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WEAKLY LASKERIAN MODULES AND WEAK COFINITENESS

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R. It is shown that if $\mathfrak{a} = (x_1, \ldots, x_t)$, and M is an R-module, then $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i iff $\operatorname{Tor}^R_i(R/\mathfrak{a}, M)$ is weakly Laskerian for all i iff the Koszul cohomology module $H^i(x_1, \ldots, x_t; M)$ is weakly Laskerian for all i. Furthermore, each of these coditions imply that $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$. In Section 3, we show that if M is an R-module with $\operatorname{Supp} M \subseteq V(\mathfrak{a})$, then M is $\mathfrak{a}$-weakly cofinite, in the following cases:
a) there exists $x \in \mathfrak{a}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-weakly cofinite.
b) there exists $x \in \sqrt{\mathfrak{a}}$ such that $0 :_M x$ and M/xM are both weakly Laskerian.

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Keywords: weakly Laskerian modules, $\mathfrak{a}$-weakly cofinite modules, local cohomology

1. INTRODUCTION

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, and $\mathfrak{a}$ will be an ideal of R. Let M be an R-module. The $\mathfrak{a}$-*torsion submodule* of M is defined as $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \geq 1} (0 :_M \mathfrak{a}^n)$. The i^{th} *local cohomology functor* $\mathrm{H}^i_{\mathfrak{a}}(.)$ is defined as the i^{th} right derived functor $\Gamma_{\mathfrak{a}}(.)$. It is known that for each $i \geq 0$ there is a natural isomorphism of R-modules

$$\mathrm{H}^i_{\mathfrak{a}}(M) \cong \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [5] or [1] for the basic properties of local cohomology.

The notions of weakly Laskerian modules and $\mathfrak{a}$-weakly cofinite modules were introduced by Divaani-Aazar and Mafi in [3] and [4]. An R module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite. An R module M is said to be $\mathfrak{a}$-*weakly cofinite* if $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $i \geq 0$.

Divaani-Aazar and Mafi in [4, Theorem 2.10] have shown using change of rings principle and spectral sequence that if M is an $\mathfrak{a}$-weakly cofinite R-module , then

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$M/\mathfrak{a}M$ is weakly Laskerian. In Section 2, without using change of rings principle and spectral sequence, we prove that if M is an R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all $i \geq 0$, then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$. One of the main results of this article is to prove that if $\mathfrak{a} = (x_1, \ldots, x_t)$, and M is an R-module, then the following statements are equivalent:
(ii) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all i.
(ii) $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all i.
(iii) The Koszul cohomology module $H^i(x_1, \ldots, x_t; M)$ is weakly Laskerian R-module for all i.

In Section 3, we obtain a sufficient condition for $\mathfrak{a}$-weakly cofinite modules. In fact, we prove that if M is an R-module with $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$, then M is $\mathfrak{a}$-weakly cofinite, in the following cases:
a) there exists $x \in \mathfrak{a}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-weakly cofinite.
b) there exists $x \in \sqrt{\mathfrak{a}}$ such that $0 :_M x$ and M/xM are both weakly Laskerian.

In Section 4, we prove that if $\mathfrak{b}$ is a second ideal of R with $\mathfrak{b} \supseteq \mathfrak{a}$ and $\mathrm{cd}(\mathfrak{b}) = 1$ and M is a weakly Laskerian R-module, then for every finitely generated R-module L with $\mathrm{Supp}\, L \subseteq V(\mathfrak{b})$, the R-module $\mathrm{Ext}^j_R(L, H^i_\mathfrak{a}(M))$ is weakly Laskerian for all i and j. In particular, the R-module $H^i_\mathfrak{a}(M)/\mathfrak{b}^n H^i_\mathfrak{a}(M)$ is weakly Laskerian for all i and n.

2. Weakly Laskerian modules and $\mathfrak{a}$-weakly cofinite modules

To prove the main results of this paper, we need to the following two lemmas.

