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SOME RING THEORY FROM JENŐ SZIGETI

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Abstract. A selection of ring theory papers by Jenő Szigeti is reviewed with an emphasis on aspects related to matrix algebras.

The present overview concentrates on three areas of Szigeti's work. "Eulerian polynomial identities" deals essentially with polynomials in several non-commuting indeterminates corresponding to directed Eulerian graphs. "Lie nilpotent determinant theory" adapts to the non-commutative case the classical concepts of determinant, adjoint and characteristic polynomial to yield analogues of well known linear algebra results, especially over Lie nilpotent rings. "Centralizers and zero-level centralizers" is about some non-commutative extensions of theorems on centralizers and double centralizers in matrix algebras, with additional considerations of two-sided annihilators.

We aim at a condensed but self contained presentation of a selection of results.

1. EULERIAN POLYNOMIAL IDENTITIES

Let Γ be a directed graph with vertex set $V(\Gamma) = \{1,2,\ldots,k\}$ and edge set $E(\Gamma) = \{x_1, x_2, \ldots, x_N\}$. For each $1 \le r \le N$, x_r is an edge from $\sigma(r)$ to $\tau(r)$, both in $V(\Gamma)$. A directed Eulerian path which starts at p and ends at q is viewed as a permutation π of the edges (or rather their indices). The polynomial $P_\Gamma(X)$ in the set $X = E(\Gamma)$ of non-commuting indeterminates, induced by Γ, is defined as follows:

$$P_\Gamma(X) = \sum_{\pi \in \Pi(\Gamma,p,q)} \operatorname{sgn}(\pi) x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(N)},$$

where $\Pi(\Gamma, p, q)$ is the set of all directed Eulerian paths of Γ from p to q. The main result of [16] states that if $N \ge 2kn$, then $P_\Gamma(X) = 0$ is a polynomial identity on the $n \times n$ matrix ring $\mathrm{M}_n(\Omega)$, where Ω is an arbitrary commutative ring with 1. This is a broad generalization of the famous Amitsur-Levitzki theorem. Surprisingly, the above graph based construction led to essentially new identities even for 3×3

Dedicated to the occasion of Jenő Szigeti's 60th birthday.

matrices. The Capelli polynomials in [4] appear as Eulerian polynomials corresponding to a particular class of graphs, where the authors proved that the inequalities conditioning the corresponding matrix identities are sharp.

Assuming certain lower bounds on out degrees the sign free (permanental) version

$$Q_{\Gamma}(X) = \sum_{\pi \in \Pi(\Gamma,p,q)} x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(N)}$$

of $P_{\Gamma}(X)$ also yields the identity $Q_{\Gamma}(X) = 0$ on $\mathrm{M}_n(\Omega)$, where Ω is a commutative ring of characteristic d. The lower bounds assumed depend on both d and n (see [7]). A variation on these assumptions on the graph Γ which ensure the validity of the identity $Q_{\Gamma}(X) = 0$ on $\mathrm{M}_n(\Omega)$ can be found in [2].

It is worth to note that the proof of the matrix identity $P_{\Gamma}(X) = 0$ is based on the use of Swan's theorem on the signs of directed Eulerian paths, while the proof of $Q_{\Gamma}(X) = 0$ is based on a formula (due to Aardenne-Ehrenfest and de Bruijn) counting the number of directed Eulerian paths. A particularly concise form of the mentioned theorem of Swan is formulated in [3] using a special adjacency matrix over the Grassmann algebra.

The form of the polynomial $P_{\Gamma}(X)$ is enriched in [8] by an involution operator $*$ applied to indeterminates traversed in opposite direction in an undirected Eulerian path, providing identities involving matrix transposition also.

2. Lie nilpotent determinant theory

The following are the outlines of a new determinant theory developed in the series of papers [5, 6, 9–12, 14, 15, 18, 19].

For an $n \times n$ matrix $A = [a_{i,j}]$ over a not necessarily commutative ring (or K-algebra) R with 1, the element

$$\mathrm{sdet}(A) = \sum_{\alpha,\beta \in \mathrm{S}_n} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) a_{\alpha(1),\beta(1)} \cdots a_{\alpha(t),\beta(t)} \cdots a_{\alpha(n),\beta(n)}$$

of R is called the *symmetric determinant* of A. The (r,s) entry of the *symmetric adjoint* $A^* = [a^*_{i,j}]$ of A is the signed symmetric determinant $a^*_{r,s} = (-1)^{r+s}\mathrm{sdet}(A_{s,r})$ of the $(n-1) \times (n-1)$ minor $A_{s,r}$ of A arising from the deletion of the s-th row and the r-th column of A. If R is commutative, then $\mathrm{sdet}(A) = n!\det(A)$ and $A^* = (n-1)!\mathrm{adj}(A)$, where $\det(A)$ and $\mathrm{adj}(A)$ denote the ordinary determinant and adjoint of A.

