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SOME GENERALIZATIONS ON THE UNIVALENCE OF AN INTEGRAL OPERATOR AND QUASICONFORMAL EXTENSIONS

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Abstract. By using the method of Loewner chains, we establish some sufficient conditions for the analyticity and univalency of functions defined by an integral operator. Also, we refine the result to a quasiconformal extension criterion with the help of Beckers's method.

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1. INTRODUCTION

Let $\mathcal{A}$ the class of functions f which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. We denote by $\mathcal{U}_r$ the open disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$, by $\mathcal{U} = \mathcal{U}_1$ the open unit disk of the complex plane and by I the interval $[0, \infty)$.

Let k be constant in $[0,1)$. Then a homeomorphism f of $G \subset \mathbb{C}$ is said to be $k-$*quasiconformal,* if $\partial_z f$ and $\partial_{\overline{z}} f$ in the distributional sense are locally integrable on G and fulfill the inequality $|\partial_{\overline{z}} f| \leq k\, |\partial_z f|$ almost everywhere in G. If we do not need to specify k, we will simply call f *quasiconformal.*

Three of the most important and known univalence criteria for analytic functions defined in the open unit disk were obtained by Nehari [14], Ozaki-Nunokawa [17] and Becker [3]. Some extensions of these three criteria were given by [15,16,21–25]. Furthermore a lot of univalence criteria have been obtained by different authors (see also [7–9]).

In the present investigation, we will obtain a number of new criteria for the functions defined by the integral operator $\mathcal{F}_\beta(z)$. Also, we obtain a refinement to a quasiconformal extension criterion of the main result.

2. PRELIMINARIES

Before proving our main theorem we present a brief summary of the method of Loewner chains and quasiconformal extension criterion.

A function $\mathcal{L}(z,t):\mathcal{U}\times[0,\infty)\to\mathbb{C}$ is said to be *subordination chain (or Loewner chain)* if:

(i) $\mathcal{L}(z,t)$ is analytic and univalent in $\mathcal{U}$ for all $t\geq 0$.
(ii) $\mathcal{L}(z,t)\prec\mathcal{L}(z,s)$ for all $0\leq t\leq s<\infty$, where the symbol " $\prec$ " stands for subordination.

To prove our results, we will need the following theorem due to Ch. Pommerenke [20].

Theorem 1. *Let $\mathcal{L}(z,t)=a_1(t)z+a_2(t)z^2+\ldots,\ a_1(t)\neq 0$ be analytic in $\mathcal{U}_r$ for all $t\in I$, locally absolutely continuous in I, and locally uniform with respect to $\mathcal{U}_r$. For almost all $t\in I$, suppose that*

$$z\frac{\partial\mathcal{L}(z,t)}{\partial z}=p(z,t)\frac{\partial\mathcal{L}(z,t)}{\partial t},\ \forall z\in\mathcal{U}_r \tag{2.1}$$

where $p(z,t)$ is analytic in $\mathcal{U}$ and satisfies the condition $\Re p(z,t)>0$ for all $z\in\mathcal{U},\ t\in I$. If $|a_1(t)|\to\infty$ for $t\to\infty$ and $\{\mathcal{L}(z,t)/a_1(t)\}$ forms a normal family in $\mathcal{U}_r$, then for each $t\in I$, the function $\mathcal{L}(z,t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$.

The method of constructing quasiconformal extension criteria is based on the following result of Becker (see [3], [4] and also [5]).

Theorem 2. *Suppose that $\mathcal{L}(z,t)$ is a Loewner chain for which the function $p(z,t)$ given in (2.1) satisfies the condition*

$$\begin{aligned} p(z,t)\in\mathcal{U}(k) &:= \left\{w\in\mathbb{C}:\left|\frac{w-1}{w+1}\right|\leq k\right\} \\ &= \left\{w\in\mathbb{C}:\left|w-\frac{1+k^2}{1-k^2}\right|\leq\frac{2k}{1-k^2}\right\},\quad(0\leq k<1) \end{aligned}$$

for all $z\in\mathcal{U}$ and $t\geq 0$. Then $\mathcal{L}(z,t)$ admits a continuous extension to $\overline{\mathcal{U}}$ for each $t\geq 0$ and the function $F(z,\overline{z})$ defined by

$$F(z,\overline{z})=\begin{cases}\mathcal{L}(z,0), & \text{if } |z|<1\\ \mathcal{L}(\frac{z}{|z|},\log|z|), & \text{if } |z|\geq 1\end{cases}$$

is a $k-$quasiconformal extension of $\mathcal{L}(z,0)$ to $\mathbb{C}$.

