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Daniela Paesano and Pasquale Vetro

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FIXED POINTS AND COMPLETENESS ON PARTIAL METRIC SPACES

DANIELA PAESANO AND PASQUALE VETRO

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Abstract. Recently, Suzuki [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008), 1861-1869] proved a fixed point theorem that is a generalization of the Banach contraction principle and characterizes the metric completeness. Paesano and Vetro [D. Paesano and P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, Topology Appl., **159** (2012), 911-920] proved an analogous fixed point result for a self-mapping on a partial metric space that characterizes the partial metric 0-completeness. In this paper we prove a fixed point result for a new class of contractions of Berinde-Suzuki type on a partial metric space. Moreover, using our results, as application we obtain a new characterization of partial metric 0-completeness. Finally, we give a typical application of fixed point methods to integral equation, by using our results.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: fixed and common fixed points, Suzuki fixed point theorem, partial metric spaces, ordered partial metric spaces, partial metric 0-completeness

1. INTRODUCTION

In 2008, in order to characterize the completeness of underlying metric spaces, Suzuki [14] introduced a weaker notion of contraction and proved the following theorem.

Theorem 1. *Define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r)=\begin{cases} 1 & \text{if } 0\le r\le(\sqrt{5}-1)/2 \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2<r<2^{-1/2} \\ (1+r)^{-1} & \text{if } 2^{-1/2}\le r<1. \end{cases}$$

Then for a metric space (X,d), the following are equivalent:

(i) *X is complete.*

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(ii) *There exists $r \in (0,1)$ such that every mapping T on X satisfying the following has a fixed point:*

$$\theta(r)d(x,Tx) \leq d(x,y) \quad implies \quad d(Tx,Ty) \leq rd(x,y) \text{ for all } x,y \in X.$$

In 1994 Matthews [9] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification.

Recently, Romaguera [13] proved that a partial metric space is 0-complete if and only if every Caristi type mapping on X has a fixed point. The result of Romaguera extended Kirk's [7] characterization of metric completeness to a kind of complete partial metric spaces.

In [11], Paesano and Vetro proved the following result.

Theorem 2. *Let (X,p) be a 0-complete partial metric space and T be a self-mapping on X. Define a nonincreasing function $\theta : [0,1) \to (1/2,1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2 \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 < r < 2^{-1/2} \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r)p(x,Tx) \leq p(x,y) \quad implies \quad p(Tx,Ty) \leq rp(x,y) \tag{1.1}$$

for all $x,y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_{n\to+\infty} T^n x = z$ for all $x \in X$.

In this paper, in the framework of partial metric spaces, we prove fixed point results for a class of weaker contractive self-mappings of Berinde-Suzuki type (see [2,3,14]). We deduce, also, common fixed point results for two self-mappings. Moreover, using our results, as application we obtain a new characterization of partial metric 0-completeness in terms of fixed point. This characterization extends Suzuki's characterization of metric completeness. Finally, we give a typical application of fixed point methods to integral equation, by using our results.

2. Preliminaries

First, we recall some definitions of partial metric spaces that can be found in [4, 6, 8–11, 13, 15] and references therein. A partial metric on a nonempty set X is a function $p : X \times X \to [0,+\infty)$ such that for all $x,y,z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(p_2) $p(x,x) \leq p(x,y)$;
(p_3) $p(x,y) = p(y,x)$;
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic examples of partial metric spaces is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$. Notice that the function $p^s : X \times X \to [0, +\infty)$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

is a metric on X, induced by the partial metric p.

Each partial metric p on X generates a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\},$$

for all $x \in X$, $\epsilon > 0$.

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$. A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$. Also, a sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s). A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$. Moreover, (X, p) is complete if and only if the metric space (X, p^s) is complete.

A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 1 (Lemma 2 in [11]). *Let (X, p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to x \in X$ and $p(x, x) = 0$, then $\lim_{n\to+\infty} p(x_n, z) = p(x, z)$ for all $z \in X$.*

Lemma 2 (Lemma 1 in [11]). *Let (X, p) be a partial metric space and $\{x_n\} \subset X$ a 0-Cauchy sequence. The sequence $\{p(x, x_n)\}$ is Cauchy in $\mathbb{R}$ for all $x \in X$.*

Let X be a non-empty set and $T, f : X \to X$. The mappings T and f are said to be weakly compatible if they commute at their coincidence point (i.e. $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. We denote with $F(T) := \{y \in X : Ty = y\}$ the set of fixed points of T, and with $PC(T, f) := \{y \in X : Tx = fx = y$ for some $x \in X\}$ the set of points of coincidence of T and f. The following lemma is obvious.

