Miskolc Mathematical Notes
Vol. 16 (2015), No 1, pp. 145-150

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1364

A note on algebraic extensions modulo I

Basma Gasmi and Noômen Jarboui

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 145–150
HU e-ISSN 1787-2413

ON ALGEBRAIC EXTENSIONS MODULO I

BASMA GASMI AND NOÔMEN JARBOUI

Received 06 October, 2014

Abstract. Let I be a nonzero ideal of a ring T, let $\varphi : T \to E := T/I$ denote the canonical projection, let D be a ring contained in E, and let $R = \varphi^{-1}(D)$. The main purpose of this paper is to characterize when the ring extension $R \subset T$ is n- (resp., universally) algebraic modulo I in case I is an intersection of finitely many maximal ideals of T.

2010 *Mathematics Subject Classification:* 13A15; 13B25; 13C15

Keywords: integral domain, prime ideal, algebraic extension, algebraic extension modulo I, residually algebraic extension, pullback

1. Introduction

All rings considered below are commutative with identity but *not necessarily integral domains*. All subrings and inclusions of rings are (unital) ring extensions; all ring/algebra homomorphisms are unital. Let A be a ring and $n \geq 1$ be an integer. We denote by $A[n]$ the ring of polynomials in n indeterminates over A (for $n = 1$, $A[1] = A[X]$ is the ring of polynomials in one indeterminate). For convenience, we write $A = A[0]$.

Let I be a nonzero ideal of a ring T, $\varphi : T \to E := T/I$ the natural projection, and D a ring contained in E. Then $R = \varphi^{-1}(D)$ is the ring arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I = E \end{array}$$

Following [4], we say that R is the ring of the (T, I, D) construction and we set $R := (T, I, D)$. We shall assume that D is properly contained in E (and hence, that R is properly contained in T), and we shall refer to this as a *pullback diagram of type* ($\square$). If I is an intersection of finitely many maximal ideals of T, we shall refer to this as a diagram ($\square_\cap$). A very good account of pullback constructions has been given in [4, 5] and [6]. It has fashionable in recent years to study rings via pullback diagrams. It is well worth noting that pullback constructions provide a rich source of examples and counterexamples in commutative algebra (see [1–5, 11, 12]). Unless otherwise specified, the symbols T, D, I, R have the above meaning throughout the paper.

In [8] the authors introduced the concept of n-algebraic extension modulo I for a diagram ($\square$) when T and D are integral domains and $n \geq 0$ is an integer. More precisely, the ring extension $R \subset T$ (of integral domains) is said to be *n-algebraic modulo I* if for every two prime ideals $Q' \subset Q$ of $T[n]$ such that $I[n] \nsubseteq Q'$, $I[n] \subseteq Q$ and $ht(Q \cap R[n]/Q' \cap R[n]) = 1$, then $R[n]/(Q \cap R[n]) \subseteq T[n]/Q$ is algebraic. This concept was first used to characterize when an integral domain R of the form $D + I$, (where I is a nonzero ideal of an integral domain T and D is a subring of T satisfying $D \cap I = (0)$) is a (stably) strong S-domain (cf. [8, Théorème 1.7]). In [2], the authors dealt with a more general situation and used this concept to characterize when a ring R arising from a diagram ($\square$) is a (stably) strong S-domain. The main purpose of this paper is to study n-algebraic extensions modulo I for a diagram ($\square_\cap$) in order to deepen our knowledge about such extensions. We first extend this notion to arbitrary commutative rings. Our motivation is an example constructed by Fontana et al (see [8, Exemple 1.8]) of a diagram ($\square_\cap$) in order to produce a ring extension $R \subset T$ which is 0-algebraic modulo I but not 1-algebraic modulo I. For this reason, M. Fontana et al (see [8]) have introduced the following definition: The ring extension $R \subset T$ is said to be *universally algebraic modulo I*, if $R \subset T$ is *n-algebraic modulo I* for each positive integer n. Our contribution (see Theorem 1) is to prove that for a diagram ($\square_\cap$), $R \subset T$ is n-algebraic modulo I if and only if $R \subset T$ is 1-algebraic modulo I if and only if $R \subset T$ is a residually algebraic extension. The key step (Lemma 1) is to show, for any diagram ($\square$), that if $R \subset T$ is n-algebraic modulo I (where $n \geq 1$), then $R \subset T$ is $(n-1)$-algebraic modulo I.