Lemma 1. *Let M be an R-module such that $0 :_M \mathfrak{a}$ is a weakly Laskerian R-module. Then $0 :_M \mathfrak{a}^n$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. Consider the exact sequence

$$0 \to 0 :_M \mathfrak{a} \to 0 :_M \mathfrak{a}^n \xrightarrow{f} a_1(0 :_M \mathfrak{a}^n) \oplus \cdots \oplus a_t(0 :_M \mathfrak{a}^n),$$

where $\mathfrak{a} = (a_1, \ldots, a_t)$ and f is defined by $f(x) = (a_1 x, \ldots, a_t x)$. The result is followed by induction on n and [3, Lemma 2.3 (i)]. Note that $a_i(0 :_M \mathfrak{a}^n)$ is a submodule of $0 :_M \mathfrak{a}^{n-1}$ for all $i = 1, 2, \ldots, t$. □

Lemma 2. *Let M be an R-module such that $M/\mathfrak{a}M$ is a weakly Laskerian R-module. Then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. Consider the exact sequence

$$(M/\mathfrak{a}^{n-1}M)^t \xrightarrow{f} M/\mathfrak{a}^n M \xrightarrow{g} M/\mathfrak{a}M \to 0,$$

where $\mathfrak{a} = (a_1, \ldots, a_t)$, g is the canonical map, and f is defined by

$$f(m_1 + \mathfrak{a}^{n-1}M, \cdots, m_t + \mathfrak{a}^{n-1}M) = a_1 m_1 + \cdots + a_t m_t + \mathfrak{a}^n M.$$

Now, the result is followed by induction on n and [3, Lemma 2.3 (i)]. $\square$

Divaani-Aazar and Mafi in [4, Theorem 2.10] have shown using change of rings principle and spectral sequence that if M is an $\mathfrak{a}$-weakly cofinite R-module , then $M/\mathfrak{a}M$ is weakly Laskerian. We generalize this result and give a direct proof without using change of rings principle and spectral sequence.

Theorem 1. *Let M be an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all $i \geq 0$. Then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. By Lemma 2, it is enough to prove that $M/\mathfrak{a}M$ is weakly Laskerian. To do this, let $\mathfrak{a} = (x_1, \ldots, x_n)$. Then $M/\mathfrak{a}M \simeq H^n(x_1, \ldots, x_n; M)$, where $H^n(x_1, \ldots, x_n; M)$ denotes the n^{th} Koszul cohomology module. Consider the co-Koszul complex

$$K^\bullet(\mathbf{x}, M) : 0 \to \operatorname{Hom}(K_0(\mathbf{x}), M) \to \operatorname{Hom}(K_1(\mathbf{x}), M) \to \cdots \\ \to \operatorname{Hom}(K_n(\mathbf{x}), M) \to 0.$$

Then $H^i(x_1, \cdots, x_n; M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $K^\bullet(\mathbf{x}, M)$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. By induction we claim that $B^j \in \mathcal{W}$ for all j. We have $B^0 = 0 \in \mathcal{W}$. Now, let $B^t \in \mathcal{W}$. Put $C^i = \operatorname{Hom}(K_i(\mathbf{x}), M)/B^i$. Since $K_t(\mathbf{x})$ is a finitely generated free R-module, it follows that $\operatorname{Hom}(K_t(\mathbf{x}), M)$ is a direct sum of finitely many copies of M. Therefore, $\operatorname{Hom}(K_t(\mathbf{x}), M) \in \mathcal{W}$ by [3, Lemma 2.3 (i)]. Now, since $B^t \in \mathcal{W}$ and $\operatorname{Hom}(K_t(\mathbf{x}), M) \in \mathcal{W}$, we have $C^t \in \mathcal{W}$ by [3, Lemma 2.3 (i)]. Hence $0 :_{C^t} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^t)$ is weakly Laskerian. But since $\mathfrak{a}H^t(x_1, \ldots, x_n; M) = 0$, it follows that $H^t(x_1, \ldots, x_n; M) \subseteq 0 :_{C^t} \mathfrak{a}$, and so $H^t(x_1, \ldots, x_n; M)$ is weakly Laskerian. Next, from the short exact sequence