Lemma 3. *Let X be a non-empty set and let $T, f : X \to X$ be weakly compatible. If T and f have a unique point of coincidence v in X, then v is a unique common fixed point of T and f.*

Let X be a nonempty set together with a partial order, that is, a binary relation $\preceq$ which is reflexive, antisymmetric and transitive. Then $(X,\preceq)$ is said to be a partially ordered set and $x,y\in X$ are called comparable if one of the elements precedes the other ($x\preceq y$ or $y\preceq x$). Also, if (X,p) is a partial metric space then $(X,p,\preceq)$ is called an ordered partial metric space. Let $f,T:X\to X$ be two mappings, T is said to be f-increasing if $fx\prec fy$ implies $Tx\prec Ty$ for all $x,y\in X$. If f is the identity mapping on X, then T is increasing. If $Tx\preceq Ty$ whenever $x,y\in X$ and $x\preceq y$, then T is said to be nondecreasing.

Lemma 4. *Let (X,p) be a 0-complete partial metric space and let $\{x_n\}\subset X$. Assume that there exists $r\in[0,1)$ such that*

$$p(x_n,x_{n+1})\le rp(x_{n-1},x_n) \tag{2.1}$$

for all $n\in\mathbb{N}$. Then there exists $z\in X$ such that $\lim_{n\to+\infty}p(x_n,z)=p(z,z)=0$. Further, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$p(z,x_{n(k)+1})\le p(z,x_{n(k)})\qquad for\ all\quad k\in\mathbb{N}. \tag{2.2}$$

Proof. From (2.1), we deduce that

$$p(x_{j-1+n},x_{j+n})\le r^n p(x_{j-1},x_j) \tag{2.3}$$

for all $n,j\in\mathbb{N}$. This implies that $\{x_n\}$ is a 0-Cauchy sequence. As (X,p) is 0-complete, there exists $z\in X$ such that $\lim_{n\to+\infty}p(x_n,z)=p(z,z)=0$.

Now, we show that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that (2.2) holds. Arguing by contradiction, we assume that there exists $\nu\in\mathbb{N}$ such that

$$p(z,x_j)>p(z,x_{j-1})\qquad \text{for all}\quad j\ge\nu. \tag{2.4}$$

By induction on n from (2.4), we deduce that

$$p(z,x_{j-1+n})>p(z,x_{j-1})\qquad\text{for all}\quad j,n\in\mathbb{N},\ \ j\ge\nu. \tag{2.5}$$

Now by (2.3), we get that

$$\begin{aligned}p(x_{j-1+n},x_{j-1})&\le p(x_{j-1},x_j)+\cdots+p(x_{j-2+n},x_{j-1+n})-\sum_{k=j}^{j+n-2}p(x_k,x_k)\\&\le(1+\cdots+r^{n-1})p(x_{j-1},x_j)\\&\le\frac{1}{1-r}\,p(x_{j-1},x_j).\end{aligned}$$

As $n\to+\infty$, by Lemma 1, we obtain that $p(z,x_{j-1})\le\dfrac{1}{1-r}\,p(x_{j-1},x_j)$ for all $j\in\mathbb{N}$. This implies that

$$p(z,x_{j-1+n})\le\frac{1}{1-r}p(x_{j-1+n},x_{j+n})\le\frac{r^n}{1-r}p(x_{j-1},x_j) \tag{2.6}$$

for all $n, j \in \mathbb{N}$. By (2.5) and (2.6) we get

$$p(z, x_{j-1}) < \frac{r^n}{1-r} p(x_{j-1}, x_j) \qquad \text{for all} \quad n \in \mathbb{N}, \ \ j \geq \nu.$$

From this relation, as $n \to +\infty$, we obtain that $p(z, x_{j-1}) \leq 0$, that is $z = x_{j-1}$ for all $j \geq \nu$, which is in contradiction with (2.4). Hence, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that (2.2) holds. □

3. Fixed point theorem in partial metric spaces

We start this section by presenting a fixed point theorem.

