Miskolc Mathematical Notes
Vol. 14 (2013), No 3, pp. 991-996

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2013.548

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V. Lampret

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AN ACCELERATION OF CONVERGENCE TO SOME GENERALIZED-EULER-CONSTANT FUNCTION

V. LAMPRET

Received 28 June, 2012

Abstract. For the generalized-Euler-constant function $\gamma(a)$,

$$\gamma(a) := \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\frac{1}{a+k} - \ln\frac{a+n-1}{a}\right) \qquad (a>0),$$

and for any positive integer $q \geq 2$, using the Bernoulli numbers B_{2m}, the sequences $n \mapsto \mathfrak{A}_n(a,q)$, $n \mapsto \mathfrak{B}_n(a,q)$ and $n \mapsto \mathfrak{C}_n(a,q)$, having the properties

$$\lim_{n\to\infty} n^{2q-2}\big[\gamma(a) - \mathfrak{A}_n(a,q)\big] = \frac{B_{2q-2}}{2q-2},$$

$$\lim_{n\to\infty} n^{2q-2}\big[\gamma(a) - \mathfrak{B}_n(a,q)\big] = -\left(1-2^{3-2q}\right)\frac{B_{2q-2}}{2q-2}$$

and

$$\lim_{n\to\infty} n^{2q-1}\big[\gamma(a) - \mathfrak{C}_n(a,q)\big] = \frac{1}{2}\,B_{2q-2},$$

are determined.

2010 *Mathematics Subject Classification:* 11B83; 40A05; 11B68; 33E20

Keywords: acceleration of convergence, generalized Euler's constant, generalized γ-sequence

1. INTRODUCTION

The gamma-sequence

$$y_n(a) = \sum_{k=0}^{n-1}\frac{1}{a+k} - \ln\frac{a+n-1}{a} \qquad (n \in \mathbb{N}), \tag{1.1}$$

considered in [2,3] is convergent for $a>0$ and defines the generalized-Euler-constant function $\gamma(a)$,

$$\gamma(a) := \lim_{n\to\infty} y_n(a) \tag{1.2}$$

The name generalized-Euler-constant function has its origin in the identity $\gamma(1) = C$, where C is the Euler-Mascheroni constant. Several results on the rate of convergence of the sequence (1.1) have been established in the literature.

Recently, A. Sîntămărian [4] accelerated the convergence (1.2) using the Stolz-Cesaro limit theorem. In this reference the sequences

$$\alpha_{n,2}(a) := \sum_{k=0}^{n-1} \frac{1}{a+k} - \frac{1}{2(a+n-1)} + \frac{1}{12(a+n-1)^2} - \ln\left(\frac{a+n-1}{a} + \frac{1}{120a(a+n-1)^3}\right)$$

and

$$\beta_{n,2}(a) := \alpha_{n,2}(a) + \frac{1}{252(a+n-1)^6}$$

were considered and in Theorem 2 the equalities

$$\lim_{n\to\infty} n^6\big[\gamma(a) - \alpha_{n,2}(a)\big] = \frac{1}{252} \tag{1.3}$$

and

$$\lim_{n\to\infty} n^8\big[\beta_{n,2}(a) - \gamma(a)\big] = \frac{121}{28800} \tag{1.4}$$

were derived. Similarly, in Theorem 3, were considered some sequences $\alpha_{n,3}(a)$, $\beta_{n,3}(a)$ and $\delta_{n,3}(a)$ such that the following limits hold:

$$\lim_{n\to\infty} n^8\big[\alpha_{n,3}(a) - \gamma(a)\big] = \frac{1}{240}, \tag{1.5}$$

$$\lim_{n\to\infty} n^{10}\big[\gamma(a) - \beta_{n,3}(a)\big] = \frac{1}{132} \tag{1.6}$$

and

$$\lim_{n\to\infty} n^{12}\big[\delta_{n,3}(a) - \gamma(a)\big] = \frac{174197}{8255520}. \tag{1.7}$$

In [4] the equalities above were demonstrated using rather tedious calculations.

The goal of this article is to complement/improve the results and the method of derivation as presented in [4]. In our paper we present an approach of incessant acceleration of the convergence (1.1) to any degree. We will present three classes of sequences converging to $\gamma(a)$ much faster than the original sequence $y_n(a)$ does.

