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# Nonlocal and integral conditions problems for a multi-term fractional-order differential equation

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# NONLOCAL AND INTEGRAL CONDITIONS PROBLEMS FOR A MULTI-TERM FRACTIONAL-ORDER DIFFERENTIAL EQUATION

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*Abstract.* In this paper we study the existence of solution for a multi-term arbitrary (fractional) order differential equation with some nonlocal and integral conditions.

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*Keywords:* fractional calculus, nonlocal condition, integral condition, fractional differential equation

## 1. INTRODUCTION

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1,2,5] and references therein. In this work we study the existence of at least one absolutely continuous solution $x \in AC[0,1]$ for the nonlocal problem of the arbitrary (fractional) order differential equation

$$x'(t) = f(t, D^{\alpha_1}x(t), D^{\alpha_2}x(t), \cdots, D^{\alpha_n}x(t)), \alpha_i \in (0, 1), \text{ a.e. } t \in (0, 1) \quad (1.1)$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k\, x(\tau_k) = \beta \sum_{j=1}^{p} b_j\, x(\eta_j) \quad (1.2)$$

where $a_k,\ b_j > 0,\ \tau_k \in (a,\ c),\ \eta_j \in (d,\ b),\ 0 < a < c \leq d < b < 1$, $\sum_{k=1}^{m} a_k \neq \beta \sum_{j=1}^{p} b_j$ and $\beta$ is parameter.
As an application, we deduce the existence of solution for the nonlocal problem of the differential equation (1.1) with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0, \quad \tau_k \in (a,c) \subset (0,1), \quad (1.3)$$

and the nonlocal integral conditions

$$\int_a^c x(s)\,ds = \beta \int_d^b x(s)\,ds,\ 0 < a < c \le d < b < 1, \tag{1.4}$$

and

$$\int_a^c x(s)\,ds = 0,\ (a,c) \subset (0,1). \tag{1.5}$$

are also considered.

## 2. PRELIMINARIES

Let $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [0,1]$,where $0 \le a < b < \infty$ and let $\Gamma(.)$ denotes the gamma function. Recall that the operator $T$ is compact if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators from the subspace $U \subset X$ into the Banach space $X$ is denoted by $C(U, X)$.

**Definition 1.** The fractional-order integral of the function $f \in L_1[a,b]$ of order $\beta > 0$ is defined by (see [7])

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,ds.$$

**Definition 2.** The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as (see [6] and [7])

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)\,ds.$$

The following theorems will be needed.

**Theorem 1** (Schauder fixed point theorem [3])**.** *Let $E$ be a Banach space and $Q$ be a convex subset of $E$, and $T : Q \longrightarrow Q$ is compact, continuous map, Then $T$ has at least one fixed point in $Q$.*

**Theorem 2** (Kolmogorov compactness criterion [4])**.** *Let $\Omega \subseteq L^p\ (0,1)$, $1 \le p < \infty$. If*

(i) *$\Omega$ is bounded in $L^p\ (0,1)$, and*

(ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$, then $\Omega$ is relatively compact in $L^p\ (0,1)$, where*

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)\,ds.$$

## 3. Main results

Consider firstly the fractional-order functional integral equation

$$y(t) = f(t,\ I^{1-\alpha_1} y(t), \cdots,\ I^{1-\alpha_n} y(t)), \tag{3.1}$$

**Definition 3.** The function $y$ is called a solution of the fractional-order functional integral equation (3.1), if $y \in L_1[0,1]$ and satisfies (3.1).

Consider the following assumption:

(i) $f : [0,1] \times R_n \to R$ be a function with the following properties:
  (a) for each $t \in [0,1]$, $f(t,.)$ is continuous,
  (b) for each $x \in R_n$, $f(.,x)$ is measurable,

(ii) there exist an integral function $a$, $a \in L_1[0,1]$ and constants $q_i > 0$, $i = 1, 2,$ such that

$$|f(t,x)| \leq a(t) + \sum_{i=1}^{n} q_i\ |x_i|, \quad \text{for each } t \in [0,1], \quad x \in R_n,$$

**Theorem 3.** *Let the assumptions (i) and (ii) be satisfied.*

$$\textit{If} \qquad \sum_{i=1}^{n} \frac{q_i}{\Gamma(2-\alpha_i)} < 1, \tag{3.2}$$

*then the fractional-order functional integral equation (3.1) has at least one solution* $y \in L_1[0,1]$, *where*

$$r \leq \frac{\|a\|}{1 - \sum_{i=1}^{n} \frac{q_i}{\Gamma(2-\alpha_i)}}$$

*Proof.* Define the operator $T$ associated with equation (3.1) by

$$Ty(t) = f(t,\ I^{1-\alpha_1} y(t), \cdots,\ I^{1-\alpha_n} y(t))$$

Let $B_r = \{y \in L_1(I) : \|y\| < r,\ r > 0\}$ and let $y$ be an arbitrary element in $B_r$. Then from the assumptions (i) and (ii), we obtain

