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ON THE ZEROS OF SOLUTIONS AND THEIR DERIVATIVES OF SECOND ORDER NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract. This paper is devoted to studying the growth and oscillation of solutions and their derivatives of equations of the type $f'' + A(z) f' + B(z) f = F(z)$, where $A(z), B(z) (\not\equiv 0)$ and $F(z) (\not\equiv 0)$ are meromorphic functions of finite order.

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1. Introduction and main results

We assume that the reader is familiar with the usual notations and basic results of the Nevanlinna theory [8, 17]. In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r, f)$ is the Nevanlinna characteristic function of f. In the following, we give the necessary notations and basic definitions.

Definition 1 ([17])**.** Let f be a meromorphic function. Then the hyper-order of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}.$$

Definition 2 ([8,11])**.** The type of a meromorphic function f of order ρ $(0 < \rho < \infty)$ is defined by

$$\tau(f) = \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\rho}}.$$

If f is entire function of order ρ $(0 < \rho < \infty)$, we can define the type by

$$\tau_M(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\rho}}.$$

Remark 1. We have not always the equality $\tau_M(f) = \tau(f)$, for example $\tau(e^z) = \frac{1}{\pi} < 1 = \tau_M(e^z)$.

Definition 3 ([7,17]). Let f be a meromorphic function. Then the hyper-exponent of convergence of zeros sequence of $f(z)$ is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| \leqslant r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| \leqslant r\}$.

The study of oscillation of solutions of linear differential equations has attracted many interests since the work of Bank and Laine [1,2], for more details, see [9]. The main subject of this research is the zeros distribution of solutions and their derivatives of linear differential equations. In this paper, we first discuss the growth of solutions of second order linear differential equation

$$f'' + A(z)f' + B(z)f = F(z), \tag{1.1}$$

where $A(z), B(z)\,(\not\equiv 0)$ and $F(z)\,(\not\equiv 0)$ are meromorphic functions of finite order. Some results on the growth of entire solutions of (1.1) have been obtained by several researchers (see [5, 6, 12, 14]). Li and Wang (see [12]) investigated the non-homogeneous linear differential equation

$$f'' + e^{-z}f' + h(z)e^{bz}f = H(z), \tag{1.2}$$

where $h(z)$ is a transcendental entire function of finite order $\rho(h) < \frac{1}{2}$, and b is a real constant. They proved that all nontrivial solutions of (1.2) are of infinite order, provided that $\rho(H) < 1$. After their, Wang and Laine (see [14]) studied the differential equation

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = H(z), \tag{1.3}$$

where $A_0(z), A_1(z), H(z)$ are entire functions of order less than one, and $a, b \in \mathbb{C}$, and obtained.

Theorem 1 ([14]). *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, H are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution f of (1.3) is of infinite order.*

J. Tu and co-authors investigated the hyper-exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)$ $(j = 0, 1, 2, \ldots)$, where f is a solution of

$$f'' + A(z) f' + B(z) f = 0 \tag{1.4}$$

and $\varphi(z)$ is an entire function satisfying $\rho(\varphi) < \rho(f)$ or $\rho_2(\varphi) < \rho_2(f)$, and obtained the following result.

Theorem 2 ([13]). *Let $A(z)$ and $B(z)$ be entire functions with finite order. If $\rho(A) < \rho(B) < \infty$ or $0 < \rho(A) = \rho(B) < \infty$ and $\tau_M(A) < \tau_M(B)$, then for every solution $f \not\equiv 0$ of (1.4) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\rho_2(\varphi) < \rho_2(f)$, we have*

$$\overline{\lambda_2}\left(f^{(j)} - \varphi\right) = \rho_2(f) = \rho(B) \;\; (j = 0, 1, 2, \ldots).$$

Recently in [15, 16], H. Y. Xu, J. Tu, X. M. Zheng and H. Y. Xu, J. Tu have investigated the relationship between small functions and the derivatives of solutions of higher order linear differential equations with entire and meromorphic functions. It is a natural to ask what about the exponent of convergence of zeros of $f^{(j)}(z)$ $(j = 0, 1, 2, \ldots)$, where f is a solution of (1.1). The main purpose of this paper is to give an answer to this question. The method used in the proofs of our theorems is quite different from the method used in the papers [13, 16]. Before we state our results we need to define the following notations

$$A_j(z) = A_{j-1}(z) - \frac{B'_{j-1}(z)}{B_{j-1}(z)} \;\text{ for }\; j = 1, 2, 3, \ldots, \tag{1.5}$$

$$B_j(z) = A'_{j-1}(z) - A_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} + B_{j-1}(z) \;\text{ for }\; j = 1, 2, 3, \ldots \tag{1.6}$$

and

$$F_j(z) = F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} \;\text{for }\; j = 1, 2, 3, \ldots, \tag{1.7}$$

where $A_0(z) = A(z)$, $B_0(z) = B(z)$ and $F_0(z) = F(z)$. We obtain the following results.

