. 65-78

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1325

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HU e-ISSN 1787-2413

A NOTE ON ANNIHILATORS IN DISTRIBUTIVE NEARLATTICES

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Received 10 September, 2014

Abstract. In this note we propose a definition of relative annihilator in distributive nearlattices with greatest element different from that given in [6] and we present some new characterizations of the distributivity. Later, we study the class of normal and p-linear nearlattices, the lattice of filters and semi-homomorphisms that preserve annihilators.

2010 *Mathematics Subject Classification:* 06A12; 03G10; 06D50

Keywords: distributive nearlattice, relative annihilator, prime ideal, semi-homomorphism

1. Introduction and preliminaries

Abbott in [1] established a correspondence between the class of Tarski algebras, or implication algebras, and join-semilattices in which every principal filter is a Boolean lattice with respect to the induced order. There is an algebraic structure that generalizes the class of Tarski algebras: *nearlattices*. A nearlattice is a join-semilattice in which every principal filter is a lattice. The class of nearlattices forms a variety that has been studied in [9] and [11] by Cornish and Hickman, and in [4], [6] and [7] by Chajda, Kolařík, Halaš and Kühr. In [2] the authors showed that the axiom systems given in [11] and [4] are dependent and that the variety of nearlattices is 2-based. An important class of nearlattices is the class of *distributive nearlattices*. Recently in [3], a full duality between distributive nearlattices with greatest element and certain topological spaces with a distinguished basis was developed.

It is well known that the notion of distributivity in a lattice can be characterized in different ways, for example, a lattice A is distributive if and only if the lattice $\mathrm{Fi}(A)$ of all filters of A is distributive. Another way is through some special subsets, called *annihilators*. In a lattice A, the *annihilator of a relative to b* is defined as the set $\langle a,b\rangle = \{x \in A : x \wedge a \leq b\}$. In [12], Mandelker studied the properties of relative annihilators and characterized the distributivity of a lattice in terms of its relative annihilators. To be more precise, a lattice A is distributive if and only if $\langle a,b\rangle$ is an ideal of A for all $a,b \in A$. These results were generalized by Varlet to the

Both authors were partially supported by PIP 112-201101-00636 CONICET.

class of distributive semilattices ([14]) and subsequently to the variety of distributive nearlattices by Chajda and Kolařík ([6]).

The main aim of this paper is to propose a definition of relative annihilator in distributive nearlattices with greatest element different from that given in [6]. In Section 2 we introduce the relative annihilators and we develop new characterizations. In Section 3 we present the class of normal and p-linear nearlattices. In Section 4 we study the lattice of filters of a distributive nearlattice. Finally, in Section 5 we characterize the semi-homomorphisms that preserve annihilators.

Given a poset $\langle X, \leq \rangle$, a set $Y \subseteq X$ is called *increasing* if it is closed under $\leq$, i.e., if for every $x \in Y$ and every $y \in X$, if $x \leq y$ then $y \in Y$. Dually, $Y \subseteq X$ is said to be *decreasing* if for every $x \in Y$ and every $y \in X$, if $y \leq x$ then $y \in Y$. The complement of a subset $Y \subseteq X$ will be denoted by $X - Y$. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \exists y \in Y \ (y \leq x)\}$ ($(Y] = \{x \in X : \exists y \in Y \ (x \leq y)\}$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively.

A *join-semilattice with greatest element* is an algebra $\langle A, \vee, 1 \rangle$ of type $(2,0)$ such that the operation $\vee$ is idempotent, commutative, associative and $x \vee 1 = 1$ for all $x \in A$. The binary relation $\leq$ defined by $x \leq y$ if and only if $x \vee y = y$ is a partial order. In what follows, we shall write simply *semilattice*.

A *filter* of a semilattice A is a subset $F \subseteq A$ such that $1 \in F$, F is increasing and if $x, y \in F$ then $x \wedge y \in F$, whenever $x \wedge y$ exists. The set of all filters of A is denoted by $\mathrm{Fi}(A)$. Let X be a non-empty subset of A. The least filter containing X is called the *filter generated by* X and will be denoted by $F(X)$. Note that if $X = \{a\}$ then $F(\{a\}) = [a)$, called the principal filter of a.

A subset I of A is called an *ideal* if I is decreasing and if $x, y \in I$ then $x \vee y \in I$. The least ideal containing X is called the *ideal generated by* X and will be denoted by $I(X)$. We shall say that a non-empty proper ideal P is *prime* if for all $x, y \in A$, if $x \wedge y \in P$, whenever $x \wedge y$ exists, then $x \in P$ or $y \in P$. We will denote by $\mathrm{Id}(A)$ and $X(A)$ the set of all ideals and prime ideals of A, respectively. Finally, we will say that a non-empty ideal I of A is *maximal* if it is proper and for every $J \in \mathrm{Id}(A)$, if $I \subseteq J$ then $J = I$ or $J = A$. We denote by $\mathrm{Idm}(A)$ the set of all maximal ideals of A. It is easy to prove that every maximal ideal is prime.

Definition 1. A *nearlattice* is a semilattice A such that for each $a \in A$ the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded lattice.

The class of nearlattices forms a variety since every nearlattice A can be described as an algebra with one ternary operation: if $x, y, a \in A$, the element $m(x, y, a) = (x \vee a) \wedge_a (y \vee a)$ is correctly defined because $x \vee a, y \vee a \in [a)$ and $[a)$ is a lattice, where $\wedge_a$ denotes the meet in $[a)$.

Proposition 1 ([2])**.** *Let A be a nearlattice. The following identities are satisfied:*

(1) $m(x, y, x) = x$,

(2) $m(m(x,y,z),m(y,m(u,x,z),z),w) = m(w,w,m(y,m(x,u,z),z))$,
(3) $m(x,x,1) = 1$.

Conversely, let $\langle A,m,1\rangle$ be an algebra of type $(3,0)$ satisfying the identities (1)–(3). If we define $x \vee y = m(x,x,y)$, then A is a semilattice and for each $a \in A$, $[a)$ is a bounded lattice, where the meet of $x,y \in [a)$ is $x \wedge_a y = m(x,y,a)$. Hence A is a nearlattice.

Definition 2. Let A be a nearlattice. We call A *distributive* if the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded distributive lattice for each $a \in A$.

Theorem 1 ([7])**.** *Let A be a nearlattice. Then A is distributive if and only if satisfies either of the following identities:*

(1) $m(x,m(y,y,z),w) = m(m(x,y,w),m(x,y,w),m(x,z,w))$,
(2) $m(x,x,m(y,z,w)) = m(m(x,x,y),m(x,x,z),w)$.