Throughout the paper, we use "$\subset$" to denote proper containment and "$\subseteq$" to denote containment. Transcendence degrees paly an important role in our study; if $A \subseteq B$ are two domains, we denote by $tr.deg[B : A]$ the transcendence degree of the quotient field of B over that of A. Any unexplained terminology is standard as in [9, 10]. Relevant terminology and results will be recalled as needed through the paper.

2. Main results

We extend Fontana-Izelgue-Kabbaj's definition, mentioned in the introduction, to arbitrary commutative rings in the following way:

Definition 1. Let $n \geq 0$ be an integer. For a diagram ($\square$), the extension $R \subset T$ is said to be *n-algebraic modulo I* if for every two prime ideals $Q' \subset Q$ of $T[n]$ such that $I[n] \nsubseteq Q'$, $I[n] \subseteq Q$ and $ht(Q \cap R[n]/Q' \cap R[n]) = 1$, then $R[n]/(Q \cap R[n]) \subseteq T[n]/Q$ is algebraic.

Definition 2. For a diagram ($\square$), the extension $R \subset T$ is said to be *universally algebraic modulo I* if $R \subset T$ is n-algebraic modulo I for each integer $n \geq 0$.

Recall that an extension of rings $A \subseteq B$ is said to be *residually algebraic* if for each prime ideal Q of B, the extension $A/(Q \cap A) \subseteq B/Q$ is algebraic. It is clear that if $R \subset T$ is a residually algebraic extension, then so is $R[n] \subset T[n]$ for any positive integer n (cf. [7, Lemme 1.4]). Hence $R \subset T$ is universally algebraic modulo I.

Recall from [10, Section 1-5] that if p is a prime ideal of a ring A, and Q is a prime ideal of $A[X]$ with $Q \cap A = p$, but with $Q \neq p[X]$, then we call Q *an upper* to p in $A[X]$ (or more simply, an upper to p, or just an upper).

The main result of this paper is the following theorem which identifies n-algebraic extensions modulo I for a diagram $(\square_\cap)$. We assume that all rings are finite-dimensional.

Theorem 1. *Let $n \geq 1$ be an integer. For a diagram $(\square_\cap)$, consider the following statements:*

(1) *$R \subset T$ is 1-algebraic modulo I.*
(2) *$tr.deg[T/M : R/(M \cap R)] = 0$ for each maximal ideal M of T containing I.*
(3) *$R \subset T$ is a residually algebraic extension.*
(4) *$R \subset T$ is universally algebraic modulo I.*
(5) *$R \subset T$ is n-algebraic modulo I.*
(6) *$R \subset T$ is 0-algebraic modulo I.*

Then:

(a) *In general, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6).*
(b) *If, in addition, $I \in Max(T)$, then the above statements $(1)-(6)$ are equivalent.*

To prove the implications (5)$\Rightarrow$(1) and (5)$\Rightarrow$(6) in Theorem 1, we need the following lemma.

Lemma 1. *Let $n \geq 1$ be an integer. For a diagram $(\square)$, if $R \subset T$ is n-algebraic modulo I, then $R \subset T$ is $(n-1)$-algebraic modulo I.*

Proof. Let $Q' \subset Q$ be two prime ideals of $T[n-1]$ such that $I[n-1] \nsubseteq Q'$ and $I[n-1] \subseteq Q$. Set $P' = Q' \cap R[n-1]$, $P = Q \cap R[n-1]$ and suppose that $P' \subset P$ are consecutive. Our task is to show that $R[n-1]/P \subseteq T[n-1]/Q$ is an algebraic extension. Let $\mathcal{Q}' = Q' + X_n T[n-1][X_n]$ and $\mathcal{Q} = Q + X_n T[n-1][X_n]$. It is obvious that $\mathcal{Q}'$ respectively $\mathcal{Q}$ are uppers to Q' respectively Q. Set $\mathcal{P}' = \mathcal{Q}' \cap R[n]$ and $\mathcal{P} = \mathcal{Q} \cap R[n]$. One can check easily that $\mathcal{P}' = P' + X_n R[n]$ and $\mathcal{P} = P + X_n R[n]$. As $X_n R[n] \subseteq \mathcal{P}' \subset \mathcal{P}$, then $\mathcal{P}' \subset \mathcal{P}$ are consecutive. On the other hand, since $R \subset T$ is n-algebraic modulo I, then $tr.deg[T[n]/\mathcal{Q} : R[n]/\mathcal{P}] = 0$. As $T[n]/\mathcal{Q} \cong T[n-1]/Q$ and $R[n]/\mathcal{P} \cong R[n-1]/P$, it follows that $tr.deg[T[n-1]/Q : R[n-1]/P] = 0$, as desired. $\square$

Before proceeding to the proof of Theorem 1 it is convenient to recall the following Cahen's lemma [4, Proposition 4]. We shall make use of this result in the proof of

Theorem 1. Note that this lemma holds even for polynomial rings since if $R := (T, I, D)$, then $R[n] := (T[n], I[n], D[n])$.