Theorem 3. *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite order such that $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \ldots)$. If f is a meromorphic solution of (1.1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \;\; (j = 0, 1, 2, \ldots)$$

and

$$\overline{\lambda}_2\left(f^{(j)}\right) = \lambda_2\left(f^{(j)}\right) = \rho \ (j = 0, 1, 2, \ldots).$$

Theorem 4. *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite order such that $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \ldots)$. If f is a meromorphic solution of (1.1) with*

$$\rho(f) > \max\{\rho(A), \rho(B), \rho(F)\},$$

then

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = \rho(f) \ (j = 0, 1, 2, \ldots).$$

Remark 2. The conditions $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \ldots)$ are necessary. For example $f(z) = e^{-z} + 1$ satisfies (1.1), where $A(z) = \frac{z}{z+1}$, $B(z) = -\frac{1}{z+1}$ and $F(z) = -\frac{1}{z+1}$. On the other hand

$$A_1 = A - \frac{B'}{B} = 1,$$

$$B_1 = A' - A\frac{B'}{B} + B \equiv 0, \ F_1 = F' - F\frac{B'}{B} \equiv 0$$

and

$$\lambda(f) = 1 > \lambda\left(f^{(j)}\right) = 0 \ (j = 1, 2, 3, \ldots).$$

Here, we will give some sufficient conditions on the coefficients which guarantee $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \ldots)$:

Theorem 5. *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be entire functions with finite order such that $\rho(B) > \max\{\rho(A), \rho(F)\}$. Then all nontrivial solutions of (1.1) satisfy*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0, 1, 2, \ldots)$$

with at most one possible exceptional solution f_0 such that

$$\rho(f_0) = \max\left\{\overline{\lambda}(f_0), \rho(B)\right\}.$$

Remark 3. The condition $\rho(B) > \max\{\rho(A), \rho(F)\}$ does not ensure that all solutions of (1.1) are of infinite order. For example we can see that $f_0(z) = e^{-z^2}$ satisfies the differential equation

$$f'' + 2zf' + (e^{z^2} + 2)f = 1,$$

where

$$\overline{\lambda}(f_0) = 0 < \rho(f_0) = \rho(B) = 2.$$

In the next, we note

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log m(r,f)}{\log r}.$$

Theorem 6. *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite order such that $\sigma(B) > \max\{\sigma(A), \sigma(F)\}$. If f is a meromorphic solution of (1.1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda_2}\left(f^{(j)}\right) = \lambda_2\left(f^{(j)}\right) = \rho \ (j = 0, 1, 2, \dots).$$

Theorem 7. *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be entire functions with finite order such that $\rho(B) = \rho(A) > \rho(F)$ and $\tau(B) > k\tau(A)$, $k \geqslant 1$ is an integer. If f is a nontrivial solution of (1.1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0, 1, \dots, k)$$

and

$$\overline{\lambda_2}\left(f^{(j)}\right) = \lambda_2\left(f^{(j)}\right) = \rho \ (j = 0, 1, \dots, k).$$

Corollary 1. *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, $H \not\equiv 0$ are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $|b| > k|a|$, $k \geqslant 1$ is an integer. Then every nontrivial solution f of (1.3) satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0, 1, \dots, k).$$

2. Preliminary lemmas

Lemma 1 ([8])**.** *Let f be a meromorphic function and let $k \geqslant 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O(\log T(r,f) + \log r)$, possibly outside of an exceptional set $E \subset (0, +\infty)$ of r with finite linear measure. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2 ([3,4])**.** *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions.*

(i) If f is a meromorphic solution of the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \dots + A_1 f' + A_0 f = F \tag{2.1}$$

with $\rho(f)=+\infty$ *, then* f *satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty.$$

(*ii*) *If* f *is a meromorphic solution of equation* (2.1) *with* $\rho(f)=+\infty$ *and* $\rho_2(f)=\rho$*, then*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty,\ \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)=\rho.$$