Theorem 3. *Let (X, p) be a 0-complete partial metric space and T be a mapping on X. Assume that there exist $r \in [0,1)$ and $L \geq 0$ such that*

$$p(y, Tx) \leq p(y, x) \text{ implies } p(Tx, Ty) \leq rp(x, y) + L(p(y, Tx) - p(y, y)) \quad (3.1)$$

for all $x, y \in X$. Then T has a fixed point in X. Moreover, if $r + L < 1$ there exists a unique fixed point z of T.

Proof. By (p_2), we have $p(Tx, Tx) \leq p(Tx, x)$ for every $x \in X$. By (3.1), we get that

$$p(Tx, T^2x) \leq rp(x, Tx) + L(p(Tx, Tx) - p(Tx, Tx)) = rp(x, Tx), \qquad (3.2)$$

for every $x \in X$. Now, we fix $x_0 \in X$ and define a sequence $\{x_n\} \subset X$ by $x_n = T^n x_0 = Tx_{n-1}$, for all $n \in \mathbb{N}$. From (3.2), we deduce that

$$p(x_n, x_{n+1}) \leq rp(x_{n-1}, x_n),$$

that is (2.1) holds for all $n \in \mathbb{N}$. By Lemma 4, there exists $z \in X$ such that $\lim_{n \to +\infty} p(x_n, z) = p(z, z) = 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$p(z, x_{n(k)+1}) \leq p(z, x_{n(k)}), \qquad \text{for all} \quad k \in \mathbb{N}.$$

Now, using (3.1) and $p(z, z) = 0$, we obtain that

$$p(Tx_{n(k)}, Tz) \leq rp(x_{n(k)}, z) + Lp(z, x_{n(k)+1}) \qquad \text{for all} \quad k \in \mathbb{N}.$$

As $k \to +\infty$, by Lemma 1, we get

$$p(z, Tz) \leq (r + L)\, p(z, z) = 0$$

and hence $Tz = z$, that is z is a fixed point of T. Now, we assume that $r + L < 1$ and let $y \in X$ be another fixed point of T with $y \neq z$. Then, we have

$$p(z, y) = p(z, Ty) \leq p(z, y)$$

and, by (3.1), we get

$$p(y, z) = p(Ty, Tz) \leq rp(y, z) + Lp(z, Ty) - Lp(z, z) \leq (r + L) p(y, z).$$

This is possible only if $p(z, y) = 0$, which is a contradiction. Therefore z is a unique fixed point of T. □

Example 1. Let $X = \{0,1\} \cup [r^{-1},4]$, where $r \in (3/4,1)$, be endowed with the partial metric $p(x,y) = \max\{x,y\}$ and $T : X \to X$ be defined by

$$Tx = \begin{cases} 0 & \text{if } x \in \{0,1\}, \\ 1 & \text{if } x \in [r^{-1},4]. \end{cases}$$

For every $x,y \in X$ we have:

$$p(y,Tx) = \begin{cases} y & \text{if } x \in \{0,1\}, \\ y & \text{if } y \in [r^{-1},4], \\ \max\{1,y\} & \text{if } x \in [r^{-1},4] \end{cases}$$

and hence $p(y,Tx) \leq p(y,x)$ and $p(Tx,Ty) \leq rp(x,y)$ for all $x,y \in X$. So T satisfies the condition (3.1) in Theorem 3 with $L = 0$.

Since (X,p) is a 0-complete partial metric space, using Theorem 3, we deduce that T has a unique fixed point.

Example 2. Let $X = [0,1] \cup [r^{-1},4]$, where $r \in (3/4,1)$, be endowed with the partial metric

$$p(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1] \\ \max\{x,y\} & \text{if } \{x,y\} \cap [r^{-1},4] \neq \varnothing \end{cases}$$

and $T : X \to X$ be defined by

$$Tx = \begin{cases} x & \text{if } x \in [0,1] \\ 1 & \text{if } x \in [r^{-1},4], \end{cases}$$

For every $x,y \in X$, we deduce easily that

$$p(y,Tx) \leq p(y,x) \quad \text{and} \quad p(Tx,Ty) \leq rp(x,y) + L(p(y,Tx) - p(y,y)),$$

where $L \geq 1-r$. So T satisfy the condition (3.1) in Theorem 3 with $L \geq 1-r$.

Since (X,p) is a 0-complete partial metric space, using Theorem 3, we deduce that T has a fixed point.

Now, we use the following lemma, that is a consequence of the axiom of choice, to obtain common fixed point results for two self-mappings defined on a partial metric space.