$$0 \to H^t(x_1, \ldots, x_n; M) \to C^t \to B^{t+1} \to 0$$

and [3, Lemma 2.3 (i)] we deduce that $B^{t+1} \in \mathcal{W}$. Hence, by induction we have proved that $B^j \in \mathcal{W}$ for all j. Now, since $B^n \in \mathcal{W}$ and $\operatorname{Hom}(K_n(\mathbf{x}), M) \in \mathcal{W}$, we obtain that $C^n \in \mathcal{W}$. Hence $0 :_{C^n} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^n)$ is weakly Laskerian. Thus $H^n(x_1, \ldots, x_n; M) \subseteq 0 :_{C^n} \mathfrak{a}$ implies that $H^n(x_1, \ldots, x_n; M)$ is weakly Laskerian. Since $M/\mathfrak{a}M = H^n(x_1, \ldots, x_n; M)$, it follows that $M/\mathfrak{a}M$ is weakly Laskerian. $\square$

Corollary 1. *Let M be a $\mathfrak{a}$-weakly cofinite R-module. Then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. The assertion follows from the definition and Theorem 1. $\square$

Corollary 2. *Let $\mathfrak{a}$ be an ideal of R, and let M be an R-module such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i, and $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. By Theorem 1, it is sufficient to prove that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i. The case $i = 0$ is clear, so let $i > 0$ and we do induction on i. We first reduce to the case $\Gamma_{\mathfrak{a}}(M) = 0$. To do this, let $\bar{M} = M/\Gamma_{\mathfrak{a}}(M)$, then we have the exact sequence

$$\ldots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \bar{M}) \\ \to \mathrm{Ext}^{i+1}_{\mathrm{R}}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \ldots$$

and isomorphism $\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(\bar{M})$ for all $i > 0$. Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite, so in view of [3, Lemma 2.3 (i)], we may assume that M is $\mathfrak{a}$-torsion free. Let E be the injective envelope of M and put $L = E/M$. Then $H^i_{\mathfrak{a}}(E) = 0$, and we therefore get the isomorphisms $\mathrm{H}^i_{\mathfrak{a}}(L) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, L) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Theorem 2. *Let $\mathfrak{a} = (x_1, \ldots, x_t)$ be an ideal of R, and let M be an R-module. Then the following statements are equivalent:*

(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is a weakly Laskerian R-module for all i.*

(ii) $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ *is a weakly Laskerian R-module for all i.*

(iii) *The Koszul cohomology module $H^i(x_1, \ldots, x_t; M)$ is weakly Laskerian R-module for all i.*

Furthermore, each of these coditions imply that $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.

Proof. (i) ⇒ (ii) Let

$$\mathbb{F}_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. We have $\mathrm{Tor}^R_i(R/\mathfrak{a}, M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $\mathbb{F}_\bullet \otimes_R M$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. By induction we claim that $Z_j \in \mathcal{W}$ for all j. We have $Z_0 = F_0 \otimes_R M \in \mathcal{W}$. Now, let $Z_t \in \mathcal{W}$. Consider the exact sequence

$$(\dagger) \qquad 0 \to C_{i+1} \to Z_i \to \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \to 0,$$

where $C_i = F_i \otimes_R M/Z_i$. Hence we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i \to \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \to 0.$$

Therefore, $\mathrm{Tor}^R_t(R/\mathfrak{a}, M)$ is a homomorphic image of $Z_t/\mathfrak{a}Z_t$. Since $Z_t \in \mathcal{W}$, it follows from Theorem 1 that $Z_t/\mathfrak{a}Z_t$ is weakly Laskerian, and so $\mathrm{Tor}^R_t(R/\mathfrak{a}, M)$

is weakly Laskerian. Hence, we deduce by (†) that $C_{t+1} \in \mathcal{W}$, and so $Z_{t+1} \in \mathcal{W}$. Hence by induction we have proved that $Z_j \in \mathcal{W}$ for all j. It follows from Theorem 1 that $Z_i/\mathfrak{a}Z_i$ is weakly Laskerian for all i, and so $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i.