Examples of quasiconformal extension criteri**a** can be found in [1], [2], [6], [13], [19] and more recently in [10–12].

3. Main Results

In this section, using Theorem 1, we obtain certain sufficient conditions for the univalence of an integral operator.

Theorem 3. *Let m be a positive real number and let α, β be complex numbers such that $\Re\alpha < 1/2$, $\Re\beta > 0$ and $f \in \mathcal{A}$. Let g and h be two analytic functions in $\mathcal{U}$, $g(z) = 1 + b_1 z + \ldots$, $h(z) = c_0 + c_1 z + \ldots$. If the following inequalities*

$$\left| \frac{f'(z)}{g(z)-\alpha} - \frac{m-1}{2} \right| < \frac{m+1}{2}, \tag{3.1}$$

and

$$\Bigg| \left(\frac{f'(z)}{g(z)-\alpha} - 1 \right) |z|^{\beta(m+1)} + \left(1 - |z|^{\beta(m+1)}\right) \left[2z^{\beta} \frac{f'(z)h(z)}{g(z)-\alpha} + \frac{1}{\beta} \frac{zg'(z)}{g(z)-\alpha} \right]$$
$$+ \frac{z^{\beta+1}\left(1-|z|^{\beta(m+1)}\right)^2}{|z|^{\beta(m+1)}} \left[\frac{z^{\beta-1} f'(z) h^2(z)}{g(z)-\alpha} + \frac{1}{\beta}\left(\frac{g'(z)h(z)}{g(z)-\alpha} - h'(z) \right) \right] - \frac{m-1}{2} \Bigg|$$
$$\leq \frac{m+1}{2} \tag{3.2}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by

$$\mathcal{F}_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\,du \right]^{1/\beta} \tag{3.3}$$

is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. We shall prove that there exists a real number r, $r \in (0,1]$ such that the function $\mathcal{L} : \mathcal{U}_r \times I \to \mathbb{C}$, defined formally by

$$\mathcal{L}(z,t) = \left[\beta \int_0^{e^{-t}z} u^{\beta-1} f'(u)\,du + \frac{\left(e^{\beta m t} - e^{-\beta t}\right) z^{\beta} \left(g\left(e^{-t}z\right) - \alpha\right)}{1 + \left(e^{\beta m t} - e^{-\beta t}\right) z^{\beta} h\left(e^{-t}z\right)} \right]^{1/\beta} \tag{3.4}$$

is analytic in $\mathcal{U}_r$ for all $t \in I$.

Because $f \in \mathcal{A}$ we have

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \quad \forall z \in \mathcal{U}.$$

Let us denote by

$$\varphi_1(z,t) = \beta \int_0^{e^{-t}z} u^{\beta-1} f'(u)\,du. \tag{3.5}$$

We obtain $\varphi_1(z,t) = \left(e^{-t}z\right)^{\beta} + \frac{2\beta a_2}{\beta+1}\left(e^{-t}z\right)^{\beta+1} + \ldots$ and we observe that

$$\varphi_1(z,t) = z^{\beta} \varphi_2(z,t) \tag{3.6}$$

where

$$\varphi_2(z,t) = e^{-\beta t} + \sum_{n=2}^{\infty} \frac{n\beta}{n+\beta-1} a_n e^{-(n+\beta-1)t} z^{n-1}. \tag{3.7}$$

The function φ_2 is analytic in $\mathcal{U}$ for all $t \in I$, since

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left| \frac{n\beta}{n+\beta-1} a_n e^{-(n+\beta-1)t} \right|} = e^{-t} \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}.$$

It is clear that if $z \in \mathcal{U}$, then $e^{-t}z \in \mathcal{U}$ for all $t \in I$ and because $f'(0) = 1$, there exists a disk $\mathcal{U}_{r_1}$, $0 < r_1 \le 1$ in which $f'(e^{-t}z) \neq 0$ for all $t \ge 0$.