Theorem 2 ([10])**.** *Let A be a distributive nearlattice. Let $I \in \mathrm{Id}(A)$ and let $F \in \mathrm{Fi}(A)$ such that $I \cap F = \varnothing$. Then there exists $P \in X(A)$ such that $I \subseteq P$ and $P \cap F = \varnothing$.*

For distributive nearlattices we have the following lemma which characterizes the generated filters and can be deduced from the results given in [9].

Lemma 1. *Let A be a distributive nearlattice. Let $X \subseteq A$ be a non-empty subset. Then*

$$F(X) = \{a \in A \ : \ \exists x_1,...,x_n \in [X) \ \exists x_1 \wedge ... \wedge x_n \ (x_1 \wedge ... \wedge x_n = a)\}.$$

A filter H is said to be *finitely generated* if $H = F(X)$ for some finite non-empty subset X of A. We will denote by $\mathrm{Fi}_f(A)$ the set of all finitely generated filters of A.

Recall that if A is a distributive nearlattice, then $\langle \mathrm{Fi}(A), \veebar, \barwedge, \{1\}, A\rangle$ is a bounded distributive lattice, where the least element is $\{1\}$, the greatest element is A, and for all $G,H \in \mathrm{Fi}(A)$ we have that $G \veebar H = F(G \cup H)$ and $G \barwedge H = G \cap H$.

Theorem 3 ([4,9])**.** *Let A be a nearlattice. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *$\langle \mathrm{Fi}(A), \veebar, \barwedge, \{1\}, A\rangle$ is a bounded distributive lattice.*
(3) *$\left\langle \mathrm{Fi}_f(A), \veebar, \barwedge, \{1\}, A\right\rangle$ is a bounded distributive lattice.*

A function $h : A \to B$ between distributive nearlattices is a *semi-homomorphism* if $h(1) = 1$ and $h(a \vee b) = h(a) \vee h(b)$ for all $a,b \in A$. A *homomorphism* is a semi-homomorphism h such that for all $a,b \in A$, if $a \wedge b$ exists, then $h(a \wedge b) = h(a) \wedge h(b)$. In [3] it was shown that there exists a duality between semi-homomorphisms of distributive nearlattices and certain binary relations.

2. Relative annihilators

In this section we will develop new characterizations of the distributivity of a nearlattice through relative annihilators and relative maximal ideals.

Definition 3. Let A be a semilattice. For $a,b \in A$, the *annihilator of a relative to b* is the set

$$a \circ b = \{x \in A : b \leq x \vee a\}.$$

Let A be a semilattice. Let $a,b \in A, I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$. We introduce the following subsets of A:

$$\begin{aligned} I \circ b &= \{x \in A : \exists i \in I \ (b \leq x \vee i)\}, \\ a \circ F &= \{x \in A : \exists f \in F \ (f \leq x \vee a)\}. \end{aligned}$$

Let $X, Y \subseteq A$. We denote by $X \circ Y$ the set

$$X \circ Y = \bigcup \{a \circ b : (a,b) \in X \times Y\}.$$

Remark 1. Note that $a \circ b = (a] \circ b = a \circ [b) = (a] \circ [b)$ for all $a,b \in A$.

The following theorem characterizes distributive nearlattices.

Theorem 4. *Let A be a nearlattice. The following conditions are equivalent:*

1. *A is distributive.*
2. *$a \circ b \in \mathrm{Fi}(A)$ for all $a,b \in A$.*
3. *$I \circ b \in \mathrm{Fi}(A)$ for all $I \in \mathrm{Id}(A)$ and $b \in A$.*
4. *$a \circ F \in \mathrm{Fi}(A)$ for all $F \in \mathrm{Fi}(A)$ and $a \in A$.*
5. *$I \circ F \in \mathrm{Fi}(A)$ for all $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$.*

Proof. (1) $\Rightarrow$ (2) It is obvious that $1 \in a \circ b$. Let $x, y \in A$ such that $x \leq y$ and $x \in a \circ b$. Then, $x \vee a \leq y \vee a$ and $b \leq x \vee a$. So, $b \leq y \vee a$ and $y \in a \circ b$. Let $x, y \in a \circ b$ such that $x \wedge y$ exists. Then, $b \leq x \vee a$ and $b \leq y \vee a$, i.e., $x \vee a$, $y \vee a \in [b)$. Since $[b)$ is a bounded distributive lattice, $b \leq (x \vee a) \wedge_b (y \vee a) = (x \wedge y) \vee a$. Then, $x \wedge y \in a \circ b$ and $a \circ b \in \mathrm{Fi}(A)$.

(2) $\Rightarrow$ (3) Let $b \in A$ and $I \in \mathrm{Id}(A)$. We note that $i \circ b \subseteq I \circ b$ for all $i \in I$. It is easy to prove that $1 \in I \circ b$ and that $I \circ b$ is increasing. Let $x, y \in I \circ b$ such that $x \wedge y$ exists. Then there exist $i_1, i_2 \in I$ such that $b \leq x \vee i_1$ and $b \leq y \vee i_2$. Let $i = i_1 \vee i_2 \in I$. So, $b \leq x \vee i$ and $b \leq y \vee i$. Since $x, y \in i \circ b$ and $i \circ b \in \mathrm{Fi}(A)$, we have that $x \wedge y \in i \circ b \subseteq I \circ b$. Therefore, $I \circ b \in \mathrm{Fi}(A)$.

(3) $\Rightarrow$ (4) Let $a \in A$ and $F \in \mathrm{Fi}(A)$. It follows easily that $1 \in a \circ F$ and that $a \circ F$ is increasing. Let $x, y \in a \circ F$ such that $x \wedge y$ exists. Then there exist $f_1, f_2 \in F$ such that $f_1 \leq x \vee a$ and $f_2 \leq y \vee a$. So, $x \vee a, y \vee a \in F$. Since $x \vee a, y \vee a \in [a)$, $(x \vee a) \wedge_a (y \vee a)$ exists and $(x \vee a) \wedge_a (y \vee a) \in F$. As $(x \vee a) \wedge_a (y \vee a) \leq x \vee a$ and $(x \vee a) \wedge_a (y \vee a) \leq y \vee a$, we have $x, y \in a \circ ((x \vee a) \wedge_a (y \vee a))$. By Remark 1, $a \circ ((x \vee a) \wedge_a (y \vee a)) = (a] \circ ((x \vee a) \wedge_a (y \vee a))$ and by hypothesis $(a] \circ ((x \vee a) \wedge_a$

$(y \vee a)) \in \mathrm{Fi}(A)$. Then $x \wedge y \in a \circ ((x \vee a) \wedge_a (y \vee a))$, but as $(x \vee a) \wedge_a (y \vee a) \in F$ we have that $x \wedge y \in a \circ F$. So, $a \circ F \in \mathrm{Fi}(A)$.