Lemma 5 (Lemma 2.1 in [5]). *Let X be a non-empty set and $f : X \to X$ a mapping. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one.*

Theorem 4. *Let (X,p) be a partial metric space and let T, f be self-mappings on X, such that $TX \subset fX$. Assume that there exists $r \in [0,1)$ and $L \geq 0$ such that*

$$p(fy,Tx) \leq p(fy,fx) \quad implies$$
$$p(Tx,Ty) \leq rp(fx,fy) + L(p(fy,Tx) - p(fy,fy)) \tag{3.3}$$

for all $x, y \in X$. Then T and f have a point of coincidence in X. Moreover, if $r + L < 1$, T and f are weakly compatible and fX is 0-complete, then T and f have a unique common fixed point.

Proof. By Lemma 5, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Define

$$S : fE \to fE \quad \text{by} \quad Sfx = Tx \qquad \text{for all} \quad fx \in fE.$$

Since f is one-to-one on E and $TX \subset fX$, S is well defined. By condition (3.3), for all $fx, fy \in fE$, we have

$$p(fy, Sfx) \leq p(fy, fx)$$

implies

$$p(Sfx, Sfy) \leq rp(fx, fy) + L(p(fy, Sfx) - p(fy, fy)).$$

Then, since fE is 0-complete, by Theorem 3, we have that S has a fixed point on fE, say fz. This implies that $u = fz = Tz$ is a point of coincidence of T and f. Now, we assume that $r + L < 1$ and we prove that T and f have a unique point of coincidence. Let fw be another point of coincidence of T and f with $fw \neq fz$. Since $p(fw, Tz) \leq p(fw, fz)$, using (3.3) we get that

$$p(Tz, Tw) \leq rp(fz, fw) + L(p(fw, Tz) - p(fw, fw)) \leq (r + L)p(Tz, Tw)$$

which is a contradiction and hence T and f have a unique point of coincidence. In conclusion, since T and f are weakly compatible, by Lemma 3, we have that fz is the unique common fixed point of T and f. □

4. Fixed point theorems on ordered partial metric spaces.

In this section we prove fixed point results on the framework of ordered partial metric spaces. For some interesting papers in this direction on metric spaces, we refer to [1, 11, 12].

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that the partial metric space (X, p) is 0-complete. Let $T : X \to X$ be an increasing mapping with respect to $\preceq$. Assume that there exists $r \in [0, 1)$ and $L \geq 0$ such that*

$$p(y, Tx) \leq p(y, x) \text{ implies } p(Tx, Ty) \leq rp(x, y) + L(p(y, Tx) - p(y, y)) \quad (4.1)$$

for all comparable $x, y \in X$ with $x \neq y$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *for an increasing sequence $\{x_n\} \subset X$ converging to $x \in X$, we have $x_n \prec x$, for all $n \in \mathbb{N}$;*

then T has a fixed point in X. Moreover, if

(iii) *for all $x, y \in F(T)$ there exists $v \in X$ such that x, v and y, v are comparable;*

(iv) $r+L<1$,

then T has a unique fixed point.

Proof. Let $x_0 \in X$ such that $x_0 \preceq Tx_0$. If $x_0 = Tx_0$, then the result is proved. Hence, we suppose $x_0 \prec Tx_0$. Define a sequence $\{x_n\} \subset X$ such that $x_{n+1} = T^{n+1}x_0 = Tx_n$, for all $n \in \mathbb{N}$. Since T is increasing and $x_0 \prec Tx_0$, we have

$$x_0 \prec Tx_0 = x_1 \prec T^2x_0 = x_2 \prec \cdots \prec x_n \prec \cdots$$

and so $\{x_n\}$ is an increasing sequence.

Note that $p(x_n, Tx_{n-1}) \leq p(x_n, x_{n-1})$ holds for all $n \in \mathbb{N}$. Since x_n and x_{n-1} are comparable for all $n \in \mathbb{N}$, from (4.1) with $x = x_{n-1}$ and $y = x_n$, we deduce that

$$p(x_n, x_{n+1}) \leq rp(x_{n-1}, x_n) + L(p(x_n, x_n) - p(x_n, x_n)) = rp(x_{n-1}, x_n).$$

This implies that (2.1) holds for all $n \in \mathbb{N}$. By Lemma 4, we deduce that there exist $z \in X$ with $p(z,z) = 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\lim_{n\to+\infty} p(x_n, z) = p(z,z)$ and

$$p(z, x_{n(k)+1}) \leq p(z, x_{n(k)}) \qquad \text{for all } k \in \mathbb{N}.$$

By (ii), we have that z is comparable with $x_{n(k)} \neq z$ for all k and, by condition (4.1) and $p(z,z) = 0$, we obtain

$$p(x_{n(k)+1}, Tz) \leq rp(x_{n(k)}, z) + Lp(z, x_{n(k)+1}) \qquad \text{for all } k \in \mathbb{N}.$$

Taking the limit as $k \to +\infty$, we deduce that $p(z, Tz) \leq (r+L)p(z,z) = 0$ and so $Tz = z$. Hence z is a fixed point of T.