Lemma 3 ([13])**.** *Let* $A_0,A_1,\dots,A_{k-1},F\not\equiv 0$ *be finite order meromorphic functions. If* f *is a meromorphic solution of equation* (2.1) *with*

$$\max\left\{\rho\left(A_j\right)\ (j=0,1,\dots,k-1),\rho(F)\right\}<\rho(f)<\infty,$$

then

$$\overline{\lambda}(f)=\lambda(f)=\rho(f).$$

Lemma 4 ([10])**.** *Let* f *and* g *be meromorphic functions in the complex plane such that* $0<\rho(f),\rho(g)<\infty$ *and* $0<\tau(f),\tau(g)<\infty$*. Then we have*
(*i*) *If* $\rho(f)>\rho(g)$*, then we obtain*

$$\tau(f+g)=\tau(fg)=\tau(f).$$

(*ii*) *If* $\rho(f)=\rho(g)$ *and* $\tau(f)\neq\tau(g)$*, then we get*

$$\rho(f+g)=\rho(fg)=\rho(f)=\rho(g).$$

Lemma 5 ([6])**.** *Let* $A,B_1,\dots,B_{k-1}$*,* $F\not\equiv 0$ *be entire functions of finite order, where* $k\geqslant 2$*. Suppose that either* (*i*) *or* (*ii*) *below holds:*
(*i*) $\rho\left(B_j\right)<\rho(A)\,(j=1,\dots,k-1)$,
(*ii*) $B_1,\dots,B_{k-1}$ *are polynomials and* A *is transcendental. Then we have*
(*a*) *All solutions of the differential equation*

$$f^{(k)}+B_{k-1}f^{(k-1)}+\cdots+B_1f'+Af=F$$

satisfy

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty$$

with at most one possible solution f_0 *of finite order.*
(*b*) *If there exists an exceptional solution* f_0 *in case* (*a*)*, then* f_0 *satisfies*

$$\rho(f_0)\leqslant\max\left\{\rho(A),\rho(F),\overline{\lambda}(f_0)\right\}<\infty. \tag{2.2}$$

Furthermore, if $\rho(A)\neq\rho(F)$ *and* $\overline{\lambda}(f_0)<\rho(f_0)$*, then*

$$\rho(f_0)=\max\{\rho(A),\rho(F)\}.$$

3. Proof of the theorems and corollary

Proof of Theorem 3. We prove this theorem by using mathematical induction. Since $B \not\equiv 0$, $F \not\equiv 0$, then by using Lemma 2, we have

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Dividing both sides of (1.1) by B, we obtain

$$\frac{1}{B} f'' + \frac{A}{B} f' + f = \frac{F}{B}. \tag{3.1}$$

Differentiating both sides of equation (3.1), we have

$$\frac{1}{B} f^{(3)} + \left(\left(\frac{1}{B}\right)' + \frac{A}{B}\right) f'' + \left(\left(\frac{A}{B}\right)' + 1\right) f' = \left(\frac{F}{B}\right)'. \tag{3.2}$$

Multiplying now (3.2) by B, we get

$$f^{(3)} + A_1 f'' + B_1 f' = F_1, \tag{3.3}$$

where

$$A_1 = A - \frac{B'}{B},$$

$$B_1 = A' - A\frac{B'}{B} + B$$

and

$$F_1 = F' - F\frac{B'}{B}.$$

Since $B_1 \not\equiv 0$, $F_1 \not\equiv 0$ are meromorphic functions with finite order, then by using Lemma 2, we obtain

$$\overline{\lambda}(f') = \lambda(f') = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f') = \lambda_2(f') = \rho_2(f) = \rho.$$

Dividing now both sides of (3.3) by B_1, we obtain

$$\frac{1}{B_1} f^{(3)} + \frac{A_1}{B_1} f'' + f' = \frac{F_1}{B_1}. \tag{3.4}$$