Lemma 2. *For a diagram (□), if $P_0 \subset \ldots \subset P_n$ is a chain of primes in R such that P_n is minimal among primes of R containing I and P_{n-1}, then this chain lifts in T.*

We now prove Theorem 1.

Proof of Theorem 1. (a) (1)⇒(2) Let Ω be the finite subset of $Max(T)$ such that $I = \bigcap_{M \in \Omega} M$. We discuss the following two cases.
Case 1. $\mid \Omega \mid \geq 2$. Since $M + \bigcap_{M' \in \Omega \setminus \{M\}} M' = T$, then there exist $u \in \bigcap_{M' \in \Omega \setminus \{M\}} M'$ and $v \in M$ such that $u + v = 1$. Let $P_1' = ((X-u)T[X]) \cap R[X]$ and $P_1 = (M[X] + (X-u)T[X]) \cap R[X]$. The prime ideals $P_1' \subset P_1$ are not necessarily consecutive. Since $T[X]$ is finite-dimensional, there exist two prime ideals P' and P of $T[X]$ such that P' is maximal among the primes such that $P_1' \subseteq P' \subset P_1$ and not containing I, and P is minimal such that $P_1' \subseteq P' \subset P \subseteq P_1$. Therefore P' does not contain I, P contains I and $P' \subset P$ are consecutive. The chain $P_1' \subseteq P' \subset P$ lifts in $T[X]$ as $Q_1' \subseteq Q' \subset Q$. Notice that $Q_1' = (X-u)T[X]$ because P_1' does not contain I and so it lifts uniquely in $T[X]$. Hence Q contains $X - u$ and I. The prime ideal Q cannot contain any prime containing u (if so, it would contain X, thus $X \in P_1$ and hence $u \in M$, which is absurd). Consequently Q is above M. Furthermore Q is an upper to M because $X - u \in Q \setminus M[X]$. The prime ideal P is above $p = M \cap R$. Next, we demonstrate that P is an upper to p. Consider the polynomial $f = (X-u)(X-v) = X^2 - X + uv$. Since $uv \in I$, then clearly f belongs to $P_1' = ((X-u)T[X]) \cap R[X]$. Thus $f \in P$. As $f \notin p[X]$, we deduce that P is an upper to p. As $R \subset T$ is 1-algebraic modulo I, it follows that $T[X]/Q$ is algebraic over $R[X]/P$. Since Q and P are uppers respectively to M and p, we deduce that T/M is algebraic over R/p.
Case 2. $\mid \Omega \mid = 1$. In this case $I = M$, where M is a maximal ideal of T. The proof in this case proceeds along the same lines as in the proof of Case 1 with some modifications. Set $P_1' = ((X-1)T[X]) \cap R[X]$ and $P_1 = (M[X] + (X-1)T[X]) \cap R[X]$. These prime ideals are not necessarily consecutive, so let P' be maximal among the primes such that $P_1' \subseteq P' \subset P_1$ and not containing I, and P be minimal such that $P_1' \subseteq P' \subset P \subseteq P_1$. Therefore P' does not contain I, P contains I, $P' \subset P$ are consecutive and the chain $P_1' \subseteq P' \subset P$ lifts in $T[X]$ as $Q_1' = (X-1)T[X] \subseteq Q' \subset Q$. It is clear that $Q \cap T$ contains I, and as I is a maximal ideal of T, then $Q \cap T = M$. Moreover, since Q contains $X - 1$, then Q is an upper to M. The prime ideal P is above $p = M \cap R$. We claim that P is an upper to p. Consider the polynomial $f = (X-1)^2 = X^2 - 2X + 1$. It is obvious that $f \in P_1' = ((X-1)T[X]) \cap R[X]$ and $f \notin p[X]$. Hence $f \in P \setminus p[X]$. Therefore P is an upper to p as claimed. Since $R \subset T$ is 1-algebraic modulo I, it results that $T[X]/Q$ is algebraic over $R[X]/P$. As Q and P are uppers respectively to M and p, it follows that T/M is algebraic