To prove the implication (ii) $\Rightarrow$ (iii), since

$$H^i(x_1, \ldots, x_t; M) \simeq H_{t-i}(x_1, \ldots, x_t; M),$$

so it is sufficient to show that $H_i(x_1, \ldots, x_t; M)$ is weakly Laskerian for all i. Let $\mathbf{x} = x_1, \ldots, x_t$. Consider the Koszul complex

$$K_\bullet(\mathbf{x}) : 0 \to K_t(\mathbf{x}) \to K_{t-1}(\mathbf{x}) \to \cdots \to K_1(\mathbf{x}) \to K_0(\mathbf{x}) \to 0,$$

We have $H_i(x_1, \ldots, x_t; M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $K_\bullet(\mathbf{x}) \otimes_R M$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $\operatorname{Tor}_i^R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to H_i(x_1, \ldots, x_t; M) \to 0,$$

where $C_i = K_i(\mathbf{x}) \otimes_R M/Z_i$. Hence we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i \to H_i(x_1, \ldots, x_t; M) \to 0.$$

By using a similar proof as in the proof of the implication (i) $\Rightarrow$ (ii), $Z_i \in \mathcal{W}$ for all i. It follows that $Z_i/\mathfrak{a}Z_i = \operatorname{Tor}_0^R(R/\mathfrak{a}, Z_i)$ is weakly Laskerian for all i, and so $H_i(x_1, \ldots, x_t; M)$ is weakly Laskerian for all i.

To prove the implication (iii) $\Rightarrow$ (i), let

$$\mathbb{F}_\bullet : \quad \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. We have $\operatorname{Ext}_R^i(R/\mathfrak{a}, M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $\operatorname{Hom}_R(\mathbb{F}_\bullet, M)$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $H^i(x_1, \ldots, x_t; N)$ is weakly Laskerian for all $i \geq 0$. Consider the short exact sequence

$$0 \to \operatorname{Ext}_R^i(R/\mathfrak{a}, M) \to C^i \to B^{i+1} \to 0,$$

where $C^i = \operatorname{Hom}_R(F_i, M)/B^i$. By using a similar proof as in the proof of the Theorem 1, $B^i \in \mathcal{W}$ for all i. Thus $C^i \in \mathcal{W}$ for all i. Now, since

$$\operatorname{Ext}_R^i(R/\mathfrak{a}, M) \subseteq 0 :_{C^i} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^i) \simeq H^0(x_1, \ldots, x_t; C^i)$$

and $H^0(x_1, \ldots, x_t; C^i)$ is weakly Laskerian, so $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is weakly Laskerian for all i.

Finally, the end part is followed by Theorem 1.

□

The first part of the next result has been proved using Gruson's Theorem by Divaani-Aazar and Mafi [4, Theorem 2.8] by using the same proof as that used in Delfino and Marley [2, Proposition 1]. We give a direct proof for this.