From the analyticity of f it follows that the function φ_3 is also analytic in $\mathcal{U}_{r_1}$, where

$$\varphi_3(z,t) = 1 + \left(e^{\beta m t} - e^{-\beta t}\right) z^{\beta} h\left(e^{-t}z\right). \tag{3.8}$$

We have $\varphi_3(0,t) = 1$ and then there exists a disk $\mathcal{U}_{r_2}$, $0 < r_2 \le r_1$ in which $\varphi_3(z,t) \neq 0$ for all $t \ge 0$.

Then the function

$$\varphi_4(z,t) = \varphi_2(z,t) + \left(e^{\beta m t} - e^{-\beta t}\right) \frac{\left(g\left(e^{-t}z\right) - \alpha\right)}{\varphi_3(z,t)} \tag{3.9}$$

is also analytic in $\mathcal{U}_{r_2}$ and $\varphi_4(0,t) = (1-\alpha)e^{\beta m t} + \alpha e^{-\beta t}$. From $\Re\alpha < 1/2$, $\Re\beta > 0$ we deduce that $\varphi_4(0,t) \neq 0$ for all $t \in I$. Therefore, there exists a disk $\mathcal{U}_r$, $0 < r \le r_2$ in which $\varphi_4(0,t) \neq 0$ for all $t \in I$ and we can choose an analytic branch of $[\varphi_4(z,t)]^{1/\beta}$, denoted by $\varphi_5(z,t)$. We choose the uniform branch which is equal to $a_1(t) = \left[(1-\alpha)e^{\beta m t} + \alpha e^{-\beta t}\right]^{1/\beta}$ at the origin, and for $a_1(t)$ we get $\lim_{t\to\infty} |a_1(t)| = \infty$. Moreover, we have $a_1(t) \neq 0$ for all $t \ge 0$.

From (3.4)-(3.9) it follows that the relation (3.4) can be written as

$$\mathcal{L}(z,t) = z\varphi_5(z,t) \tag{3.10}$$

and hence we obtain that the function $\mathcal{L}(z,t)$ is analytic in $\mathcal{U}_r$,

$$\mathcal{L}(z,t) = a_1(t)z + \ldots, \ \forall z \in \mathcal{U}_r, \ \forall t \in I.$$

$\mathcal{L}(z,t)$ is an analytic function in $\mathcal{U}_r$ for all $t \in I$ and then it follows that there is a number r_3, $0 < r_3 < r$ and a positive constant $K = K(r_3)$ such that

$$\left| \frac{\mathcal{L}(z,t)}{a_1(t)} \right| < K, \ \forall z \in \mathcal{U}_{r_3}, \ t \ge 0.$$

Then, by Montel's theorem, it follows that $\left\{ \frac{\mathcal{L}(z,t)}{a_1(t)} \right\}_{t \ge 0}$ is a normal family in $\mathcal{U}_{r_3}$.

From (3.10) we have

$$\frac{\partial \mathcal{L}(z,t)}{\partial t} = z \frac{\partial \varphi_5(z,t)}{\partial t}. \tag{3.11}$$

It is clear that $\frac{\partial \varphi_5(z,t)}{\partial t}$ is an analytic function in $\mathcal{U}_{r_3}$ and then $\frac{\partial \mathcal{L}(z,t)}{\partial t}$ is also an analytic function in $\mathcal{U}_{r_3}$. Then, for all fixed numbers $T > 0$ and r_4, $0 < r_4 < r_3$, there exists a constant $K_1 > 0$ (which depends on T and r_4) such that

$$\left|\frac{\partial \mathcal{L}(z,t)}{\partial t}\right| < K_1, \ \forall z \in \mathcal{U}_{r_4} \text{ and } t \in [0,T].$$

Therefore, the function $\mathcal{L}(z,t)$ is locally absolutely continuous in $[0,\infty)$ and is locally uniform with respect to $\mathcal{U}_{r_4}$.

Since $\frac{\partial \mathcal{L}(z,t)}{\partial t}$ is analytic in $\mathcal{U}_{r_4}$, from (3.11) it follows that there is a number r_0, $0 < r_0 < r_4$, such that $\frac{1}{z}\frac{\partial \mathcal{L}(z,t)}{\partial t} \neq 0$, $\forall z \in \mathcal{U}_{r_0}$, so the function

$$p(z,t) = z\frac{\partial \mathcal{L}(z,t)}{\partial z}\bigg/\frac{\partial \mathcal{L}(z,t)}{\partial t}$$

is analytic in $\mathcal{U}_{r_0}$ for all $t \geq 0$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in $\mathcal{U}$ for all $t \geq 0$, it is sufficient to prove that the function $w(z,t)$ defined in $\mathcal{U}_{r_0}$ by

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}$$

can be extended analytically in $\mathcal{U}$, $|w(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \geq 0$.