(4) $\Rightarrow$ (5) Let $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$. It is easy to see that $1 \in I \circ F$ and that $I \circ F$ is increasing. Let $x, y \in I \circ F$ such that $x \wedge y$ exists. Then there exist $(i_1, f_1), (i_2, f_2) \in I \times F$ such that $x \in i_1 \circ f_1$ and $y \in i_2 \circ f_2$, i.e., $f_1 \leq x \vee i_1$ and $f_2 \leq y \vee i_2$. Let $i = i_1 \vee i_2 \in I$. On the other hand, $x \vee f_1, y \vee f_2 \in F$ and $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2)$ exists in $[x \wedge y)$. It follows that $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2) \in F$. We consider $i \circ F$. We note that $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2) \leq x \vee i$ and $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2) \leq y \vee i$. So, $x, y \in i \circ F$. By hypothesis $i \circ F \in \mathrm{Fi}(A)$ and $x \wedge y \in i \circ F$, i.e., there exists $f \in F$ such that $f \leq (x \wedge y) \vee i$. Then $x \wedge y \in i \circ f$ and $x \wedge y \in I \circ F$. Thus, $I \circ F \in \mathrm{Fi}(A)$.

(5) $\Rightarrow$ (1) Let $a \in A$ and $x, y, z \in [a)$. We know that the inequality $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ always holds. We prove the other inequality. As $(x \vee y) \wedge (x \vee z) \leq y \vee x$ and $(x \vee y) \wedge (x \vee z) \leq z \vee x$ then $y, z \in x \circ ((x \vee y) \wedge (x \vee z))$. By Remark 1, $x \circ ((x \vee y) \wedge (x \vee z)) = (x] \circ [(x \vee y) \wedge (x \vee z))$ and by hypothesis $(x] \circ [(x \vee y) \wedge (x \vee z)) \in \mathrm{Fi}(A)$. So, $y \wedge z \in (x] \circ [(x \vee y) \wedge (x \vee z))$, i.e., there exist $i \in (x]$ and $f \in [(x \vee y) \wedge (x \vee z))$ such that $y \wedge z \in i \circ f$. So, $f \leq (y \wedge z) \vee i$. It follows that $(x \vee y) \wedge (x \vee z) \leq x \vee (y \wedge z)$ and $[a)$ is a bounded distributive lattice. □

In lattice theory, a lattice is distributive if and only if every proper ideal is an intersection of prime ideals. Here we present a generalization of this characterization.

Theorem 5. *Let A be a nearlattice. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *Every proper ideal of A is an intersection of prime ideals.*

Proof. (1) $\Rightarrow$ (2) See Corollary 2.9 of [3].

(2) $\Rightarrow$ (1) Let $a, b \in A$. We prove that $a \circ b \in \mathrm{Fi}(A)$. It is easy to see that $1 \in a \circ b$ and that $a \circ b$ is increasing. Let $x, y \in a \circ b$ such that $x \wedge y$ exists. Let $Q = ((x \wedge y) \vee a]$ and suppose that $b \notin Q$. So, Q is a proper ideal and by hypothesis we have that $Q = \bigcap\{P \in X(A) : Q \subseteq P\}$. Then there exists $P \in X(A)$ such that $(x \wedge y) \vee a \in P$ and $b \notin P$. So, $x \wedge y \in P$ and $a \in P$. As P is prime, $x \in P$ or $y \in P$. Suppose that $x \in P$. Then $x \vee a \in P$ and since $x \in a \circ b$, i.e., $b \leq x \vee a$, we have that $b \in P$ which is a contradiction. The same reasoning applies when $y \in P$. Then $b \in Q$ and $b \leq (x \wedge y) \vee a$. Therefore, $x \wedge y \in a \circ b$ and $a \circ b \in \mathrm{Fi}(A)$. It follows from Theorem 4 that A is distributive. □

We study a new characterization of distributive nearlattices in terms of the notion of relative maximal ideal with respect to a set.

Definition 4. Let A be a semilattice. Let S be an increasing subset of A. An ideal I of A is called a *relative maximal ideal with respect to S*, when I is maximal among all the ideals which are disjoint to S.

Lemma 2. *Let A be a semilattice. Let $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$. Then I is a relative maximal ideal with respect to F if and only if $(H \circ F) \cap I \neq \varnothing$ for all $H \in \mathrm{Id}(A)$ such that $H \nsubseteq I$.*

Proof. Suppose that I is a relative maximal ideal with respect to F. Let $H \in \mathrm{Id}(A)$ such that $H \nsubseteq I$. We consider the ideal $I \vee H$. Since I is a relative maximal ideal with respect to F and $I \subseteq I \vee H$ then $(I \vee H) \cap F \neq \varnothing$, i.e., there exist $f \in F, i \in I$ and $h \in H$ such that $f \leq i \vee h$. So, $i \in h \circ f$ and $i \in H \circ F$. Therefore, $(H \circ F) \cap I \neq \varnothing$.

Assume that $(H \circ F) \cap I \neq \varnothing$ for all $H \in \mathrm{Id}(A)$ such that $H \nsubseteq I$. Suppose that I is not a relative maximal ideal with respect to F. Then there exists $J \in \mathrm{Id}(A)$ such that $I \subset J$ and $J \cap F = \varnothing$. Since $J \nsubseteq I$, by hypothesis we get $(J \circ F) \cap I \neq \varnothing$. Then there exist $i \in I$ and $(j, f) \in J \times F$ such that $i \in j \circ f$, i.e., $f \leq i \vee j$. As $i \in I \subset J$ and $i \vee j \in J$, we have that $f \in J$. So, $J \cap F \neq \varnothing$ which is a contradiction. $\square$

Theorem 6. *Let A be a nearlattice. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *Every relative maximal ideal I with respect to $a \circ b$ is prime for all $a, b \in A$.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in A$ and $I \in \mathrm{Id}(A)$ such that I is a relative maximal ideal with respect to $a \circ b$. We prove that I is prime. Let $x, y \in A$ such that $x \wedge y$ exists and $x \wedge y \in I$. Suppose that $x \notin I$ and $y \notin I$. Let $I_x = I \vee (x]$ and $I_y = I \vee (y]$. Then $I_x \cap a \circ b \neq \varnothing$ and $I_y \cap a \circ b \neq \varnothing$, i.e., there exist $f_1, f_2 \in a \circ b$ and $i_1, i_2 \in I$ such that $f_1 \leq x \vee i_1$ and $f_2 \leq y \vee i_2$. Let $i = i_1 \vee i_2 \in I$. So, $x \vee i, y \vee i \in a \circ b$ and $(x \vee i) \wedge_i (y \vee i)$ exists in $[i)$. From Theorem 4, it follows that $a \circ b \in \mathrm{Fi}(A)$ and $(x \vee i) \wedge_i (y \vee i) = (x \wedge y) \vee i \in a \circ b$. On the other hand, as I is an ideal, $(x \wedge y) \vee i \in I$. Thus, $I \cap a \circ b \neq \varnothing$ which is a contradiction. Then I is prime.