Theorem 3. *Let M be an R-module such that* $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is a weakly Laskerian R-module for all $i \geq 0$. Then for any finitely generated R-module L with* $\operatorname{Supp} L \subseteq V(\mathfrak{a})$*, the R-modules* $\operatorname{Ext}^i_R(L, M)$ *and* $\operatorname{Tor}^R_i(L, M)$ *are weakly Laskerian for all $i \geq 0$.*

Proof. We have $V(\operatorname{Ann}_R L) = \operatorname{Supp} L \subseteq V(\mathfrak{a})$. Hence there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n L = 0$. It follows that $\mathfrak{a}^n \operatorname{Ext}^i_R(L, M) = 0$ and $\mathfrak{a}^n \operatorname{Tor}^R_i(L, M) = 0$ for all i. Let

$$\mathbb{F}_\bullet : \quad \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be a free resolution of finitely generated R-modules for L. Then $\operatorname{Ext}^i_R(L, M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $\operatorname{Hom}_R(\mathbb{F}_\bullet, M)$, respectively. Let $\mathcal{C}$ be the class of all R modules N such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. Consider the short exact sequence

$$0 \to \operatorname{Ext}^i_R(L, M) \to C^i \to B^{i+1} \to 0,$$

where $C^i = \operatorname{Hom}_R(F_i, M)/B^i$. By using a similar proof as in the proof of the Theorem 1 and using Lemma 1, we have that $B^i \in \mathcal{C}$ for all i. (Note that $\operatorname{Ext}^i_R(L, M) \subseteq 0 :_{C^i} \mathfrak{a}^n$.) Thus $C^i \in \mathcal{C}$ for all i. Hence $0 :_{C^i} \mathfrak{a}$ is weakly Laskerian for all i, and so it follows from Lemma 1 that $0 :_{C^i} \mathfrak{a}^n$ is weakly Laskerian for all i. Now, since $\operatorname{Ext}^i_R(L, M) \subseteq 0 :_{C^i} \mathfrak{a}^n$, $\operatorname{Ext}^i_R(L, M)$ is weakly Laskerian for all i.

Also, we have $\operatorname{Tor}^R_i(L, M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $\mathbb{F}_\bullet \otimes_R M$, respectively. Let $\mathcal{C}'$ be the class of all R modules N such that $\operatorname{Tor}^R_i(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. In view of Theorem 2 and our assumption, $M \in C'$. Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to \operatorname{Tor}^R_i(L, M) \to 0,$$

where $C_i = F_i \otimes_R M/Z_i$. As $\mathfrak{a}^n \operatorname{Tor}^R_i(L, M) = 0$ for all i, we obtain the exact sequence

$$Z_i/\mathfrak{a}^n Z_i \to \operatorname{Tor}^R_i(L, M) \to 0.$$

Now, by using a similar proof as in the proof of the Theorem 2((i) $\Rightarrow$ (ii)) and using Lemma 2, we have $Z_i \in \mathcal{C}$ for all i. Therefore, it follows from Lemma 2 that $Z_i/\mathfrak{a}^n Z_i$ is weakly Laskerian for all i, and $\operatorname{Tor}^R_i(L, M)$ is weakly Laskerian for all i. $\square$

The change of ring principle for weak cofiniteness has been proved by using a spectral sequence argument by Divaani-Aazar and Mafi [4, Theorem 2.9]. We give a direct proof for it.

Theorem 4. *Let the ring T be a homomorphic image of R, and let M be a T-module. Then M is an $\mathfrak{a}T$-weakly cofinite as a T-module if and only if M is an $\mathfrak{a}$-weakly cofinite as an R-module.*

Proof. Assume that $T = R/I$ for some ideal I of R and let N be a T-module. Then $\mathfrak{p} \in \operatorname{Ass}_R N$ if and only if $\mathfrak{p}/I \in \operatorname{Ass}_T N$, and so N is weakly Laskerian as a T-module if and only if N is weakly Laskerian as an R-module. Also, since $\mathfrak{p} \in \operatorname{Supp}_R N$ if and only if $\mathfrak{p}/I \in \operatorname{Supp}_T N$, it follows that $\operatorname{Supp}_T M \subseteq V(\mathfrak{a}T)$ if and only if $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$.