After some calculations we obtain:

$$w(z,t) = \frac{2}{m+1}\mathcal{G}(z,t) - \frac{m-1}{m+1}, \tag{3.12}$$

where

$$\begin{aligned}
\mathcal{G}(z,t) = {} & e^{-\beta(m+1)t}\left(\frac{f'(e^{-t}z)}{g(e^{-t}z)-\alpha} - 1\right) \\
& + \left(1 - e^{-\beta(m+1)t}\right)\left[2e^{-\beta t}z^{\beta}\frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)-\alpha} + \frac{e^{-t}z}{\beta}\frac{g'(e^{-t}z)}{g(e^{-t}z)-\alpha}\right] \\
& + \frac{e^{-\beta t}z^{\beta}\left(1-e^{-\beta(m+1)t}\right)^2}{e^{-\beta(m+1)t}} \\
& \times\left[e^{-\beta t}z^{\beta}\frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z)-\alpha}\right. \\
& \left. + \frac{e^{-t}z}{\beta}\left(\frac{h(e^{-t}z)g'(e^{-t}z)}{g(e^{-t}z)-\alpha} - h'(e^{-t}z)\right)\right].
\end{aligned} \tag{3.13}$$

for $z \in \mathcal{U}$ and $t \geq 0$.

The inequality $|w(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \geq 0$, where $w(z,t)$ defined by (3.12), is equivalent to

$$\left|\mathcal{G}(z,t) - \frac{m-1}{2}\right| < \frac{m+1}{2}, \ \forall z \in \mathcal{U} \text{ and } t \geq 0. \tag{3.14}$$

Define

$$\mathcal{H}(z,t) = \mathcal{G}(z,t) - \frac{m-1}{2}, \ \forall z \in \mathcal{U} \text{ and } t \geq 0. \tag{3.15}$$

In view of (3.1) and (3.2), from (3.13) and (3.15) we have

$$|\mathcal{H}(z,0)| = \left|\left(\frac{f'(z)}{g(z)-\alpha} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2}. \tag{3.16}$$

Let $t > 0$, $z \in \mathcal{U} - \{0\}$. In this case the function $\mathcal{H}(z,t)$ is analytic in $\overline{\mathcal{U}}$ because $\left|e^{-t}z\right| \leq e^{-t} < 1$, for all $z \in \overline{\mathcal{U}}$. Using the maximum principle for $z \in \mathcal{U}$ and $t > 0$ we have

$$|\mathcal{H}(z,t)| < \max_{|\xi|=1} |\mathcal{H}(\xi,t)| = \left|\mathcal{H}(e^{i\theta},t)\right|,$$

where $\theta = \theta(t)$ is a real number.

Let $u = e^{-t}e^{i\theta}$. We have $|u| = e^{-t}$ and $e^{-\beta(m+1)t} = \left(e^{-t}\right)^{\beta(m+1)} = |u|^{\beta(m+1)}$. From (3.13), we have

$$\begin{aligned}
\left|\mathcal{G}(e^{i\theta},t)\right| = \Bigg| & |u|^{\beta(m+1)}\left(\frac{f'(u)}{g(u)-\alpha} - 1\right) \\
& + \left(1 - |u|^{\beta(m+1)}\right)\left[\frac{2u^{\beta} f'(u)h(u)}{g(u)-\alpha} + \frac{u}{\beta}\frac{g'(u)}{g(u)-\alpha}\right] \\
& + \frac{u^{\beta}\left(1-|u|^{\beta(m+1)}\right)^2}{|u|^{\beta(m+1)}} \\
& \times \left[\frac{u^{\beta} f'(u)h^2(u)}{g(u)-\alpha} + \frac{u}{\beta}\left(\frac{h(u)g'(u)}{g(u)-\alpha} - h'(u)\right)\right] - \frac{m-1}{2}\Bigg|.
\end{aligned}$$

Since $u \in \mathcal{U}$, the inequality (3.2) implies that

$$\left|\mathcal{H}(e^{i\theta},t)\right| \leq \frac{m+1}{2}, \tag{3.17}$$

and from (3.16) and (3.17) it follows that the inequality (3.14)

$$|\mathcal{H}(z,t)| = \left|\mathcal{G}(z,t) - \frac{m-1}{2}\right| < \frac{m+1}{2}$$

is satisfied for all $z \in \mathcal{U}$ and $t \in I$. Therefore $|w(z,t)| < 1$, for all $z \in \mathcal{U}$ and $t \geq 0$.