Now, we assume that hypotheses (iii) and (iv) hold and we prove the uniqueness of the fixed point. Suppose that there exists $y \in X$ such that $Ty = y$, with $y \neq z$. We have two possible cases:

Case 1. y and z are comparable. Then, since

$$p(z,y) = p(z,Ty) \leq p(z,y)$$

by (4.1), it follows that

$$p(y,z) = p(Ty,Tz) \leq rp(y,z) + L(p(z,Ty) - p(z,z)) \leq (r+L)p(y,z),$$

which is possible only if $p(y,z) = 0$, which leads to contradiction.

Case 2. y and z are not comparable. In this case, there exists $v \in X \setminus \{y,z\}$, comparable with y and with z. Since T is increasing with respect to $\preceq$, it follows that $T^{n-1}y$, $T^{n-1}v$ are comparable and $T^{n-1}y \neq T^{n-1}v$ for all $n \in \mathbb{N}$. Now, since

$$p(T^{n-1}v, T^n y) \leq p(T^{n-1}v, T^{n-1}y) \qquad \text{for all } n \in \mathbb{N},$$

by (4.1), we obtain that

$$\begin{aligned} p(T^n y, T^n v) &\leq rp(T^{n-1}y, T^{n-1}v) + L(p(T^{n-1}v, T^n y) - p(T^{n-1}v, T^{n-1}v)) \\ &\leq (r+L)p(T^{n-1}y, T^{n-1}v) \end{aligned}$$

for all $n \in \mathbb{N}$. Iterating this process, it follows that

$$p(T^n y, T^n v) \leq (r+L) p(T^{n-1} y, T^{n-1} v) \leq \cdots \leq (r+L)^n p(y,v),$$

that is $p(T^n y, T^n v) \to 0$ as $n \to +\infty$. With similar arguments, we deduce that $p(T^n z, T^n v) \to 0$ as $n \to +\infty$. Hence,

$$0 < p(y,z) = p(T^n y, T^n z) \leq p(T^n y, T^n v) + p(T^n v, T^n z) \to 0$$

as $n \to +\infty$, which is a contradiction.

In both cases, we deduce that T has a unique fixed point. □

From the above theorem, we deduce the following consequence.

Corollary 1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that the partial metric space (X, p) is 0-complete. Let $T : X \to X$ be an increasing mapping with respect to $\preceq$. Assume that there exist $r \in [0, 1)$ and $L \geq 0$ such that*

$$p(y, Tx) \leq p(y, x) \text{ implies } p(Tx, Ty) \leq r p(x, y) + L(p(y, Tx) - p(y, y)) \quad (4.2)$$

for all comparable $x, y \in X$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq T x_0$;*
(ii) *for an increasing sequence $\{x_n\} \subset X$ converging to $x \in X$, we have $x_n \prec x$, for all $n \in \mathbb{N}$;*

then T has a fixed point in X. Moreover, if

(iii) *for all $x, y \in F(T)$ there exists $v \in X$ such that x, v and y, v are comparable;*
(iv) *$r + L < 1$,*

then T has a unique fixed point.

Example 3. Let $X = [0,1] \cup [r^{-1}, 4]$, where $r \in (3/4, 1)$, be endowed with the partial metric

$$p(x,y) = \begin{cases} |x-y| & \text{if } x, y \in [0,1], \\ \max\{x,y\} & \text{if } \{x,y\} \cap [r^{-1}, 4] \neq \varnothing, \end{cases}$$

and a partial order $\preceq$ on X defined by $x \preceq y$ if:

$$x = y, \ x \leq y \text{ and } x, y \in [0,1] \cap \mathbb{Q} \ \text{ or } \ x \in [0,1] \text{ and } y \in [r^{-1}, 4].$$

Then $(X, p, \preceq)$ is a 0-complete ordered partial metric space. The mapping $T : X \to X$ defined by

$$Tx = \begin{cases} x/2 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ \frac{x}{1+x} & \text{if } x \in [0,1] \setminus \mathbb{Q}, \\ 1 & \text{if } x \in [r^{-1}, 4], \end{cases}$$

is increasing with respect to $\preceq$. In order to show that T satisfies the condition (4.2) in Corollary 1, we consider the following cases:

Case 1. $x \leq y$ and $x, y \in [0,1] \cap \mathbb{Q}$, then $p(Tx, Ty) = \frac{1}{2} p(x, y)$.