The *right adjoint sequence* $(P_k)_{k \geq 1}$ of A is defined by the following recursion: $P_1 = A^*$ and $P_{k+1} = (AP_1 \cdots P_k)^*$ for $k \geq 1$. The *k-th right determinant* of A is the trace

$$\mathrm{rdet}_{(k)}(A) = \mathrm{tr}(AP_1 \cdots P_k)$$

and the *k-th right adjoint* of A is the product

$$\mathrm{radj}_{(k)}(A) = nP_1 \cdots P_k.$$

An important observation is that

$$\operatorname{rdet}_{(1)}(A) = \operatorname{tr}(AA^*) = \operatorname{sdet}(A) = \operatorname{tr}(A^*A).$$

For 2×2 and 3×3 matrices we have the following symmetric analogues of the Newton trace formulae : if $A \in \mathrm{M}_2(R)$, then

$$\operatorname{sdet}(A) = \operatorname{tr}^2(A) - \operatorname{tr}(A^2)$$

and if $A \in \mathrm{M}_3(R)$, then

$$\begin{aligned}\operatorname{sdet}(A) = {} & \operatorname{tr}^3(A) - [\operatorname{tr}(A)\cdot\operatorname{tr}(A^2) + \operatorname{tr}(A\cdot\operatorname{tr}(A)\cdot A) + \operatorname{tr}(A^2)\cdot\operatorname{tr}(A)] \\ & + [\operatorname{tr}(A^3) + \operatorname{tr}\left((A^\top)^3\right)],\end{aligned}$$

where $A^\top$ denotes the transpose of A. The term preceded by the negative sign can also be expressed more symmetrically, with the notation $\operatorname{tr}(A) = a$, as the trace of $aAA + AaA + AAa$.

The k-th *right characteristic polynomial* of A is the k-th right determinant of the $n\times n$ matrix $xI_n - A$ in $\mathrm{M}_n(R[x])$:

$$p_{A,k}(x) = \operatorname{rdet}_{(k)}(xI_n - A)$$

and $p_{A,k}(x)$ is of the form

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)}x + \cdots + \lambda_{n^k-1}^{(k)}x^{n^k-1} + \lambda_{n^k}^{(k)}x^{n^k},$$

where $\lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{n^k-1}^{(k)}, \lambda_{n^k}^{(k)} \in R$ and $\lambda_{n^k}^{(k)} = n\{(n-1)!\}^{1+n+n^2+\cdots+n^{k-1}}$. Under additional assumptions on A it is proved in a recent work [14] that the coefficients of $p_{A,k}(x)$ are in the fixed ring of a certain group of automorphisms of R.

If R satisfies the commutator identity

$$[[[\ldots[[x_1,x_2],x_3],\ldots],x_k],x_{k+1}] = 0$$

(i.e. if R is Lie nilpotent of index k), then the product $A\cdot\operatorname{radj}_{(k)}(A)$ is a scalar matrix

$$A\cdot\operatorname{radj}_{(k)}(A) = nAP_1\cdots P_k = \operatorname{rdet}_{(k)}(A)\cdot I_n$$

and the Cayley-Hamilton identity with right scalar coefficients

$$(A)p_{A,k} = I_n\lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0$$

holds for A. We also have $(A)u = 0$, where $u(x) = p_{A,k}(x)h(x)$ and $h(x) \in R[x]$ is arbitrary. Clearly, these results can be viewed as the index k Lie nilpotent version of classical determinant theory.

At this stage the application of the new theory culminates in the following theorem (see [14]). Let R be a Lie nilpotent algebra (over $\mathbb{Q}$) of index $k \geq 1$. If $\delta : R \longrightarrow R$ is an automorphism with $\delta^n = \mathrm{id}_R$, then the skew polynomial algebra $R[w,\delta]$ is right integral over $\mathrm{Fix}(\delta)[w^n]$ of degree n^k. In other words, for any $f(w) \in R[w,\delta]$ we have

$$g_0(w^n) + f(w)g_1(w^n) + \cdots + f^{n^k-1}(w)g_{n^k-1}(w^n) + f^{n^k}(w) = 0$$

for some $g_t(w^n) \in \mathrm{Fix}(\delta)[w^n]$, $0 \le t \le n^k - 1$.