Differentiating both sides of equation (3.4) and multiplying by B_1, we get

$$f^{(4)} + A_2 f^{(3)} + B_2 f'' = F_2, \tag{3.5}$$

where $A_2, B_2 \not\equiv 0$ and $F_2 \not\equiv 0$ are meromorphic functions defined in (1.5)-(1.7). By using Lemma 2, we obtain

$$\overline{\lambda}(f'') = \lambda(f'') = \rho(f) = +\infty$$

and
$$\overline{\lambda}_2\left(f''\right)=\lambda_2\left(f''\right)=\rho_2\left(f\right)=\rho.$$
We suppose now that
$$\overline{\lambda}\left(f^{(k)}\right)=\lambda\left(f^{(k)}\right)=\rho\left(f\right)=+\infty,\ \overline{\lambda}_2\left(f^{(k)}\right)=\lambda_2\left(f^{(k)}\right)=\rho_2\left(f\right)=\rho \quad (3.6)$$
for all $k=0,1,2,\ldots,j-1$, and we prove that (3.6) is true for $k=j$. By the same method as before, we can obtain
$$f^{(j+2)}+A_j f^{(j+1)}+B_j f^{(j)}=F_j,$$
where $A_j, B_j\not\equiv 0$ and $F_j\not\equiv 0$ are meromorphic functions defined in (1.5) -(1.7). By using Lemma 2, we obtain
$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=\rho\left(f\right)=+\infty$$
and
$$\overline{\lambda}_2\left(f^{(j)}\right)=\lambda_2\left(f^{(j)}\right)=\rho_2\left(f\right)=\rho.$$
Thus, the proof of Theorem 3 is completed. □

Proof of Theorem 4. By a similar reasoning as in the proof of Theorem 3, and by using Lemma 3, we obtain
$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=\rho\left(f\right)\quad (j=0,1,2,\ldots).$$
□

Proof of Theorem 5. By Lemma 5, all nontrivial solutions of (1.1) are of infinite order with at most one exceptional solution f_0 of finite order. By using (1.5) and Lemma 1 we have
$$m\left(r,A_j\right)\leqslant m\left(r,A_{j-1}\right)+O\left(\log r\right)$$
for all $j=1,2,3,\ldots$, which we can rewrite as
$$m\left(r,A_j\right)\leqslant m\left(r,A\right)+O\left(\log r\right)\quad (j=1,2,3,\ldots). \tag{3.7}$$
On the other hand, we have from (1.6)
$$B_j=A_{j-1}\left(\frac{A'_{j-1}}{A_{j-1}}-\frac{B'_{j-1}}{B_{j-1}}\right)+B_{j-1}$$
$$=A_{j-1}\left(\frac{A'_{j-1}}{A_{j-1}}-\frac{B'_{j-1}}{B_{j-1}}\right)+A_{j-2}\left(\frac{A'_{j-2}}{A_{j-2}}-\frac{B'_{j-2}}{B_{j-2}}\right)+B_{j-2}$$
$$=\sum_{k=0}^{j-1}A_k\left(\frac{A'_k}{A_k}-\frac{B'_k}{B_k}\right)+B. \tag{3.8}$$

Now we prove that $B_j \not\equiv 0$ for all $j = 1,2,3,\dots$. Suppose there exists an integer $j = 1,2,3,\dots$ such that $B_j \equiv 0$. By (3.7) and (3.8)

$$T(r,B) = m(r,B) \leqslant \sum_{k=0}^{j-1} m(r,A_k) + O(\log r)$$
$$\leqslant jm(r,A) + O(\log r) = jT(r,A) + O(\log r) \tag{3.9}$$

which implies the contradiction $\rho(B) \leqslant \rho(A)$. Hence $B_j \not\equiv 0$ for all $j = 1,2,3,\dots$. Suppose now there exists an integer $j = 1,2,3,\dots$ that is the first index for which $F_j \equiv 0$. Then, by (1.7) and $F_{j-1}(z) \not\equiv 0$ we have

$$F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} = 0$$

which implies

$$F_{j-1}(z) = cB_{j-1}(z),$$

where $c \in \mathbb{C}\setminus\{0\}$. By (3.8) we have

$$\frac{1}{c}F_{j-1} = \sum_{k=0}^{j-2} A_k\left(\frac{A'_k}{A_k} - \frac{B'_k}{B_k}\right) + B. \tag{3.10}$$

On the other hand, we have from (1.7)

$$m\left(r,F_j\right) \leqslant m(r,F) + O(\log r) \quad (j = 1,2,3,\dots). \tag{3.11}$$

By (3.10), (3.11) and Lemma 1, we obtain

$$T(r,B) = m(r,B) \leqslant \sum_{k=0}^{j-2} m(r,A_k) + m\left(r,F_{j-1}\right) + O(\log r)$$
$$\leqslant (j-1)m(r,A) + m(r,F) + O(\log r)$$
$$= (j-1)T(r,A) + T(r,F) + O(\log r) \tag{3.12}$$

which implies the contradiction $\rho(B) \leqslant \max\{\rho(A),\rho(F)\}$. Since $B_j \not\equiv 0$ and $F_j \not\equiv 0$ $(j = 1,2,3,\dots)$, then by applying Theorem 3 and Lemma 5 we have