Since all the conditions of Theorem 1 are satisfied, we obtain that the function $\mathcal{L}(z,t)$ has an analytic and univalent extension to the whole unit disk $\mathcal{U}$, for all $t \in I$.

For $t = 0$ we have $\mathcal{L}(z,0) = \mathcal{F}_\beta(z)$, for $z \in \mathcal{U}$ and therefore, the function $\mathcal{F}_\beta(z)$ is analytic and univalent in $\mathcal{U}$. □

For $g = f'$ in Theorem 3, we obtain another univalence criterion as follows.

Corollary 1. *Let m be a positive real number and let α, β be complex numbers such that $\Re\alpha < 1/2$, $\Re\beta > 0$ and $f \in \mathcal{A}$. Let h be an analytic functions in $\mathcal{U}$, $h(z) = c_0 + c_1 z + \ldots$. If the following inequalities*

$$\left|\frac{f'(z)}{f'(z)-\alpha} - \frac{m+1}{2}\right| < \frac{m+1}{2}, \tag{3.18}$$

and

$$\Bigg|\left(\frac{f'(z)}{f'(z)-\alpha}-1\right)|z|^{\beta(m+1)} + \left(1-|z|^{\beta(m+1)}\right)\left[2z^\beta \frac{f'(z)h(z)}{f'(z)-\alpha} + \frac{1}{\beta}\frac{zf''(z)}{f'(z)-\alpha}\right]$$

$$+\frac{z^{\beta+1}\left(1-|z|^{\beta(m+1)}\right)^2}{|z|^{\beta(m+1)}}\left[\frac{z^{\beta-1}f'(z)h^2(z)}{f'(z)-\alpha} + \frac{1}{\beta}\left(\frac{f''(z)h(z)}{f'(z)-\alpha} - h'(z)\right)\right] \tag{3.19}$$

$$-\frac{m-1}{2}\Bigg| \le \frac{m+1}{2} \tag{3.20}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

If we choose $h = f''$ in Corollary 1, we have another univalence criterion as follows.

Corollary 2. *Let m be a positive real number and let α, β be complex numbers such that $\Re\alpha < 1/2$, $\Re\beta > 0$ and $f \in \mathcal{A}$. Let h be an analytic functions in $\mathcal{U}$, $h(z) = c_0 + c_1 z + \ldots$. If the following inequalities*

$$\left|\frac{f'(z)}{f'(z)-\alpha} - \frac{m+1}{2}\right| < \frac{m+1}{2}, \tag{3.21}$$

and

$$\Bigg|\left(\frac{f'(z)}{f'(z)-\alpha}-1\right)|z|^{\beta(m+1)} + \left(1-|z|^{\beta(m+1)}\right)\left[2z^\beta \frac{f'(z)h(z)}{f'(z)-\alpha} + \frac{1}{\beta}\frac{zf''(z)}{f'(z)-\alpha}\right]$$

$$+\frac{z^{\beta+1}\left(1-|z|^{\beta(m+1)}\right)^2}{|z|^{\beta(m+1)}}\left[\frac{z^{\beta-1}f'(z)h^2(z)}{f'(z)-\alpha}+\frac{1}{\beta}\left(\frac{f''(z)h(z)}{f'(z)-\alpha}-h'(z)\right)\right] \tag{3.22}$$

$$-\frac{m-1}{2}\Bigg|$$

$$\leq\frac{m+1}{2} \tag{3.23}$$

are true for all $z\in\mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Corollary 3. *Let m be a positive real number and let α, β be complex numbers such that $\Re\alpha<1/2$, $\Re\beta>0$ and $f\in\mathcal{A}$. If the following inequalities*

$$\left|\frac{f'(z)}{f'(z)-\alpha}-\frac{m+1}{2}\right|<\frac{m+1}{2}, \tag{3.24}$$

and

$$\left|\left(\frac{f'(z)}{f'(z)-\alpha}-1\right)|z|^{\beta(m+1)}+\left(1-|z|^{\beta(m+1)}\right)\left[\frac{1}{\beta}\frac{zf''(z)}{f'(z)-\alpha}\right]-\frac{m-1}{2}\right|$$

$$\leq\frac{m+1}{2} \tag{3.25}$$

are true for all $z\in\mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. It results from Corollary 1 with $g=f'$ and $h=0$. □

If we consider $g(z)=f'$, $h(z)=-\frac{1}{2}\frac{f''}{f'}$, $\alpha=0$, $\beta=1$ in Theorem 3, we obtain another univalence criterion as follows.