Now, let $\mathfrak{a} = (x_1, \dots, x_t)$ and let $\varphi : R \to T$ be the natural epimorphism. As $\mathfrak{a}T = (\varphi(x_1), \dots, \varphi(x_t))$, it follows from Theorem 2 that $\operatorname{Ext}^i_T(T/\mathfrak{a}T, M)$ is weakly Laskerian T-module for all i if and only if the Koszul cohomology modules $H^i(\varphi(x_1), \dots, \varphi(x_t); M)$ are weakly Laskerian T-modules for all i. But, by above $H^i(\varphi(x_1), \dots, \varphi(x_t); M)$ is a weakly Laskerian T-module if and only if $H^i(\varphi(x_1), \dots, \varphi(x_t); M)$ is a weakly Laskerian R-module. On the other hand,

$$H^i(\varphi(x_1), \dots, \varphi(x_t); M) \cong H^i(x_1, \dots, x_t; M).$$

Now, the result follows from Theorem 2. □

3. A SUFFICIENT CONDITION FOR $\mathfrak{a}$-WEAKLY COFINITE MODULES

Theorem 5. *Let $f : M \to N$ be an R-homomorphism such that the modules $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Ker} f)$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Coker} f)$ are both weakly Laskerian for all i. Then $\operatorname{Ker}\operatorname{Ext}^i_R(\mathrm{id}_{R/\mathfrak{a}}, f)$ and $\operatorname{Coker}\operatorname{Ext}^i_R(\mathrm{id}_{R/\mathfrak{a}}, f)$ are also weakly Laskerian for all i.*

Proof. Consider the exact sequences

$$0 \to \operatorname{Ker} f \to M \xrightarrow{g} \operatorname{Im} f \to 0 \ \text{ and } \ 0 \to \operatorname{Im} f \xrightarrow{\iota} N \to \operatorname{Coker} f \to 0,$$

where $\iota \circ g = f$. Hence we obtain the following two exact sequences

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Ker} f) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Im} f) \to \cdots$$

and

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Im} f) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, N) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Coker} f) \to \cdots$$

Now, since $\operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, \operatorname{Ker} f)$ is weakly Laskerian, it follows from the first exact sequence that $\operatorname{Coker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, g)$ and $\operatorname{Ker}\operatorname{Ext}^{i+1}_R(id_{R/\mathfrak{a}}, g)$ are both weakly Laskerian for all i. Also, as $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Coker} f)$ is weakly Laskerian, the second exact sequence implies that the R-modules $\operatorname{Coker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, \iota)$ and $\operatorname{Ker}\operatorname{Ext}^{i+1}_R(id_{R/\mathfrak{a}}, \iota)$ are weakly Laskerian for all i. Therefore, the assertion follows from the exact sequences

$$0 \to \operatorname{Ker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, g) \to \operatorname{Ker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, f) \to \operatorname{Ker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, \iota)$$

and

$$\mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, g) \to \mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, f) \to \mathrm{Coker}\,\mathrm{Ext}^i_R(id_{R/\mathfrak{a}}, \iota) \to 0. \quad \square$$

Corollary 3. *Let M be an R-module with $\operatorname{Supp} M \subseteq V(\mathfrak{a})$. Suppose that $x \in \mathfrak{a}$ is such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-weakly cofinite. Then M is also $\mathfrak{a}$-weakly cofinite.*

Proof. Put $f = x1_M$. Then $\operatorname{Ker} f = 0 :_M x$ and $\operatorname{Coker} f = M/xM$. Hence in view of Theorem 5, the R-module $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f)$ is weakly Laskerian. But since $\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = 0$, so $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = \operatorname{Ext}^i_R(R/\mathfrak{a}, M)$. This completes the proof. $\square$

Corollary 4. *Let M be an R-module. Suppose that $x \in \sqrt{\mathfrak{a}}$ is such that $0 :_M x$ and M/xM are both weakly Laskerian. Then $\operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_x(M))$ is also weakly Laskerian for all i.*