In case of an arbitrary R and $A \in \mathrm{M}_n(R)$, we have

$$n(xI_n - A)(xI_n - A)^* = p_{A,1}(x)I_n + C_0 + C_1x + \cdots + C_nx^n,$$

where the matrices $C_i \in \mathrm{M}_n(R)$ are uniquely determined by A, $\mathrm{tr}(C_i) = 0$ and each entry of C_i is in the additive subgroup $[R,R]$ generated by the commutators, i.e. $C_i \in \mathrm{M}_n([R,R])$ for all $0 \le i \le n$. Now the identity

$$(\lambda_0^{(1)} I_n + C_0) + A(\lambda_1^{(1)} I_n + C_1) + \cdots + A^{n-1}(\lambda_{n-1}^{(1)} I_n + C_{n-1}) + A^n(n!I_n + C_n) = 0$$

with right matrix coefficients holds for A. Since commutativity of R would imply $C_0 = C_1 = \cdots = C_{n-1} = C_n = 0$, this is a direct right sided generalization of the classical Cayley-Hamilton theorem.

All the results above have a natural left sided version.

Matrix algebras over the infinitely generated Grassmann algebra, which is Lie nilpotent of index 2, are particularly in the purview of this non-commutative determinant theory. Polynomial identities satisfied by these matrix algebras play a central role in Kemer's classification of T-prime T-ideals.

3. Centralizers and zero level centralizers

For a linear transformation of a finite dimensional vector space over an algebraically closed field the existence of the Jordan normal base is well known in elementary linear algebra. An astonishing fact is that a far reaching generalization of this fundamental result can be formulated for nilpotent complete join homomorphisms of lattices (see [13]). The description of the centralizers and zero level centralizers (two sided annihilators) in Szigeti's papers ([1, 17]) depends on the use of the following remarkable consequence of this lattice theoretical Jordan normal base theorem.

A doubly indexed subset $X = \{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \le i \le k_\gamma\}$ of a (unitary) left R-module ${}_RM$ is called a *nilpotent Jordan normal base with respect to* $\varphi \in \mathrm{End}_R(M)$ if each R-submodule $Rx_{\gamma,i} \le M$ is simple,

$$\bigoplus_{\gamma \in \Gamma, 1 \le i \le k_\gamma} Rx_{\gamma,i} = M$$

is a direct sum, $\varphi(x_{\gamma,i}) = x_{\gamma,i+1}$, for all $\gamma \in \Gamma$, $1 \le i \le k_\gamma - 1$, $\varphi(x_{\gamma,k_\gamma}) = 0$ and the set $\{k_\gamma \mid \gamma \in \Gamma\}$ of integers is bounded (Γ is called the set of *Jordan blocks* and the *size* of the block $\gamma \in \Gamma$ is the integer k_γ). The existence of such Jordan normal base implies that $\varphi^n = 0 \ne \varphi^{n-1}$, where $n = \max\{k_\gamma \mid \gamma \in \Gamma\}$.

For $\varphi \in \mathrm{End}_R(M)$ the following are equivalent:
(i) ${}_RM$ is semisimple and φ is nilpotent.
(ii) ${}_RM$ has a nilpotent Jordan normal base with respect to φ.

The rest of the section follows literally the exposition in [1] and [17].

If R is a local ring with Jacobson radical J and ${}_RM$ is a finitely generated semisimple left R-module, then the centralizer $\mathrm{Cen}(\varphi)$ of a nilpotent $\varphi \in \mathrm{End}_R(M)$ is isomorphic to the opposite of the factor $\mathcal{N}(X)/\mathcal{I}(X)$ as an algebra over the center $\mathrm{Z}(R)$:

$$\mathrm{Cen}(\varphi) = \{\psi \in \mathrm{End}_R(M) \mid \psi \circ \varphi = \varphi \circ \psi\} \cong (\mathcal{N}(X)/\mathcal{I}(X))^{\mathrm{op}},$$