2. PRELIMINARIES

Referring to (1.1), (1.2) and [1, Theorems 1–3], we have the following equalities[1]

$$\gamma(a) = S_n(a,q) + R_n(a,q) \qquad (n,q \in \mathbb{N}) \tag{2.1}$$
$$= \sigma_n(a,q) + \rho_n(a,q) \qquad (n,q \in \mathbb{N}) \tag{2.2}$$
$$= S_n^*(a,q) + R_n^*(a,q) \qquad (n,q \in \mathbb{N}) \tag{2.3}$$

with[2]

$$S_n(a,q) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln\frac{a+n}{a} + \frac{1}{2(a+n)} + \sum_{j=1}^{q-1} \frac{B_{2j}}{2j\,(a+n)^{2j}}, \tag{2.4}$$

$$\sigma_n(a,q) = \sum_{k=0}^{n-1} \frac{1}{a+k} + \ln\left(\frac{a}{a+n-\frac{1}{2}}\right) - \sum_{i=1}^{q-1}\left(\frac{1-2^{1-2i}}{2i} \cdot \frac{B_{2i}}{(a+n-\frac{1}{2})^{2i}}\right) \tag{2.5}$$

and

$$S_n^*(a,q) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln\frac{a+n}{a}$$
$$-1 - \ln\left(1 - \frac{1}{a+n+1}\right)^{a+n+1} + \frac{1}{2(a+n)} - \frac{1}{2}\ln\left(1+\frac{1}{a+n}\right)$$
$$-\sum_{j=1}^{q-1} \frac{B_{2j}}{(2j)(2j-1)}\left[\frac{1}{(a+n)^{2j-1}} - \frac{1}{(a+n+1)^{2j-1}} - \frac{2j-1}{(a+n)^{2j}}\right]. \tag{2.6}$$

The remainders are estimated as

$$|R_n(a,q)| < \frac{|B_{2q}|}{q(a+n)^{2q}}, \tag{2.7}$$

$$|\rho_n(a,q)| < \frac{|B_{2q}|}{q(a+n-\frac{1}{2})^{2q}} \tag{2.8}$$

[1]The sequence $\sigma_n(a,q)$ in the expression (2.5) is given in the corrected form appearing in the proof of [1, Theorem 2], where in the first sum the start "$k = 1$" should be replaced by "$k = 0$" and where the summands in the third sum of $\sigma_n(a,q)$ are written incorrectly.

[2]By definition $\sum_{k=1}^{m} x_k = 0$ for $m < 1$.

and

$$\left|R_n^*(a,q)\right| < \frac{\left|B_{2q}\right|}{(a+n)^{2q+1}}\,. \tag{2.9}$$

Here, the symbol B_k means the k-th Bernoulli number,

$$\frac{t\,e^{xt}}{e^t-1} \equiv \sum_{k=0}^{\infty} B_k(x)\frac{t^j}{j!} \qquad (x \in \mathbb{R},\ |t| < 2\pi),$$

$B_k \equiv B_k(0)$, $B_k(x)$ is k-th Bernoulli polynomial.

3. An acceleration of convergence

Referring to (2.4)–(2.6) we make the following definition.

Definition 1. For any $a > 0$ and any integer $q \geq 2$ we consider the following sequences:

$$n \mapsto \mathfrak{A}_n(a,q) := S_n(a,q-1), \tag{3.1}$$

$$n \mapsto \mathfrak{B}_n(a,q) := \sigma_n(a,q-1) \tag{3.2}$$

and

$$n \mapsto \mathfrak{C}_n(a,q) := S_n^*(a,q-1). \tag{3.3}$$

Now, we are in the position to formulate the following result.

Theorem 1. *For any positive a and any integer $q \geq 2$ we have the following limits:*

$$\lim_{n\to\infty} n^{2q-2}\left[\gamma(a) - \mathfrak{A}_n(a,q)\right] = \frac{B_{2q-2}}{2q-2} \qquad =: L_{\mathfrak{A}}(q), \tag{3.4}$$

$$\lim_{n\to\infty} n^{2q-2}\left[\gamma(a) - \mathfrak{B}_n(a,q)\right] = -\left(1-2^{3-2q}\right)\frac{B_{2q-2}}{2q-2} \qquad =: L_{\mathfrak{B}}(q) \tag{3.5}$$

and

$$\lim_{n\to\infty} n^{2q-1}\left[\gamma(a) - \mathfrak{C}_n(a,q)\right] = \frac{1}{2}\,B_{2q-2} \qquad =: L_{\mathfrak{C}}(q). \tag{3.6}$$

Note that the limits are independent of a.