Proof. We have $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. Put $f = x^n 1_{\Gamma_x(M)}$. Then, $\operatorname{Ker} f = 0 :_{\Gamma_x(M)} x^n = 0 :_M x^n$ and $\operatorname{Coker} f = \Gamma_x(M)/x^n\Gamma_x(M)$. Consider the exact sequence

$$0 \to \operatorname{Coker} f \to M/x^n M.$$

As M/xM is weakly Laskerian, it follows from Lemma 2 that M/x^nM is weakly Laskerian, and so $\operatorname{Coker} f$ is weakly Laskerian. Therefore, in view of [3, Lemma 2.3 (i)] and Theorem 5, $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f)$ is weakly Laskerian. But $x^n \in \mathfrak{a}$ implies that $\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = 0$, and so $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = \operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_x(M))$. This completes the proof. $\square$

Corollary 5. *Let M be an R-module with $\operatorname{Supp} M \subseteq V(\mathfrak{a})$. Suppose that $x \in \sqrt{\mathfrak{a}}$ is such that $0 :_M x$ and M/xM are both weakly Laskerian. Then M is $\mathfrak{a}$-weakly cofinite.*

Proof. The result follows from Corollary 4. $\square$

4. Cohomological dimension and weakly Laskerian modules

Before bringing the next result we recall that, for an R-module M, the *cohomological dimension of M with respect to an ideal* $\mathfrak{a}$ of R is defined as

$$\operatorname{cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

Proposition 1. *Let $\operatorname{cd}(\mathfrak{a}) = 1$, and let M be a weakly Laskerian R-module. Then $\mathrm{H}^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i.*

Proof. Since $\mathrm{H}^0_{\mathfrak{a}}(M)$ is a submodule of M, it follows that $\mathrm{H}^0_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite. Also, $\mathrm{cd}(\mathfrak{a}) = 1$ implies that $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i > 1$. Therefore, the result follows from [4, Theorem 3.1]. □

Proposition 2. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be two ideals of R with $\mathrm{cd}(\mathfrak{b}) = 1$, and let M be an R-module with $\Gamma_{\mathfrak{a}}(M) = 0$. Then*

$$\mathrm{H}^j_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}}(M)) \cong \begin{cases} \mathrm{H}^1_{\mathfrak{b}}(M), & \text{if } j = 0, i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See the proof of [6, Proposition 3.15]. □

Corollary 6. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be two ideals of R with $\mathrm{cd}(\mathfrak{b}) = 1$, and let M be a weakly Laskerian R-module. Then $\mathrm{H}^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-weakly cofinite for all i and j.*

Proof. Since $\mathrm{cd}(\mathfrak{b}) = 1$, it follows from Proposition 1 that $\mathrm{H}^j_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-weakly cofinite for all j. Now, let $i > 0$. As $\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$, we may therefore assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Thus, the result follows from Propositions 1 and 2. □

Corollary 7. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be two ideals of R with $\mathrm{cd}(\mathfrak{b}) = 1$, and let M be a weakly Laskerian R-module. Then for every finitely generated R-module L with $\mathrm{Supp}\, L \subseteq V(\mathfrak{b})$, the R-modules $\mathrm{Ext}^j_R(L, \mathrm{H}^i_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(L, \mathrm{H}^i_{\mathfrak{a}}(M))$ are weakly Laskerian for all i and j. In particular, the R-modules $H^i_{\mathfrak{a}}(M)/\mathfrak{b}^n H^i_{\mathfrak{a}}(M)$ are weakly Laskerian for all i and n.*

Proof. By Corollary 6, $\mathrm{H}^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-weakly cofinite for all i and j. Therefore, it follows from Corollary 2 that the R-modules $\mathrm{Ext}^j_R(R/\mathfrak{b}, \mathrm{H}^i_{\mathfrak{a}}(M))$ are weakly Laskerian for all i and j. Thus, the result follows from Theorem 3. □

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