(2) $\Rightarrow$ (1) By Theorem 4, it is sufficient to prove that $a \circ b \in \mathrm{Fi}(A)$. It is easy to see that $1 \in a \circ b$ and that $a \circ b$ is increasing. Let $x, y \in a \circ b$ such that $x \wedge y$ exists. Suppose that $x \wedge y \notin a \circ b$. Then $(x \wedge y] \cap a \circ b = \varnothing$. We consider the following family

$$\mathcal{F} = \{I \in \mathrm{Id}(A) \, : \, (x \wedge y] \subseteq I \text{ and } I \cap a \circ b = \varnothing\}.$$

So, $\mathcal{F} \neq \varnothing$. By Zorn's Lemma there exists a maximal element $M \in \mathcal{F}$. It is not difficult to show that M is a relative maximal ideal with respect to $a \circ b$. So, $x \wedge y \in M$ and by hypothesis M is prime. Then $x \in M$ or $y \in M$. Thus, $M \cap a \circ b \neq \varnothing$ which is a contradiction. Therefore, $x \wedge y \in a \circ b$ and $a \circ b \in \mathrm{Fi}(A)$. $\square$

3. Normal and p-linear nearlattices

Let A be a semilattice and $a \in A$. From Definition 3 we have the following relative annihilator

$$a^{\top} = a \circ 1 = \{x \in A \, : \, x \vee a = 1\},$$

called the *annihilator of a*. We have the following result.

Lemma 3. *Let A be a distributive nearlattice. Let $a,b \in A$ and $I \in \mathrm{Id}(A)$. Then*

(1) *$I \cap a \circ b = \varnothing$ if and only if there exists $Q \in X(A)$ such that $I \subseteq Q$, $a \in Q$ and $b \notin Q$.*
(2) *$I \cap a^{\top} = \varnothing$ if and only if there exists $Q \in X(A)$ such that $I \subseteq Q$ and $a \in Q$.*
(3) *$I \cap a^{\top} = \varnothing$ if and only if there exists $U \in \mathrm{Idm}(A)$ such that $I \subseteq U$ and $a \in U$.*
(4) *$U \in \mathrm{Idm}(A)$ if and only if for all $a \in A$, $a \notin U$ if and only if $U \cap a^{\top} \neq \varnothing$.*

Proof. (1) Let $J \in \mathrm{Id}(A)$ such that $J \cap a \circ b = \varnothing$. Let $H = I(J \cup \{a\})$. We prove that $H \cap [b) = \varnothing$. If $x \in H \cap [b)$ then there exists $j \in J$ such that $x \leq j \vee a$ and $b \leq x$. So, $b \leq j \vee a$ and $j \in a \circ b$ which is a contradiction. Then $H \cap [b) = \varnothing$ and by Theorem 2 there exists $Q \in X(A)$ such that $J \subseteq Q$, $a \in Q$ and $b \notin Q$.

The other direction is immediate.

(2) It follows from (1).

(3) If $I \cap a^{\top} = \varnothing$ then there exists $Q \in X(A)$ such that $I \subseteq Q$ and $a \in Q$. We consider the family

$$\mathcal{Z} = \{R \in \mathrm{Id}(A) - \{A\} \ : \ I \subseteq R \text{ and } a \in R\}.$$

So, $\mathcal{Z} \neq \varnothing$ because $Q \in \mathcal{Z}$. Then, by Zorn's Lemma, there exists a maximal element $U \in \mathcal{Z}$. It is clear that U is proper. We prove that U is a maximal ideal. Let $b \in A$ such that $b \notin U$. If $U \cap b^{\top} = \varnothing$ then $H = I(U \cup \{b\})$ is a proper ideal. Otherwise, if $1 \in H$ then there exists $p \in U$ such that $p \vee b = 1$, i.e., $p \in U \cap b^{\top}$ which is a contradiction. So $U \subset H$ and $H \in \mathcal{Z}$, which is a contradiction because U is maximal. Then $U \cap b^{\top} \neq \varnothing$ and there exists $c \in U$ such that $c \vee b = 1$. Therefore, $H = A$ and U is maximal.

Suppose that $I \cap a^{\top} \neq \varnothing$. Then there exists $i \in I$ such that $i \vee a = 1$. So, there exists $U \in \mathrm{Idm}(A)$ such that $I \subseteq Q$ and $a \in Q$. Then $i \vee a = 1 \in U$, which is a contradiction because U is maximal.

(4) Let $U \in \mathrm{Idm}(A)$. Suppose that $a \notin U$. As U is maximal, $I(U \cup \{a\}) = A$. Then $1 \in I(U \cup \{a\})$, i.e., there exists $p \in U$ such that $p \vee a = 1$. So, $p \in a^{\top}$ and $U \cap a^{\top} \neq \varnothing$.

If $U \cap a^{\top} \neq \varnothing$ and $a \in U$ then there exists $p \in U$ such that $p \vee a = 1$. Thus, $1 \in U$ which is a contradiction.

Conversely, let $I \in \mathrm{Id}(A)$ such that $U \subset I$. Then there exists $a \in I$ such that $a \notin U$. So, $U \cap a^{\top} \neq \varnothing$, i.e., there exists $p \in U$ such that $p \vee a = 1$. Since $U \subset I$, we have that $p \in I$ and $a \vee p = 1 \in I$. Therefore, $I = A$ and U is maximal. □

We recall that a bounded distributive lattice is *normal* if each prime ideal contains a unique minimal prime ideal. This concept was introduced by Cornish in [8] and extended to the class of distributive semilattices in [13]. Now, we introduce a generalization of this notion.

Definition 5. Let A be a distributive nearlattice. We say that A is *normal* if each prime ideal is contained in a unique maximal ideal.