over R/p.
(2)⇒(3) Let $q \in Spec(T)$. Our purpose is to show that $R/(q \cap R) \subseteq T/q$ is an algebraic extension. If $I \not\subseteq q$, then $T_q \simeq R_{q \cap R}$ (see [4, Proposition 0]). So $tr.deg[T/q : R/(q \cap R)] = 0$. If $I \subseteq q$, then $q \in \Omega$. Hence $tr.deg[T/q : R/(q \cap R)] = 0$.
(3)⇒(4)⇒(5) are trivial.
(5)⇒(1) The conclusion is clear if $n = 1$. So assume that $n \geq 2$. The conclusion follows readily from Lemma 1.
(5)⇒(6) Follows readily from Lemma 1.
(b) We now assume that $I \in Max(T)$. We will prove that (6)⇒(2). To this end, we have only to show that $tr.deg[T/I : R/I] = 0$. Let q' be a prime ideal of T such that $q' \subset I$ are consecutive in T (such ideal exists since T is finite-dimensional). Let $p' = q' \cap R$, then $p' \subset I$ are also consecutive in R. Indeed, assume that there exists a prime ideal p of R such that $p' \subset p \subset I$. This chain lifts in T to $q' \subset q \subset I$ (notice that the unique prime ideal of T lying over I is I itself since $I \in Max(T)$). The desired contradiction since $q' \subset I$ are consecutive. As $R \subset T$ is 0-algebraic modulo I, then $tr.deg[T/I : R/I] = 0$, as asserted. □

Remark 1. If we leave out the assumption "$I \in Max(T)$" in the statement of Theorem 1 (b), the conclusion does not hold. More precisely, Fontana et al (see [8, Exemple 1.8]) have already constructed a diagram ($\square_\cap$), where I is an intersection of two maximal ideals of T, such that $R \subset T$ is 0-algebraic modulo I, whereas $R \subset T$ is not 1-algebraic modulo I.

ACKNOWLEDGEMENT

The authors would like to thank the referee for many valuable suggestions.

REFERENCES

[1] A. Ayache, "About a conjecture on Nagata rings," *J. Pure. Appl. Algebra*, vol. 98, pp. 1–5, 1995.
[2] A. Ayache and N. Jarboui, "On questions related to stably strong S-domains," *J. Algebra*, vol. 291, no. 3, pp. 164–170, 2005.
[3] M. Ben Nasr and N. Jarboui, "A counterexample for a conjecture about the catenarity of polynomial rings," *J. Algebra*, vol. 248, pp. 785–789, 2002.
[4] P.-J. Cahen, "Couple d'anneaux partageant un idéal," *Arch. Math.*, vol. 51, pp. 505–514, 1988.
[5] P.-J. Cahen, "Construction B, I, D et anneaux localement ou résiduellement de Jaffard," *Arch. Math.*, vol. 54, pp. 125–141, 1990.
[6] M. Fontana, "Topologically defined classes of commutative rings," *Ann. Mat. Pura Appl.*, vol. 123, no. 4, pp. 331–355, 1980.
[7] Fontana, M., Izelgue, L. and Kabbaj, S., "Quelques propriétés des chaines d'idéaux dans les anneaux $A + XB[X]$," *Commun. Algebra*, vol. 22, no. 1, pp. 9–27, 1994.
[8] Fontana, M., Izelgue, L. and Kabbaj, S., "Sur quelques propriétés des sous-anneaux de la forme $D + I$ d'un anneau intègre," *Commun. Algebra*, vol. 23, no. 11, pp. 4189–4210, 1995.
[9] R. Gilmer, *Multiplicative ideal theory*, ser. Pure and Applied Mathematics. New York: Marcel Dekker, Inc., 1972, vol. 12.
[10] I. Kaplansky, *Commutative Rings*. Chicago: University Press, 1974.

[11] I. Yengui, "Two counterexamples about the Nagata and Serre conjecture rings," *J. Pure Appl. Algebra*, vol. 153, no. 2, pp. 191–195, 2000.
[12] I. Yengui, "On questions related to saturated chains of primes in polynomial rings," *J. Pure Appl. Algebra*, vol. 178, no. 2, pp. 215–224, 2003.

Authors' addresses

Basma Gasmi
University of Sfax, Faculty of Sciences of Sfax, Department of Mathematics, P.O.Box: 1171, 3000 Sfax, Tunisia
E-mail address: basmagasmi@gmail.com

Noômen Jarboui
University of Sfax, Faculty of Sciences of Sfax, Department of Mathematics, P.O.Box: 1171, 3000 Sfax, Tunisia
E-mail address: noomenjarboui@yahoo.fr