Proof. According to (2.1), (2.4) and (3.1), we have

$$\gamma(a) = \mathfrak{A}_n(a,q) + \frac{B_{2q-2}}{(2q-2)(a+n)^{2q-2}} + R_n(a,q),$$

for $n \geq 1$, $a > 0$ and $q \geq 2$. Consequently, using (2.7), the equality (3.4) follows. Similarly, referring to (2.2), (2.5) and (3.2), we get

$$\gamma(a) = \mathfrak{B}_n(a,q) - \frac{1-2^{3-2q}}{2q-2} \cdot \frac{B_{2q-2}}{(a+n-\frac{1}{2})^{2q-2}} + \rho_n(a,q),$$

for $n \geq 1$, $a > 0$ and $q \geq 2$. Thus, considering (2.8), we confirm (3.5). Finally, referring to (2.3), (2.6) and (3.3) we obtain

$$\gamma(a) = \mathfrak{C}_n(a,q) + R_n^*(a,q) + \frac{B_{2q-2}}{(2q-2)(2q-3)} \left[\frac{1}{(a+n+1)^{2q-3}} - \frac{1}{(a+n)^{2q-3}} + \frac{2q-3}{(a+n)^{2q-2}} \right], \tag{3.7}$$

for $n \geq 1$, $a > 0$ and $q \geq 2$. Denoting $a+n = b$, $2q-3 = m$ and using Taylor's formula of order 1 around b for the function $f(x) \equiv x^{-m}$, $(b+1)^{-m} = b^{-m} - m\,b^{-(m+1)} + \frac{1}{2}m(m+1)(b+\vartheta)^{-(m+2)}$, we obtain the equality

$$\frac{1}{(a+n+1)^{2q-3}} = \frac{1}{(a+n)^{2q-3}} - \frac{2q-3}{(a+n)^{2q-2}} + \frac{(2q-3)(2q-2)}{2(a+n+\vartheta)^{2q-1}}, \tag{3.8}$$

for some $\vartheta = \vartheta_n(a,q) \in (0,1)$. From (3.7) and (3.8) we get the expression

$$\gamma(a) - \mathfrak{C}_n(a,q) = \frac{B_{2q-2}}{(2q-2)(2q-3)} \cdot \frac{(2q-3)(2q-2)}{2(a+n+\vartheta)^{2q-1}} + R_n^*(a,q),$$

which, recalling (2.9), demonstrates the relation (3.6). □

Example 1. Referring to (3.4)–(3.6) and using [5] we obtain the following tables:

q	2	3	4	5	6	7
$L_{\mathfrak{A}}(q)$	$\frac{1}{12}$	$-\frac{1}{120}$	$\frac{1}{252}$	$-\frac{1}{240}$	$\frac{1}{132}$	$-\frac{691}{32760}$

TABLE 1. The type $\mathfrak{A}$–limits; Theorem 1, Eq. (3.4).

q	2	3	4	5	6	7
$L_{\mathfrak{B}}(q)$	$-\frac{1}{24}$	$\frac{7}{960}$	$-\frac{31}{8064}$	$\frac{127}{30720}$	$-\frac{511}{67584}$	$\frac{1414477}{67092480}$

TABLE 2. The type $\mathfrak{B}$–limits; Theorem 1, Eq. (3.5).

q	2	3	4	5	6	7
$L_{\mathfrak{C}}(q)$	$\frac{1}{12}$	$-\frac{1}{60}$	$\frac{1}{84}$	$-\frac{1}{60}$	$-\frac{5}{132}$	$-\frac{691}{5460}$

TABLE 3. The type $\mathfrak{C}$–limits; Theorem 1, Eq. (3.6).

ACKNOWLEDGEMENT

The author express sincere thanks to the referee whose comments resulted in a significant improvement of the paper.

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Author's address

V. Lampret
University of Ljubljana, Faculty of Civil and Geodetic Engineering, Jamova 2, 1000 Ljubljana, Slovenia
E-mail address: vito.lampret@fgg.uni-lj.si