Corollary 4. *Let m be a positive real number and $f\in\mathcal{A}$. If the following inequality*

$$\left|\frac{z^2\left(1-|z|^{m+1}\right)^2}{|z|^{m+1}}\left(\frac{1}{2}\{f;z\}\right)-\frac{m-1}{2}\right|\leq\frac{m+1}{2} \tag{3.26}$$

where

$$\{f;z\}=\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is true for all $z\in\mathcal{U}$, then the function $f(z)$ is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Setting $\alpha=0$ in Corollary 3 we have another univalence criterion as follows.

Corollary 5. *Let m be a positive real number and let β be complex number such that $\Re\beta > 0$ and $f \in \mathcal{A}$. If the following inequality*

$$\left| \frac{\left(1-|z|^{\beta(m+1)}\right)}{\beta} \left(\frac{zf''(z)}{f'(z)} \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3.27}$$

is true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Corollary 6. *Let m be a positive real number and let β be complex number with $\Re\beta > 0$ and $f \in \mathcal{A}$. If the following inequality*

$$\left| \frac{\left(1-|z|^{(m+1)\Re\beta}\right)}{\Re\beta} \left(\frac{zf''(z)}{f'(z)} \right) \right| \leq 1 \tag{3.28}$$

is true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. It can be proved (see [18]) that for $z \in \mathcal{U} \setminus \{0\}$, $\Re\beta > 0$ and $m \in \mathbb{R}_+$

$$\left| \frac{1-|z|^{(m+1)\beta}}{\beta} \right| \leq \frac{1-|z|^{(m+1)\Re\beta}}{\Re\beta}.$$

For $m \geq 1$, we have

$$\begin{aligned} &\left| \frac{1-|z|^{(m+1)\beta}}{\beta} \left(\frac{zf''(z)}{f'(z)} \right) - \frac{m-1}{2} \right| \\ &\leq \left| \frac{1-|z|^{(m+1)\beta}}{\beta} \left(\frac{zf''(z)}{f'(z)} \right) \right| + \frac{m-1}{2} \\ &\leq \frac{1-|z|^{(m+1)\Re\beta}}{\Re\beta} \left| \frac{zf''(z)}{f'(z)} \right| + \frac{m-1}{2} \\ &\leq 1 + \frac{m-1}{2} = \frac{m+1}{2}. \end{aligned}$$

Since inequalities (3.1) and (3.2) are satisfied, making use of Theorem 3, we can conclude that the function $\mathcal{F}_\beta$ is analytic and univalent in $\mathcal{U}$. □

Putting $g(z) = \left(\frac{f(z)}{z}\right)^2$, $h(z) = 0$, $\alpha = 0$, in Theorem 3, we get the univalence criterion as follows.

Corollary 7. *Let m be a positive real number and let β be complex number such that $\Re\beta > 0$ and $f \in \mathcal{A}$. If the following inequalities*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - \frac{m+1}{2} \right| < \frac{m+1}{2}, \tag{3.29}$$

and

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)}-1\right)|z|^{\beta(m+1)}+\frac{2\left(1-|z|^{\beta(m+1)}\right)}{\beta}\left(\frac{zf'(z)}{f(z)}-1\right)-\frac{m-1}{2}\right|$$
$$\leq \frac{m+1}{2} \tag{3.30}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Corollary 8. *Let m be a positive real number and $f \in \mathcal{A}$. If the following inequality*

$$\left|z\left(1-|z|^{m+1}\right)\left(2f''(z)+\frac{f''(z)}{f'(z)}\right)\right.$$
$$\left.+\frac{z^2\left(1-|z|^{m+1}\right)^2}{|z|^{m+1}}\left(\frac{(f''(z))^2}{f'(z)}+\left(f''(z)\right)^2-f'''(z)\right)-\frac{m-1}{2}\right|$$
$$\leq \frac{m+1}{2} \tag{3.31}$$

is true for all $z \in \mathcal{U}$, then the function $f(z)$ is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. It results from Corollary 2 with $\alpha = 0$, $\beta = 1$. □

Remark 1. (1) Putting $g(z) = f'(z)$, $h(z) = 0$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we have Becker's criterion [3].