We say that A is *p-linear* if the family of prime ideals which contain a prime ideal is a chain.

Remark 2. We note that every normal nearlattice is p-linear.

Remark 3. If A is a bounded distributive lattice then the Definition 5 is equivalent to saying that every prime filter contains a unique minimal filter, which is a concept dual to the definition given by Cornish in [8].

The following results characterize normal nearlattices through annihilators.

Lemma 4. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is normal.*
(2) *For every $P \in X(A)$ and for all $a, b \in A$ with $a \vee b = 1$, $P \cap a^{\top} \neq \varnothing$ or $P \cap b^{\top} \neq \varnothing$.*

Proof. (1) $\Rightarrow$ (2) Let $P \in X(A)$ and $a, b \in A$ such that $a \vee b = 1$. Suppose that $P \cap a^{\top} = \varnothing$ and $P \cap b^{\top} = \varnothing$. So, by Lemma 3, there exist $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$, $P \subseteq U_2$, $a \in U_1$ and $b \in U_2$. Since A is normal, $U_1 = U_2$. Then $a, b \in U_1$, but $a \vee b = 1 \in U_1$ which is a contradiction. Thus, $P \cap a^{\top} \neq \varnothing$ or $P \cap b^{\top} \neq \varnothing$.

(2) $\Rightarrow$ (1) Let $P \in X(A)$ and $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$ and $P \subseteq U_2$. If $U_1 \neq U_2$ then there exists $a \in U_1$ such that $a \notin U_2$. As U_2 is maximal, $I(U_2 \cup \{a\}) = A$. Then $1 \in I(U_2 \cup \{a\})$, i.e., there exists $b \in U_2$ such that $a \vee b = 1$. On the other hand, by Lemma 3, $P \cap a^{\top} = \varnothing$ and $P \cap b^{\top} = \varnothing$, which contradicts the assumption. □

Lemma 5. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is normal.*
(2) *For all $a, b \in A$, $(a \vee b)^{\top} = F(a^{\top} \cup b^{\top})$.*
(3) *For all $a, b \in A$ with $a \vee b = 1$, $F(a^{\top} \cup b^{\top}) = A$.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in A$. Note that the inclusion $F(a^{\top} \cup b^{\top}) \subseteq (a \vee b)^{\top}$ always holds. Let us prove the other inclusion. Suppose that there exists $x \in (a \vee b)^{\top}$ such that $x \notin F(a^{\top} \cup b^{\top})$. So, by Theorem 2, there exists $P \in X(A)$ such that $x \in P$ and $P \cap F(a^{\top} \cup b^{\top}) = \varnothing$. Since $a^{\top}, b^{\top} \subseteq F(a^{\top} \cup b^{\top})$, we have that $P \cap a^{\top} = \varnothing$ and $P \cap b^{\top} = \varnothing$. Then, by Lemma 3, there exist $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$, $P \subseteq U_2$, $a \in U_1$ and $b \in U_2$. As A is normal, $U_1 = U_2$ and $a, b \in U_1$. Also, $x \in U_1$. Then $x \vee (a \vee b) = 1 \in U_1$ which is a contradiction. Therefore, $(a \vee b)^{\top} = F(a^{\top} \cup b^{\top})$.

(2) $\Rightarrow$ (3) It is immediate.

(3) $\Rightarrow$ (1) Let $P \in X(A)$ and $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$ and $P \subseteq U_2$. Suppose that there exists $a \in U_1$ such that $a \notin U_2$. Since U_2 is maximal, there exists $b \in U_2$ such that $a \vee b = 1$. Then $F(a^{\top} \cup b^{\top}) = A$. Let $x \in P$. Hence, $x \in F(a^{\top} \cup b^{\top})$ and there exist $x_1, ..., x_n \in a^{\top} \cup b^{\top}$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$.

So, $x_1 \wedge ... \wedge x_n \in P$ and by the primality of P there exists $x_i \in \{x_1,...,x_n\}$ such that $x_i \in P$. If $x_i \in a^{\top}$ then $x_i \in U_1$ and $x_i \vee a = 1 \in U_1$, which is a contradiction because U_1 is proper. If $x_i \in b^{\top}$, we arrive at a contradiction. Thus, $U_1 \subseteq U_2$ and consequently $U_1 = U_2$. Therefore, A is normal. □

Corollary 1. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is normal.*
(2) *The application $\rho : A \to \mathrm{Fi}(A)$ defined by $\rho(a) = a^{\top}$ is a homomorphism of distributive nearlattices.*

Proof. It follows from Lemma 5. □

Theorem 7. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is p-linear.*
(2) *For every $P \in X(A)$ and for all $a,b \in A$, $P \cap a \circ b \neq \varnothing$ or $P \cap b \circ a \neq \varnothing$.*
(3) *For all $a,b \in A$, $F((a \circ b) \cup (b \circ a)) = A$.*

Proof. (1) $\Rightarrow$ (2) Let $P \in X(A)$ and $a,b \in A$. Assume that $P \cap a \circ b = \varnothing$ and $P \cap b \circ a = \varnothing$. By Lemma 3 there exist $Q_1, Q_2 \in X(A)$ such that $P \subseteq Q_1$, $a \in Q_1$, $b \notin Q_1$, $P \subseteq Q_2$, $b \in Q_2$, and $a \notin Q_2$. As A is p-linear, $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. If $Q_1 \subseteq Q_2$ then $a \in Q_2$, which is impossible. If $Q_2 \subseteq Q_1$, then $b \in Q_1$ which is a contradiction. Thus, $P \cap a \circ b \neq \varnothing$ or $P \cap b \circ a \neq \varnothing$.

(2) $\Rightarrow$ (3) Suppose that there exist $a,b \in A$ such that $F((a \circ b) \cup (b \circ a)) \neq A$. Then there exists $c \in A$ such that $c \notin F((a \circ b) \cup (b \circ a))$. Since $F((a \circ b) \cup (b \circ a)) \in \mathrm{Fi}(A)$, by Theorem 2 there exists $P \in X(A)$ such that $c \in P$ and $P \cap F((a \circ b) \cup (b \circ a)) = \varnothing$. So, $P \cap a \circ b = \varnothing$ and $P \cap b \circ a = \varnothing$ which is impossible. Thus, $F((a \circ b) \cup (b \circ a)) = A$.