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0,1,2,\dots)$$

with at most one exceptional solution f_0 of finite order. Since $\rho(B) > \max\{\rho(A),\rho(F)\}$, then by (2.2) we obtain

$$\rho(f_0) \leqslant \max\left\{\rho(B),\overline{\lambda}(f_0)\right\}. \tag{3.13}$$

On the other hand by (1.1), we can write

$$B = \frac{F}{f_0} - \left(\frac{f''_0}{f_0} + A\frac{f'_0}{f_0}\right).$$

It follows that

$$\begin{aligned} T(r,B) = m(r,B) &\leqslant m\left(r,\frac{F}{f_0}\right) + m(r,A) + O(\log r) \\ &\leqslant T(r,f_0) + T(r,F) + T(r,A) + O(\log r) \end{aligned}$$

which implies

$$\rho(B) \leqslant \max\{\rho(f_0), \rho(A), \rho(F)\} = \rho(f_0). \tag{3.14}$$

Since $\overline{\lambda}(f_0) \leqslant \rho(f_0)$, then by using (3.13) and (3.14) we obtain

$$\rho(f_0) = \max\left\{\rho(B), \overline{\lambda}(f_0)\right\}.$$

This completes the proof of Theorem 5. □

Proof of Theorem 6. By using the same reasoning as in the proof of Theorem 5, we can prove Theorem 6. □

Proof of Theorem 7. First, we prove that $B_j \not\equiv 0$ for all $j = 1,2,\dots,k$. Suppose there exists an integer s, $1 \leqslant s \leqslant k$ such that $B_s \equiv 0$. By (3.7) and (3.8), we have

$$\begin{aligned} T(r,B) = m(r,B) &\leqslant \sum_{k=0}^{s-1} m(r,A_k) + O(\log r) \\ &\leqslant sm(r,A) + O(\log r) = sT(r,A) + O(\log r) \end{aligned} \tag{3.15}$$

which implies the contradiction $k\tau(A) < \tau(B) \leqslant s\tau(A)$. Hence $B_j \not\equiv 0$ for all $j = 1,2,\dots,k$. Now, we prove that $F_j \not\equiv 0$ for all $j = 1,2,\dots,k$. Suppose there exists an integer s, $1 \leqslant s \leqslant k$ such that $F_s \equiv 0$. From (3.12), we have

$$\begin{aligned} T(r,B) &\leqslant (s-1)m(r,A) + m(r,F) + O(\log r) \\ &= (s-1)T(r,A) + T(r,F) + O(\log r) \end{aligned} \tag{3.16}$$

which implies the contradiction $k\tau(A) < \tau(B) \leqslant (s-1)\tau(A)$. Hence $F_j \not\equiv 0$ for all $j = 1,2,\dots,k$. Since $B_j \not\equiv 0$ and $F_j \not\equiv 0$ $(j = 1,\dots,k)$, then by Theorem 3 we have

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0,1,\dots,k)$$

and

$$\overline{\lambda_2}\left(f^{(j)}\right) = \lambda_2\left(f^{(j)}\right) = \rho \ (j = 0,1,\dots,k).$$

□

Proof of Corollary 1. Since $ab \neq 0$, $|b| > k|a|$, then by Theorem 1, every non-trivial solution f of (1.3) is of infinite order. By using Lemma 4 we have

$$\tau\left(A_0 e^{bz}\right) = \tau\left(e^{bz}\right) = \frac{|b|}{\pi} > k\frac{|a|}{\pi} = k\tau\left(A_1 e^{az}\right).$$

Then by Theorem 7, we obtain

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=\rho(f)=+\infty \ (j=0,1,\ldots,k).$$

□

4. Open problem

It's interesting to study whether the condition $|b|>k|a|$, where $k\geqslant 1$ is an integer is necessary in Corollary 1. For that, we pose the following problem.

Conjecture 1. *Suppose that $A_0\not\equiv 0$, $A_1\not\equiv 0$, $H\not\equiv 0$ are entire functions of order less than one, and the complex constants a,b satisfy $ab\neq 0$ and $|b|>|a|$. Then every nontrivial solution f of (1.3) satisfies*

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=+\infty \ (j=0,1,\ldots).$$

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