(2) If we consider $g(z) = f'(z)$, $h(z) = -\frac{1}{2}\frac{f''(z)}{f'(z)}$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we obtain the univalence criterion due to Nehari [14].

(3) Setting $g(z) = \left(\frac{f(z)}{z}\right)^2$, $h(z) = \frac{1}{z} - \frac{f(z)}{z^2}$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we get the univalence criterion due to Ozaki-Nunokawa [17].

(4) For $g(z) = f'(z)$, $h(z) = \frac{1}{z} - \frac{f(z)}{f(z)}$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we arrive at Goluzin's criterion for univalence [9].

(5) For $m = 1$ in Corollary 6, we obtain the univalence criterion due to Pascu [18].

(6) If we consider $g(z) = f'(z)$, $h(z) = 0$, $\beta = 1$ in Theorem 3, we have results of Raducanu et al. [23].

(7) Putting $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we get the univalence criterion due to Ovesea-Tudor and Owa [16].

Example 1. Let the function

$$f(z) = \frac{z}{1-\frac{z^2}{2}}. \tag{3.32}$$

Then f is univalent in $\mathcal{U}$ and the function

$$\mathcal{F}_2(z) = \left(2 \int_0^z u f'(u) du \right)^{\frac{1}{2}} \tag{3.33}$$

is analytic and univalent in $\mathcal{U}$.

Infact, from equality (3.29) for $m = 1$, we have

$$\frac{z^2 f'(z)}{f^2(z)} - 1 = \frac{z^2}{2}. \tag{3.34}$$

It is clear that the condition (3.29) of the Corollary 7 is satisfied for $m = 1$, and then the function f is univalent in $\mathcal{U}$.

Taking into account (3.34), the condition (3.30) of Corollary 7 becomes for $\beta = 2$, $m = 1$,

$$\begin{aligned} \left| \frac{z^2}{2} |z|^4 + \left(1 - |z|^4\right) \frac{2z^2}{2 - z^2} \right| &\le \frac{|z|^6}{2} + 2\left(1 - |z|^4\right) |z|^2 \\ &= \frac{1}{2}\left(4|z|^2 - 3|z|^6\right) < 1 \end{aligned}$$

because the greatest value of the function $g(x) = 4x^2 - 3x^6$, for $x \in [0,1]$ is taken for $x = \sqrt{\frac{2}{3}}$ and $g(\sqrt{\frac{2}{3}}) = \frac{24}{27}$. Therefore the function $\mathcal{F}_2(z)$ defined by (3.33) is analytic and univalent in $\mathcal{U}$.

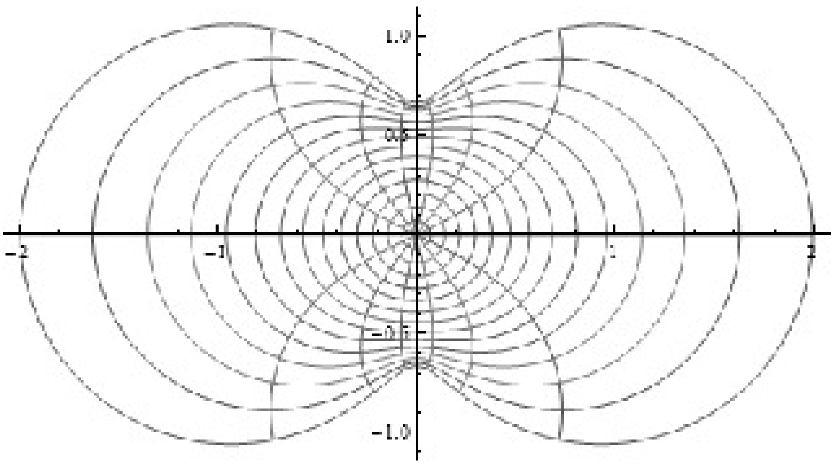

FIGURE 1. $f(z) = \frac{z}{1 - \frac{z^2}{2}}$

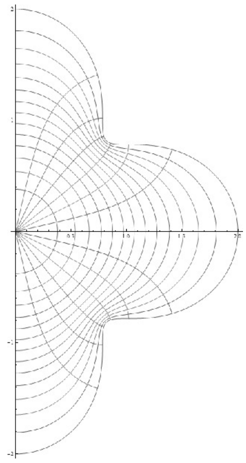

FIGURE 2. $\mathcal{F}_2(z) = \left(4\int_0^z \frac{2+u^2}{(2-u^2)^2}du\right)^{\frac{1}{2}}$

4. Quasiconformal Extension Criterion

In this section we will generalize the univalence condition given in Theorem 3 to a quasiconformal extension criterion.