Case 2. $x \in [0,1]$ and $y \in [r^{-1},4]$, then

$$p(Tx,Ty) = 1 - Tx \leq 1 = rr^{-1} \leq ry = r\,p(x,y).$$

So T satisfies the condition (4.2) in Corollary 1 with $L = 0$.

Hence Corollary 1 applies and T has a unique fixed point.

Note that the mapping T does not satisfy the condition (3.1) of Theorem 3, in fact for $x \in [0,1] \setminus \mathbb{Q}$ and $y = Tx$, we have $p(y,Tx) = 0 < p(x,y)$. If there exist $r \in [0,1]$ and $L \geq 0$ such that (4.1) holds, then

$$\begin{aligned} p(Tx,Ty) &= \frac{x-y}{(1+x)(1+y)} \\ &\leq r(x-y) + L(p(y,Tx) - p(y,y)) \\ &= r(x-y). \end{aligned}$$

This implies that

$$\frac{1}{(1+x)(1+y)} \leq r$$

and letting $x \to 0$ we get that $1 \leq r$, which is a contradiction.

Theorem 6. *Let $(X,\preceq)$ be a partially ordered set and let T,f be self-mappings on X, such that T is f-increasing and $TX \subset fX$. Suppose that there exists a partial metric p on X such that fX is a 0-complete subset of X. Assume that there exists $r \in [0,1)$ and $L \geq 0$ such that*

$$\begin{aligned} p(fy,Tx) \leq p(fy,fx) \quad implies \\ p(Tx,Ty) \leq rp(fx,fy) + L(p(fy,Tx) - p(fy,fy)) \end{aligned} \tag{4.3}$$

for all comparable $fx, fy \in X$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$;*
(ii) *for every increasing sequence $\{x_n\} \subset X$ converging to $x \in X$ we have $x_n \preceq x$, for all $n \in \mathbb{N}$;*
(iii) *for all $x,y \in PC(T,f)$ there exists $v \in fX$ such that x,v and y,v are comparable and T and f are weakly compatible;*
(iv) *$r + L < 1$,*

then T and f have a unique common fixed point.

Proof. By Lemma 5, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Define

$$S : fE \to fE \quad \text{by} \quad Sfx = Tx \qquad \text{for all } fx \in fE.$$

Since f is one-to-one on E and since $TX \subset fX$, S is well defined. By hypothesis (4.3), for all comparable $fx, fy \in fE$, we have

$$p(fy,Sfx) \leq p(fy,fx)$$

implies

$$p(Sfx, Sfy) \leq rp(fx, fy) + L(p(fy, Sfx) - p(fy, fy)).$$

Furthermore, since T is f-increasing, we obtain that for all $x, y \in E$,

$$fx \prec fy \quad \text{implies} \quad Sfx = Tx \prec Ty = Sfy,$$

that is S is increasing on fE. Now, we note that if fz is a fixed point of S, then $fz \in PC(T, f)$. So, condition (iii) of Theorem 6 ensures that condition (iii) of Corollary 1 holds. Then, since fE is 0-complete, by Corollary 1, we have that S has a unique fixed point on fE, say fz, if $r + L < 1$.

Now, we assume that $r + L < 1$ and we prove that T and f have a unique point of coincidence. Let $fy \in PC(T, f)$ with $fy \neq fz$. Proceeding as in Corollary 1, Case 1 if fy and fz are comparable or Case 2 if fy and fz are not comparable, we deduce that $fy = fz$. In conclusion, since T and f are weakly compatible, by Lemma 3, we have that fz is the unique common fixed point of T and f. □

5. Completeness in partial metric spaces and fixed points

In this section we characterize those partial metric spaces for which every self-mapping of Berinde-Suzuki type has a fixed point in the style of Suzuki's characterization of metric completeness. This will be done by means of the notion of 0-complete partial metric space which was introduced by Romaguera in [13].