where

$$\mathcal{N}(X) = \begin{bmatrix} R[z] & R[z] & R[z] & \cdots & R[z] \\ J[z]+(z^{k_1-k_2}) & R[z] & R[z] & \cdots & R[z] \\ J[z]+(z^{k_1-k_3}) & J[z]+(z^{k_2-k_3}) & R[z] & \cdots & R[z] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J[z]+(z^{k_1-k_m}) & J[z]+(z^{k_2-k_m}) & J[z]+(z^{k_3-k_m}) & \cdots & R[z] \end{bmatrix}$$

is a subalgebra of $\mathrm{M}_m(R[z])$ and

$$\mathcal{I}(X) = \begin{bmatrix} J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m}) \\ J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m}) \\ \vdots & \vdots & \ddots & \vdots \\ J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m}) \end{bmatrix}$$

is an ideal of $\mathcal{N}(X)$. The number of blocks $m = \dim_R(\ker(\varphi))$, $\Gamma = \{1,2,\dots,m\}$ and the finite sequence of block sizes $k_1 \ge k_2 \ge \dots \ge k_m \ge 1$ in a nilpotent Jordan normal base $X = \{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \le i \le k_\gamma\}$ with respect to φ are independent of the choice of X and uniquely determined by φ.

For another not necessarily nilpotent endomorphism $\sigma \in \mathrm{End}_R(M)$, the containment $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ holds if and only if there is an R-generating set $\{y_j \in M \mid 1 \le j \le d\}$ of ${}_RM$ and there are elements $a_1, a_2, \dots, a_n \in R$ such that

$$a_1\psi(y_j) + a_2\varphi(\psi(y_j)) + \cdots + a_n\varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j))$$

for all $1 \le j \le d$ and all $\psi \in \mathrm{Cen}(\varphi)$.

Since $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ is equivalent to $\sigma \in \mathrm{Cen}(\mathrm{Cen}(\varphi))$, the last statement can be viewed as a non-commutative generalization of Schur's double centralizer theorem in the nilpotent case.

Keeping the above conditions and notations, the zero-level centralizer $\mathrm{Cen}_0(\varphi)$ of the nilpotent φ is isomorphic to the opposite of the factor $\mathcal{N}_0(X)/\mathcal{I}(X)$ as a $\mathrm{Z}(R)$-algebra:

$$\mathrm{Cen}_0(\varphi) = \{\psi \in \mathrm{End}_R(M) \mid \psi \circ \varphi = \varphi \circ \psi = 0\} \cong (\mathcal{N}_0(X)/\mathcal{I}(X))^{\mathrm{op}},$$

where

$$\mathcal{N}_0(X)=\begin{bmatrix} J[z]+(z^{k_1-1}) & J[z]+(z^{k_2-1}) & \cdots & J[z]+(z^{k_m-1}) \\ J[z]+(z^{k_1-1}) & J[z]+(z^{k_2-1}) & \cdots & J[z]+(z^{k_m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ J[z]+(z^{k_1-1}) & J[z]+(z^{k_2-1}) & \cdots & J[z]+(z^{k_m-1}) \end{bmatrix}$$

is a subalgebra of $\mathrm{M}_m(R[z])$, still containing $\mathcal{J}(X)$ as an ideal.

Now let $\varphi \in \mathrm{End}_R(M)$ be an arbitrary R-endomorphism of a finitely generated semisimple left R-module ${}_RM$. Then there exist R-submodules W_1, W_2 and V of M such that $W = W_1 \oplus W_2$ and $M = V \oplus W$ are direct sums, $\ker(\varphi) \subseteq W$, $\varphi(W) = W_2$, $\varphi(V) = V$, $\dim_R(W_1) = \dim_R(\ker(\varphi))$, $(\varphi \restriction W) \in \mathrm{End}_R(W)$ is nilpotent and for the zero-level centralizer of φ we have $\mathrm{Cen}_0(\varphi) \cong T$, where

$$T = \{\theta \in \mathrm{End}_R(W) \mid \theta(W_1) \subseteq \ker(\varphi) \text{ and } \theta(W_2) = \{0\}\} = \mathrm{Cen}_0(\varphi \restriction W).$$

The double zero-centralizer theorem states that for another endomorphism $\sigma \in \mathrm{End}_R(M)$ the following conditions are equivalent:
(i) $\sigma \in \mathrm{Cen}_0(\mathrm{Cen}_0(\varphi))$,
(ii) $\mathrm{Cen}_0(\varphi) \subseteq \mathrm{Cen}_0(\sigma)$,
(iii) $\ker(\varphi) \subseteq \ker(\sigma)$ and $\mathrm{im}(\sigma) \subseteq \mathrm{im}(\varphi)$.

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