(3) $\Rightarrow$ (1) Let $P, Q_1, Q_2 \in X(A)$ such that $P \subseteq Q_1$ and $P \subseteq Q_2$. If Q_1 and Q_2 are incomparable, then there exist $a,b \in A$ such that $a \in Q_1 - Q_2$ and $b \in Q_2 - Q_1$. Let $x \in P$. Since $A = F((a \circ b) \cup (b \circ a))$, there exist $x_1,...,x_n \in (a \circ b) \cup (b \circ a)$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. So, $x_1 \wedge ... \wedge x_n \in P$ and by the primality of P there exists $x_i \in \{x_1,...,x_n\}$ such that $x_i \in P$. If $x_i \in a \circ b$, then $b \leq x_i \vee a$. As $x_i, a \in Q_1$, we have $x_i \vee a \in Q_1$ and $b \in Q_1$ which is a contradiction. Similarly, if $x_i \in b \circ a$ we arrive at a contradiction. Thus, Q_1 and Q_2 are comparable and A is p-linear. □

4. THE LATTICE OF FILTERS

In this section we study the structure of the lattice of filters of a distributive nearlattice. Recall that a Heyting algebra is an algebra $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ of type $(2,2,2,0,0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the operation $\Rightarrow$ satisfies the following condition: $a \wedge b \leq c$ if and only if $a \leq b \Rightarrow c$ for all $a,b,c \in A$. The *pseudocomplement* of an element $a \in A$ is the element $a^* = a \Rightarrow 0$.

Let A be a distributive nearlattice. For each pair $F,H \in \mathrm{Fi}(A)$ let us define the subset $F \rhd H$ of A as follows:

$$F \rhd H = \{a \in A : [a) \cap F \subseteq H\}.$$

Theorem 8. *Let A be a distributive nearlattice. Let $F,H \in \mathrm{Fi}(A)$. Then*

(1) $F \rhd H \in \mathrm{Fi}(A)$.
(2) $F \rhd H = \{a \in A : \forall f \in F\ \exists h \in H\ (h \leq a \vee f)\}$.
(3) *The structure* $\langle \mathrm{Fi}(A), \veebar, \barwedge, \rhd, \{1\}, A\rangle$ *is a Heyting algebra.*

Proof. (1) Let $F,H \in \mathrm{Fi}(A)$. We prove that $F \rhd H \in \mathrm{Fi}(A)$. Since $[1) \cap F = \{1\} \subseteq H$, then $1 \in F \rhd H$. Let $a,b \in A$ such that $a \leq b$ and $a \in F \rhd H$. Thus, $[b) \subseteq [a)$ and $[a) \cap F \subseteq H$. It follows that $[b) \cap F \subseteq H$, i.e., $b \in F \rhd H$. Let $a,b \in F \rhd H$ and suppose that $a \wedge b$ exists. By Theorem 3, the lattice $\mathrm{Fi}(A)$ is distributive and

$$\begin{aligned}[a \wedge b) \cap F &= ([a) \veebar [b)) \cap F \\ &= ([a) \cap F) \veebar ([b) \cap F) \\ &\subseteq H.\end{aligned}$$

Therefore, $a \wedge b \in F \rhd H$ and $F \rhd H \in \mathrm{Fi}(A)$.

(2) Let $F,H \in \mathrm{Fi}(A)$. Let $X = \{a \in A : \forall f \in F\ \exists h \in H\ (h \leq a \vee f)\}$ and $a \in X$. We prove that $a \in F \rhd H$, i.e., $[a) \cap F \subseteq H$. If $x \in [a) \cap F$, then $a \leq x$ and $x \in F$. Since $a \in X$ and $x \in F$, there exists $h \in H$ such that $h \leq a \vee x = x$. As H is a filter, $x \in H$. So, $a \in F \rhd H$. Conversely, let $a \in F \rhd H$ and $f \in F$. Then $a \vee f \in [a) \cap F$ and by hypothesis, $a \vee f \in H$. It follows that $a \in X$. Therefore, $F \rhd H = X$.

(3) By Theorem 3, $\langle \mathrm{Fi}(A), \veebar, \barwedge, \{1\}, A\rangle$ is a bounded distributive lattice. Let $F,G,H \in \mathrm{Fi}(A)$. We prove that $F \cap G \subseteq H$ if and only if $F \subseteq G \rhd H$. Suppose that $F \cap G \subseteq H$ and let $a \in F$. If $x \in [a) \cap G$, then $a \leq x$ and $x \in G$. Therefore $x \in F \cap G$ and by hypothesis $x \in H$, i.e., $F \subseteq G \rhd H$. Conversely, let $x \in F \cap G$. By hypothesis $F \subseteq G \rhd H$, then $x \in G \rhd H$. As $[x) \cap G \subseteq H$, we have that $x \in H$. $\square$

Remark 4. As a particular case, we have $a \circ b = [a) \rhd [b)$. Indeed

$$\begin{aligned} x \in a \circ b \quad &\text{iff} \quad b \leq x \vee a && \text{iff} \quad [x \vee a) \subseteq [b) \\ &\text{iff} \quad [x) \cap [a) \subseteq [b) && \text{iff} \quad x \in [a) \rhd [b). \end{aligned}$$

Then we can write the annihilator of a relative to b in terms of the binary operation $\rhd$.

Note that if $F \in \mathrm{Fi}(A)$, then $F^* = F \rhd \{1\} = \{a \in A : [a) \cap F = \{1\}\}$. The following result describes the filter F^* in a different manner.

Proposition 2. *Let A be a distributive nearlattice. Then for every $F \in \mathrm{Fi}(A)$,*

$$F^* = \{a \in A : \forall f \in F\ (a \vee f = 1)\}.$$

Proof. Let $C = \{a \in A : \forall f \in F\ (a \vee f = 1)\}$ and $a \in C$. We prove that $[a) \cap F = \{1\}$. Let $x \in A$ such that $a \leq x$ and $x \in F$. Then $x = a \vee x = 1$. Thus, $a \in F^*$. Conversely, let $a \in F^*$. Then $[a) \cap F = \{1\}$. Since $a \leq a \vee f$ and $f \leq a \vee f$ for every $f \in F$, and as F is a filter, $a \vee f \in [a) \cap F$. So, $a \vee f = 1$ for each $f \in F$. Therefore, $a \in C$ and $F^* = C$. □

Remark 5. We note that $[a)^* = \{x \in A : x \vee a = 1\}$, i.e., $a^{\top} = [a)^*$.

We prove that the pseudocomplement of a subset X is the pseudocomplement of the filter generated by X. This result was proved for Tarski algebras in [5].