$$\begin{aligned}
||Ty||_{L_1} &= \int_0^1 |Ty(t)|\, dt \\
&\leq \int_0^1 |f(t,\ I^{1-\alpha_1} y(t), \cdots,\ I^{1-\alpha_n} y(t))| dt \\
&\leq \int_0^1 |a(t)| dt + \sum_{i=1}^{n} q_i \int_0^1 \int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} |y(s)|\, ds\, dt \\
&\leq \|a\| + \sum_{i=1}^{n} q_i \int_0^1 \int_s^1 \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} dt |y(s)|\, ds
\end{aligned}$$

$$\leq \|a\| + \sum_{i=1}^{n} q_i \int_0^1 -\frac{(t-s)^{1-\alpha_i}}{(1-\alpha_i)\Gamma(1-\alpha_i)}|_s^1 |y(s)|\, ds$$

$$\leq \|a\| + \sum_{i=1}^{n} q_i \int_0^1 \frac{1}{\Gamma(2-\alpha_i)}|y(s)|\, ds$$

$$\leq \|a\| + \sum_{i=1}^{n} \frac{q_i}{\Gamma(2-\alpha_i)} \|y\|_{L_1} \leq r,$$

which implies that the operator $T$ maps $B_r$ into itself.
Assumption (i) implies that $T$ is continuous.
Now, we will show that $T$ is compact, applying Theorem 1. So, let $\Omega$ be a bounded subset of $B_r$. Then $T(\Omega)$ is bounded in $L_1[0,1]$, i.e. condition (i) of Theorem 2 is satisfied. It remains to show that $(Ty)_h \to Ty$ in $L_1[0,1]$ as $h \to 0$, uniformly with respect to $Ty \in T\,\Omega$. Now

$$||(Ty)_h - Ty|| = \int_0^1 |(Ty)_h(t) - (Ty)(t)|\, dt$$
$$= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Ty)(s)\, ds - (Ty)(t)|\, dt$$
$$\leq \int_0^1 \left( \frac{1}{h}\int_t^{t+h} |(Ty)(s) - (Ty)(t)|\, ds \right) dt$$
$$\leq \int_0^1 -\frac{1}{h}\int_t^{t+h} |f(s, I^{1-\alpha_1}y(t), ..., I^{1-\alpha_n}y(t))$$
$$- f(t, I^{1-\alpha_1}y(t), ..., I^{1-\alpha_n}y(t))|\, ds\, dt.$$

Now, $y \in \Omega$ implies (by assumption (ii)) that $f \in L_1(0,1)$, then

$$\frac{1}{h} - \int_t^{t+h} -|f(s, I^{1-\alpha_1}y(t), \cdots, I^{1-\alpha_n}y(t)) -$$
$$- f(t, I^{1-\alpha_1}y(t), \cdots, I^{1-\alpha_n}y(t))|ds \to 0$$

Therefore, by Theorem 2, we have that $T(\Omega)$ is relatively compact, that is, $T$ is a compact operator, then the operator $T$ has a fixed point $B_r$, which proves the existence of a positive solution $y \in (0,\ 1)$ of equation (3.1). □

**Theorem 4.** *Let the assumptions of Theorem 3 be satisfied. Then the nonlocal problem (1.1)- (1.2) has at least one solution* $x \in AC[0,1]$.

*Proof.* Consider the nonlocal fractional differential equation

$$x'(t) \ = \ f(t,\ D^{\alpha_1}x(t),\ D^{\alpha_2}x(t),\ \cdots,\ D^{\alpha_n}x(t))\ ,\alpha_i \in (0,\ 1),\ \text{a.e. } t \in (0,\ 1),$$

$$\sum_{k=1}^{m} a_k\ x(\tau_k) \ = \ \beta \sum_{j=1}^{p} b_j\ x(\eta_j), \qquad a_k,\ b_j > 0,\ \tau_k \in (0,\ c),\ \eta_j \in (d,\ 1),\ c \leq\ d.$$

Let $y(t) \ = \ x'(t),$ then

$$x(t) \ = \ x(0) \ + \ I\,y(t) \tag{3.3}$$

and $y$ is the solution of the fractional-order integral equation (3.1).
Let $t = \tau_k$ in equation (3.3), we get

$$x(\tau_k) = \int_0^{\tau_k} y(s)\ ds + x(0)$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + x(0) \sum_{k=1}^{m} a_k$$

And let $t = \eta_j$ in equation (3.3), we get

$$x(\eta_j) = \int_0^{\eta_j} y(s)\ ds + x(0)$$

$$\sum_{j=1}^{p} b_j x(\eta_j) = \sum_{j=1}^{p} bj \int_0^{\eta_j} y(s)\ ds + x(0) \sum_{j=1}^{p} b_j$$

From equation (1.2), we get

$$\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + x(0) \sum_{k=1}^{m} a_k = \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\ ds + x(0)\beta \sum_{j=1}^{p} b_j$$