Theorem 4. *Let m be a positive real number and let α, β be complex numbers such that $\Re\alpha < 1/2$, $\Re\beta > 0$, $f \in \mathcal{A}$ and $k \in [0,1)$. Let g and h be two analytic functions in $\mathcal{U}$, $g(z) = 1 + b_1 z + \ldots$, $h(z) = c_0 + c_1 z + \ldots$. If the following inequalities*

$$\left|\frac{f'(z)}{g(z)-\alpha} - \frac{m+1}{2}\right| < k\frac{m+1}{2}, \tag{4.1}$$

and

$$\begin{aligned}\Bigg| & \left(\frac{f'(z)}{g(z)-\alpha} - 1\right)|z|^{\beta(m+1)} \\ & + \left(1-|z|^{\beta(m+1)}\right)\left[2z^\beta \frac{f'(z)h(z)}{g(z)-\alpha} + \frac{1}{\beta}\frac{zg'(z)}{g(z)-\alpha}\right] \\ & + \frac{z^{\beta+1}\left(1-|z|^{\beta(m+1)}\right)^2}{|z|^{\beta(m+1)}}\left[\frac{z^{\beta-1}f'(z)h^2(z)}{g(z)-\alpha} + \frac{1}{\beta}\left(\frac{g'(z)h(z)}{g(z)-\alpha} - h'(z)\right)\right] - \frac{m-1}{2}\Bigg| \\ & \leq k\frac{m+1}{2}\end{aligned} \tag{4.2}$$

is true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ given by (3.3) has a $k-$quasiconformal extension to $\mathbb{C}$.

Proof. Set

$$\mathcal{L}(z,t)=\left[\beta\int_{0}^{e^{-t}z}u^{\beta-1}f'(u)du+\frac{\left(e^{\beta mt}-e^{-\beta t}\right)z^{\beta}\left(g\left(e^{-t}z\right)-\alpha\right)}{1+\left(e^{\beta mt}-e^{-\beta t}\right)z^{\beta}h\left(e^{-t}z\right)}\right]^{1/\beta} \tag{4.3}$$

In the proof of Theorem 3 has been shown that the function $\mathcal{L}(z,t)$ given by (4.3) is a subordination chain in $\mathcal{U}$. Then we have

$$\begin{aligned}\left|\frac{p(z,t)-1}{p(z,t)+1}\right| &= \left|\frac{2}{m+1}\left\{e^{-\beta(m+1)t}\left(\frac{f'(e^{-t}z)}{g(e^{-t}z)-\alpha}-1\right)\right.\right.\\ &+\left(1-e^{-\beta(m+1)t}\right)\left[2e^{-\beta t}z^{\beta}\frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)-\alpha}+\frac{e^{-t}z}{\beta}\frac{g'(e^{-t}z)}{g(e^{-t}z)-\alpha}\right]\\ &+\frac{e^{-\beta t}z^{\beta}\left(1-e^{-\beta(m+1)t}\right)^{2}}{e^{-\beta(m+1)t}}\\ &\times\left.\left[e^{-\beta t}z^{\beta}\frac{f'(e^{-t}z)h^{2}(e^{-t}z)}{g(e^{-t}z)-\alpha}+\frac{e^{-t}z}{\beta}\left(\frac{h(e^{-t}z)g'(e^{-t}z)}{g(e^{-t}z)-\alpha}-h'(e^{-t}z)\right)\right]\right\}\\ &\left.-\frac{m-1}{m+1}\right|\\ &\leq k. \end{aligned} \tag{4.4}$$

The right hand of (4.4) always less than or equal to k from (4.2) and therefore $\mathcal{F}_{\beta}$ can be extended to k quasiconformal mapping to $\mathbb{C}$ by Theorem 1 and Theorem 2. □

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