Theorem 9. *Let A be a distributive nearlattice. Then for every subset $X \subseteq A$, we have $X^* = F(X)^*$.*

Proof. Since $X \subseteq F(X)$, we have that $F(X)^* \subseteq X^*$. Conversely, let $x \in X^*$. We prove that for every $a \in F(X)$, $x \vee a = 1$. Suppose that there exists $a \in F(X)$ such that $x \vee a \neq 1$. Then there exist $x_1, ..., x_n \in [X)$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = a$. So, there exist $y_1, ..., y_n \in X$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. As $x \in X^*$, $x \vee y_i = 1$ for all $y_i \in \{y_1, ..., y_n\}$. Then $x \vee x_i = 1$ for all $x_i \in \{x_1, ..., x_n\}$. Since $x \vee a \neq 1$, by Theorem 2, there exists $P \in X(A)$ such that $x \vee a \in P$ and $1 \notin P$. Then $x_1 \wedge ... \wedge x_n \in P$ and as P is a prime ideal, there exists $x_i \in \{x_1, ..., x_n\}$ such that $x_i \in P$. On the other hand, $x \vee a \in P$ and $x \in P$. So, $x \vee x_i = 1 \in P$ which is a contradiction. Thus, $x \vee a = 1$ for all $a \in F(X)$ and consequently $x \in F(X)^*$. □

5. Semi-homomorphisms preserving annihilators

Our next aim is to study a particular class of semi-homomorphisms: semi-homomorphisms preserving annihilators. We give some characterizations in terms of prime and maximal ideals.

Definition 6. Let A, B be two distributive nearlattices and let $h : A \to B$ be a semi-homomorphism. We say that h is a *semi-homomorphism preserving annihilators*, or $\top$-*semi-homomorphism*, if $F(h(a^{\top})) = h(a)^{\top}$ for all $a \in A$.

Remark 6. We note that $F(h(a^{\top})) \subseteq h(a)^{\top}$ for all $a \in A$. If $x \in F(h(a^{\top}))$ then there exist $x_1, ..., x_n \in [h(a^{\top}))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. As $x_1, ..., x_n \in [h(a^{\top}))$, there exist $y_1, ..., y_n \in h(a^{\top})$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. So, there exist $t_1, ..., t_n \in a^{\top}$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Thus, $t_1 \vee a = \cdots = t_n \vee a = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(a) = \cdots = y_n \vee h(a) = 1$. Then $x \vee h(a) = [(y_1 \vee x_1) \wedge ... \wedge (y_n \vee x_n)] \vee h(a)$. As $[x)$ is a bounded distributive lattice, $x \vee h(a) = (y_1 \vee x_1 \vee h(a)) \wedge ... \wedge (y_n \vee x_n \vee h(a)) = 1$. Therefore, $x \vee h(a) = 1$ and $x \in h(a)^{\top}$.

Let $h : A \to B$ be a semi-homomorphism between distributive nearlattices. In general, $h^{-1}(P) \notin X(A)$ for each $P \in X(B)$. Now, we prove that if P is maximal and h is a $\top$-semi-homomorphism, then $h^{-1}(P)$ is maximal and therefore prime.

Lemma 6. *Let A, B be two distributive nearlattices and let $h : A \to B$ be a $\top$-semi-homomorphism. Then $h^{-1}(P) \in \mathrm{Idm}(A)$ for every $P \in \mathrm{Idm}(B)$.*

Proof. Let $P \in \mathrm{Idm}(B)$. Since h is a semi-homomorphism, $h^{-1}(P)$ is an ideal. As $h(1) = 1 \notin P$, we have that $h^{-1}(P)$ is proper. Let $a \in A$ such that $a \notin h^{-1}(P)$. Then $h(a) \notin P$ and as P is maximal, by Lemma 3, $P \cap h(a)^\top \neq \varnothing$. So, there exists $x \in A$ such that $x \in P \cap F(h(a^\top)) \neq \varnothing$, i.e., there exist $x_1, ..., x_n \in [h(a^\top))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. Thus, there exist $y_1, ..., y_n \in h(a^\top)$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. It follows that there exist $t_1, ..., t_n \in a^\top$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Then $t_1 \vee a = ... = t_n \vee a = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(a) = ... = y_n \vee h(a) = 1$. As

$$x = x_1 \wedge ... \wedge x_n = (x_1 \vee y_1) \wedge ... \wedge (x_n \vee y_n) \in P$$

and P is prime, there exists $i \in \{1, ..., n\}$ such that $x_i \vee y_i \in P$. So, $y_i = h(t_i) \in P$, i.e., $t_i \in h^{-1}(P)$ and $h^{-1}(P) \cap a^\top \neq \varnothing$. Conversely, it is easy to prove that if $h^{-1}(P) \cap a^\top \neq \varnothing$, then $a \notin h^{-1}(P)$. Therefore, by Lemma 3, $h^{-1}(P) \in \mathrm{Idm}(A)$. $\square$

Theorem 10. *Let A, B be two distributive nearlattices and let $h : A \to B$ be a semi-homomorphism. Then the following conditions are equivalent:*

(1) *h is a $\top$-semi-homomorphism.*
(2) *For all $P \in X(B)$ and for every $Q \in X(A)$ such that $h^{-1}(P) \subseteq Q$, there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}(D)$.*
(3) $\mathrm{Idm}(A) \cap [h^{-1}(P)) \subseteq h^{-1}[X(B) \cap [P)]$ *for all $P \in X(B)$.*

Proof. (1) $\Rightarrow$ (2) Let $P \in X(B)$ and $Q \in X(A)$ such that $h^{-1}(P) \subseteq Q$. Let us consider the ideal $H = I(P \cup h(Q))$. We note that H is a proper ideal. Indeed, if we assume otherwise, there exists $p \in P$ and $q \in Q$ such that $p \vee h(q) = 1$. So, $p \in h(q)^\top = F(h(q^\top))$. Then, there exist $x_1, ..., x_n \in [h(q^\top))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = p$. So, there exist $y_1, ..., y_n \in h(q^\top)$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. It follows that there exist $t_1, ..., t_n \in q^\top$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Then $t_1 \vee q = ... = t_n \vee q = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(q) = ... = y_n \vee h(q) = 1$. As $x = x_1 \wedge ... \wedge x_n = (x_1 \vee y_1) \wedge ... \wedge (x_n \vee y_n) \in P$ and P is prime, there exists $i \in \{1, ..., n\}$ such that $x_i \vee y_i \in P$. So, $y_i = h(t_i) \in P$, i.e., $t_i \in h^{-1}(P) \subseteq Q$ and since $q \in Q$, $t_i \vee q = 1 \in Q$ which is a contradiction. Therefore, H is a proper ideal and there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}(D)$.