Theorem 7. *For a partial metric space (X, p) the following are equivalent:*

(i) *(X, p) is 0-complete;*

(ii) *There exsist $r \in (0, 1)$ and $L \geq 0$ such that every mapping T on X satisfying the following has a fixed point:*

$$p(y, Tx) \leq p(y, x) \quad \textit{implies} \quad p(Tx, Ty) \leq rp(x, y) + L(p(y, Tx) - p(y, y))$$

for all $x, y \in X$.

Proof. By Theorem 3, (i) implies (ii). Let us prove that (ii) implies (i). We assume (ii). Arguing by contradiction, we also assume that (X, p) is not 0-complete, that is, there exists a 0-Cauchy sequence $\{y_n\}$ which does not converge.

Define a function from X into $[0, +\infty)$ by $fx = \lim_{n \to +\infty} p(x, y_n)$ for $x \in X$. Since, by Lemma 2, the sequence $\{p(x, y_n)\}$ is Cauchy in $\mathbb{R}$ for all $x \in X$, then the function f is well defined. The following are obvious:

P1) $fx > 0$ for all $x \in X$.
In fact, if $fx = \lim_{n \to +\infty} p(x, y_n) = 0$ then $p(x, x) = 0$ and $\lim_{n \to +\infty} y_n = x$, which is a contradiction.

P2) $\lim_{n \to +\infty} fy_n = \lim_{n,m \to +\infty} p(y_n, y_m) = 0$.

P3) $fx - fy \leq p(x, y) \leq fx + fy \quad$ for all $x, y \in X$.
It follows from

- $p(x,y) \le p(x,y_n) + p(y_n,y) - p(y_n,y_n)$;
- $p(x,y_n) \le p(x,y) + p(y,y_n) - p(y,y)$.

Now, by $P1$) and $P2$), there exists $\nu : X \to \mathbb{N}$ such that $f y_{\nu(x)} \le \frac{r}{3+2r} f x$ for each $x \in X$. Define $T : X \to X$, by $Tx = y_{\nu(x)}$. Then it is obvious that

$$fTx \le \frac{r}{3+2r} fx \quad \text{and} \quad Tx \in \{y_n : n \in \mathbb{N}\},$$

for all $x \in X$. T does not have a fixed point, since $Tx \ne x$ for all $x \in X$ because $fTx < fx$. Fix $x, y \in X$ with $p(y,Tx) \le p(x,y)$. In the case where $fx > 2fy$, by $P3$), we have

$$\begin{aligned} p(Tx,Ty) &\le fTx + fTy < \frac{r}{3}(fx+fy) \\ &< \frac{r}{3}(fx+fy) + \frac{2r}{3}(fx-2fy) \\ &= r(fx-fy) \le rp(x,y). \end{aligned}$$

In the other case, where $fx \le 2fy$, we have

$$\begin{aligned} p(x,y) &\ge p(y,Tx) \ge fy - fTx \\ &\ge fy - \frac{r}{3+2r} fx \ge \left(1 - \frac{2r}{3+2r}\right) fy \\ &= \frac{3+2r-2r}{3+2r} fy = \frac{3}{3+2r} fy \end{aligned}$$

and hence

$$\begin{aligned} p(Tx,Ty) &\le fTx + fTy \le \frac{r}{3+2r}(fx+fy) \\ &\le \frac{3r}{3+2r} fy \le rp(x,y). \end{aligned}$$

By (ii), T has a fixed point which leads to contradiction. Hence we obtain that X is 0-complete. This completes the proof. □

6. Application to integral equation

In this section we give a typical application of fixed point methods to integral equation, by using our theorems in ordered partial metric spaces. Here, we consider the following integral equation:

$$x(t) = g(t) + \int_0^t K(t,s,x(s))ds, \qquad \text{for all } t \in [0,I], \tag{6.1}$$

where $I > 0$. Let $X = C([0,I],\mathbb{R})$ be the space of all continuous functions defined on $[0,I]$. Notice that $C([0,I])$ endowed with the metric

$$d(x,y) = \|x-y\|_\infty = \max_{t\in[0,I]} |x(t)-y(t)|$$

is a complete metric space. In the sequel, we consider X equipped with the partial order $\preceq$ given by

$$x, y \in X, \qquad x \preceq y \Longleftrightarrow (x(t) \leq y(t) \text{ and } \|x\|_\infty, \|y\|_\infty \leq 1) \text{ or } (x(t) = y(t)),$$

for all $t \in [0, I]$.