Then we get

$$x(0) = A \left( \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\ ds \right),$$

where $A = (\beta\ \sum_{j=1}^{p} b_j - \sum_{k=1}^{m}\ a_k)^{-1}$
and

$$x(t) \ = A \left( \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\ ds \right) + \int_0^{t} y(s)\ ds \tag{3.4}$$

which, by Theorem 3, has at least one solution $x \in AC(0,1)$.
Now, from equation (3.4), we have

$$x(0) = \lim_{t \to 0^+} x(t) = A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - A\beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\, ds.$$

Also

$$x(1) = \lim_{t \to 1^-} x(t) = A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - A\beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\, ds$$
$$+ \int_0^1 y(s)\, ds$$

from which we deduce that equation (3.4) has at least one solution $x \in AC[0,1]$.
To complete the proof

$$\frac{dx}{dt} = y(t),$$
$$D^{\alpha_i} x(t) = I^{1-\alpha_i} \frac{d}{dt} x(t) = I^{1-\alpha_i} y(t)$$

where

$$x'(t) = f(t,\ x(t),\ D^{\alpha_1} x(t),\ D^{\alpha_2} x(t),\ \cdots,\ D^{\alpha_n} x(t)).$$

□

Now letting $\beta = 0$ in (1.2), we can easily prove the following theorem.

**Theorem 5.** *Let the assumptions (i) - (ii) be satisfied. Then the nonlocal problem*

$$x'(t) = f(t,\ D^{\alpha_1} x(t),\ D^{\alpha_2} x(t),\ \cdots,\ D^{\alpha_n} x(t)),\alpha_i \in (0,\ 1),\ a.e.\ t \in (0,\ 1),$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0,\ \ \tau_k \in (a,c) \subset (0,1).$$

*has at least one solution* $x \in AC[0,1]$ *represented by*

$$x(t) = A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - \int_0^t y(s)\, ds,\ \ where\ \ A = (\sum_{k=1}^{m} a_k)^{-1}.$$

## 4. Nonlocal integral condition

Let $x \in AC[0,1]$ be the solution of the nonlocal problem (1.1-1.2).
Let $a_k = \tau_k - \tau_{k-1},\ \ t_k \in (\tau_{k-1}, \tau_k),\ a = \tau_0 < \tau_1 < \tau_2, \ldots < \tau_m = c$ and $b_j = \eta_j - \eta_{j-1},\ \ t_j \in (\eta_{j-1}, \eta_j),\ d = \eta_0 < \eta_1 < \eta_2, \ldots < \eta_p = b$ then the nonlocal condition (1.2) will be

$$\sum_{k=1}^{m} (\tau_k - \tau_{k-1})\, x(t_k) = \beta \sum_{j=1}^{p} (\eta_j - \eta_{j-1})\, x(t_j).$$

From the continuity of the solution $x$ of the nonlocal problem (1.1-1.2) we can obtain

$$\lim_{m\to\infty}\sum_{k=1}^{m}(\tau_k-\tau_{k-1})\,x(t_k) \;=\; \beta\lim_{p\to\infty}\sum_{j=1}^{p}(\eta_j-\eta_{j-1})\,x(t_j).$$

and the nonlocal condition (1.2) is transformed to the integral one

$$\int_a^c x(s)\,ds \;=\; \beta\int_d^b x(s)\,ds\,. \tag{4.1}$$

Also from the continuity of the function $Iy(t)$, where $y$ is the solution of the functional integral equation (3.1), we deduce that the solution (3.4) will be

$$x(t) = (\beta\,(b-d)-(c-a))^{-1}\left(\int_a^c\int_0^s y(\theta)d\theta\,ds-\beta\int_d^b\int_0^s y(\theta)\,d\theta\,ds\right)$$
$$+\int_0^t y(s)ds.$$

Now, we have the following theorem.

**Theorem 6.** *Let the assumptions of Theorem 4 be satisfied. Then there exist at least one solution $x\in AC[0,1]$ of the nonlocal problem with integral condition,*

$$x'(t) \;=\; f(t,\,D^{\alpha_1}x(t),\,D^{\alpha_2}x(t),\,\cdots,\,D^{\alpha_n}x(t))\,,\alpha_i\in(0,\,1),\;a.e.\;t\in(0,\,1),$$

$$\int_a^c x(s)\,ds \;=\; \beta\int_d^b y(s)\,ds\,,\;0\le\,a<c\,\le\,d\,<b\le\,1,\;\;\beta\,(b-d)\neq(c-a).$$

Letting $\beta\;=\;0$ in (4.1), we can easily prove the following corollary.

**Corollary 1.** *Let the assumptions (i) - (ii) be satisfied. Then the nonlocal problem*

$$x'(t) \;=\; f(t,\,D^{\alpha_1}x(t),\,D^{\alpha_2}x(t),\,\cdots,\,D^{\alpha_n}x(t))\,,\alpha_i\in(0,\,1),\;a.e.\;t\in(0,\,1),$$

$$\int_a^c x(s)\,ds \;=\; 0,\quad(a,c)\subset(0,1),$$

*has at least one solution $x\in AC[0,1]$ represented by*

$$x(t) \;=\; \int_0^t y(s)\,ds-(c-a)^{-1}\int_a^c\int_0^s y(\theta)\,d\theta\,ds.$$

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