(2) $\Rightarrow$ (3) Let $P \in X(B)$ and $Q \in \mathrm{Idm}(A) \cap [h^{-1}(P))$. Then $Q \in X(A)$ and $h^{-1}(P) \subseteq Q$. By hypothesis, there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}(D)$. Since $h^{-1}(D)$ is an ideal and Q is maximal, $Q = h^{-1}(D)$. So, $D \in h^{-1}[X(B) \cap [P)]$ and $\mathrm{Idm}(A) \cap [h^{-1}(P)) \subseteq h^{-1}[X(B) \cap [P)]$.

(3) $\Rightarrow$ (1) Let $a \in A$. We prove that $h(a)^\top \subseteq F(h(a^\top))$. Suppose that there exists $x \in h(a)^\top$ such that $x \notin F(h(a^\top))$. By Theorem 2, there exists $P \in X(A)$ such that

$x \in P$ and $P \cap F(h(a^{\top})) = \varnothing$. Then,

$$P \cap h(a)^{\top} \neq \varnothing \text{ and } P \cap h(a^{\top}) = \varnothing.$$

Thus, $h^{-1}(P) \cap a^{\top} = \varnothing$ and $h^{-1}(P) \in \mathrm{Id}(A)$. By Lemma 3, there exists $U \in \mathrm{Idm}(A)$ such that $h^{-1}(P) \subseteq U$ and $a \in U$. So, we get that

$$U \in \mathrm{Idm}(A) \cap [h^{-1}(P)) \subseteq h^{-1}[X(B) \cap [P)].$$

Then, there exists $D \in X(B)$ such that $P \subseteq D$ and $U = h^{-1}(D)$. Since $a \in U$, $h(a) \in D$ and $D \cap h(a)^{\top} = \varnothing$. On the other hand, $P \cap h(a)^{\top} \neq \varnothing$ and $P \subseteq D$. Thus, $D \cap h(a)^{\top} \neq \varnothing$ which is a contradiction. Therefore, $h(a)^{\top} \subseteq F(h(a^{\top}))$ and h is a $\top$-semi-homomorphism. □

It is possible to give another characterization of the $\top$-semi-homomorphisms in normal nearlattices.

Proposition 3. *Let A, B be two distributive nearlattices and let $h : A \to B$ be a semi-homomorphism. Suppose that B is normal. Then h is a $\top$-semi-homomorphism if and only if:*

1. *For all $P \in X(B)$ and for all $Q_1, Q_2 \in \mathrm{Idm}(A)$, if $h^{-1}(P) \subseteq Q_1 \cap Q_2$ then $Q_1 = Q_2$.*
2. *$h^{-1}(P) \in \mathrm{Idm}(A)$ for every $P \in \mathrm{Idm}(B)$.*

Proof. $\Rightarrow$) Suppose that h is a $\top$-semi-homomorphism. By Lemma 6, we only need to prove (1). Let $P \in X(B)$ and $Q_1, Q_2 \in \mathrm{Idm}(A)$ such that $h^{-1}(P) \subseteq Q_1 \cap Q_2$. Suppose that there exists $a \in Q_1$ such that $a \notin Q_2$. Since Q_2 is maximal, there exists $b \in Q_2$ such that $a \vee b = 1$. Then $h(a \vee b) = h(a) \vee h(b) = h(1) = 1$. As B is normal, by Lemma 4, we have that $P \cap h(a)^{\top} \neq \varnothing$ or $P \cap h(b)^{\top} \neq \varnothing$, and since h is $\top$-semi-homomorphism, $P \cap F(h(a^{\top})) \neq \varnothing$ or $P \cap F(h(b^{\top})) \neq \varnothing$. If $P \cap F(h(a^{\top})) \neq \varnothing$, then there exists $x \in P$ such that $x \in F(h(a^{\top}))$, i.e., there exist $x_1, ..., x_n \in [h(a^{\top}))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. So, there exist $y_1, ..., y_n \in h(a^{\top})$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. It follows that there exist $t_1, ..., t_n \in a^{\top}$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Then, $t_1 \vee a = ... = t_n \vee a = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(a) = ... = y_n \vee h(a) = 1$. As $x = x_1 \wedge ... \wedge x_n = (x_1 \vee y_1) \wedge ... \wedge (x_n \vee y_n) \in P$ and P is prime, there exists $i \in \{1, ..., n\}$ such that $x_i \vee y_i \in P$. So, $y_i = h(t_i) \in P$ and $t_i \in h^{-1}(P) \subseteq Q_1 \cap Q_2$. Since $a, t_i \in Q_1$, we have that $t_i \vee a = 1 \in Q_1$, which is a contradiction because Q_1 is maximal. If $P \cap F(h(b^{\top})) \neq \varnothing$, we get also a contradiction. Therefore, $Q_1 \subseteq Q_2$ and consequently $Q_1 = Q_2$.

$\Leftarrow$) Let $a \in A$. We prove that $h(a)^{\top} \subseteq F(h(a^{\top}))$. Suppose that there exists $x \in h(a)^{\top}$ such that $x \notin F(h(a^{\top}))$. Then there exists $P \in X(B)$ such that $x \in P$ and $P \cap F(h(a^{\top})) = \varnothing$, i.e., $P \cap h(a)^{\top} \neq \varnothing$ and $P \cap F(h(a^{\top})) = \varnothing$. Since B is normal, there exists a unique $Q \in \mathrm{Idm}(B)$ such that $P \subseteq Q$. We note that $h(a) \notin Q$. Indeed, if $h(a) \in Q$ then $h(a)^{\top} \cap Q \neq \varnothing$ and there exists $x \in Q$ such that $h(a) \vee x = 1 \in Q$, which is a contradiction. Since $a \notin h^{-1}(Q)$ and by (2) we have $h^{-1}(Q) \in \mathrm{Idm}(A)$, then, by Lemma 3, $h^{-1}(Q) \cap a^{\top} \neq \varnothing$, i.e., $Q \cap F(h(a^{\top})) \neq \varnothing$. On the other hand, since $P \cap F(h(a^{\top})) = \varnothing$ then $h^{-1}(P) \cap a^{\top} = \varnothing$ and, by Lemma 3, there exists $U \in \mathrm{Idm}(A)$ such that $h^{-1}(P) \subseteq U$ and $a \in U$. Then $h^{-1}(P) \subseteq h^{-1}(Q) \cap U$ and by (1), we have $h^{-1}(Q) = U$, which is a contradiction because $a \in U$ and $a \notin h^{-1}(Q)$. Therefore, $h(a)^{\top} = F(h(a^{\top}))$ and h is a $\top$-semi-homomorphism. □

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