Now, we state and prove our theorem:

Theorem 8. *Let $T : C([0,I]) \to C([0,I])$ be the operator defined by*

$$T(x)(t) = g(t) + \int_0^t K(t,s,x(s))ds, \qquad \text{for all } t \in [0,I].$$

Suppose that the following conditions hold:

(i) *$K \in C([0,I] \times [0,I] \times \mathbb{R}, \mathbb{R})$ and $g \in X$;*
(ii) *$K(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ is increasing, for all $t, s \in [0,I]$;*
(iii) *there exists $x_0 \in X$ such that*

$$x_0(t) \leq g(t) + \int_0^t K(t,s,x_0(s))ds, \qquad \text{for all } t \in [0,I];$$

(iv) *there exist $r \in [0,1)$, $L > 0$ and a continuous function $G : [0,I] \times [0,I] \to [0,+\infty)$ such that*

$$p(y, T(x)) \leq p(y,x)$$

implies

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq rI^{-1}|x(s) - y(s)| + G(t,s)|y(s) - T(x)(s)|,$$

for all $t, s \in [0,I]$ and $x, y \in \mathbb{R}$ with $x \preceq y$;
(v) *$\max_{t\in[0,I]} \int_0^t G(t,s)ds = L$;*
(vi) *$\|T(x)\|_\infty \leq 1$ whenever $\|x\|_\infty \leq 1$ for $x \in X$.*

Then (6.1) has at least one solution $x^* \in X$. Moreover, if $r + L < 1$, then T has a unique fixed point.

Proof. First, condition (iii) implies that $x_0 \preceq T(x_0)$, that is condition (i) of Theorem 5 holds true. Clearly, by condition (ii) the operator T is increasing. Also, in [6], the authors showed that $(X, \preceq)$ satisfies condition (ii) of Theorem 5. Now, let X be endowed with the partial metric defined by

$$p(x,y) = \begin{cases} d(x,y) & \text{if } \|x\|_\infty, \|y\|_\infty \leq 1, \\ d(x,y) + L & \text{otherwise.} \end{cases}$$

Notice that (X, p) is 0-complete but is not complete. Indeed we have

$$p^s(x,y) = \begin{cases} 2d(x,y) & \text{if } (\|x\|_\infty, \|y\|_\infty \leq 1) \text{ or } (\|x\|_\infty, \|y\|_\infty > 1), \\ 2d(x,y) + L & \text{otherwise,} \end{cases}$$

and so (X, p^s) is not complete.
By definition of T it is clear that $x^* \in X$ is a solution of (6.1) if and only if x^* is a fixed point of T. Then, let $x, y \in X$ such that $x \preceq y$. From condition (iv), we deduce that

$$\begin{aligned}|T(x)(t) - T(y)(t)| &= \left|\int_0^t K(t,s,x(s)) - K(t,s,y(s))ds\right| \\ &\leq \int_0^t |K(t,s,x(s)) - K(t,s,y(s))|ds \\ &\leq \int_0^t [rI^{-1}|x(s) - y(s)| + G(t,s)|y(s) - T(x)(s)|]ds.\end{aligned}$$

On routine calculations, for all $x, y \in X$ with $x \prec y$, since $\|x\|_\infty, \|y\|_\infty \leq 1$, we have

$$p(T(x), T(y)) \leq rp(x,y) + Lp(y, T(x)).$$

Therefore, without loss of generality, we write

$$p(y, T(x)) \leq p(y,x) \Longrightarrow p(T(x), T(y)) \leq rp(x,y) + Lp(y, T(x)),$$

for all comparable $x, y \in X$ with $x \neq y$. Therefore, (4.1) holds true for all comparable $x, y \in X$ with $x \neq y$, since $p(y, y) = d(x, y) = 0$. Thus, Theorem 5 applies to this case and so we deduce that the operator T has a fixed point $x^* \in X$, which is a solution of the integral equation (6.1). Now, if $r + L < 1$, since x and $\max\{x, y\}$ as well as y and $\max\{x, y\}$ are comparable for all $x, y \in X$, we get that the operator T has a unique fixed point $x^* \in X$, which is a unique solution of the integral equation (6.1). □

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Authors' addresses

Daniela Paesano
Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi, 34, 90123 Palermo, Italy
E-mail address: daniela_paesano@libero.it

Pasquale Vetro
Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi, 34, 90123 Palermo, Italy
E-mail address: vetro@math.unipa.it