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GALILEI INVARIANCE OF THE HELMHOLTZ MORPHISM

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Abstract. We study the Helmholtz morphism in terms of the variational sequences. We find conditions for the Helmholtz form to be invariant with respect to the Galilei group.

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1. Introduction

In this paper, we shall use the framework of the theory of variational sequences on fibred manifolds introduced by Krupka [2, 3]. The variational sequence is a quotient sequence of the de Rham sequence such that one of the morphisms is the *Euler–Lagrange morphism* $\mathcal{E}_1\colon \lambda \to E_\lambda$, assigning to a Lagrangian, i. e. one-form $\lambda = L\,\mathrm{d}t$, its Euler–Lagrange form, i. e. two-form $E_\lambda = E_\sigma(L)\,\mathrm{d}q^\sigma \wedge \mathrm{d}t$, where $E_\sigma(L)$ are the Euler–Lagrange expressions

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\sigma}\,.$$

The next morphism $\mathcal{E}_2\colon E \to H_E$, called the *Helmholtz morphism*, assigns to a two-form $E = E_\sigma\,\mathrm{d}q^\sigma \wedge \mathrm{d}t$ a three-form H_E

$$\begin{aligned} H_E &= \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\right)\right)\omega^\nu \wedge \omega^\sigma \wedge \mathrm{d}t \\ &+ \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} + \frac{\partial E_\nu}{\partial \dot{q}^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial E_\nu}{\partial \ddot{q}^\sigma}\right)\right)\dot{\omega}^\nu \wedge \omega^\sigma \wedge \mathrm{d}t \qquad (1.1) \\ &+ \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial E_\nu}{\partial \ddot{q}^\sigma}\right)\ddot{\omega}^\nu \wedge \omega^\sigma \wedge \mathrm{d}t. \end{aligned}$$

called *Helmholtz form*.

While the Euler–Lagrange morphism $\mathcal{E}_1$ is well-understood, much less is known about the Helmholtz morphism. In the present paper, we study the Galilei invariance of the Helmholtz morphism.

In Section 2, we recall the basic structures and notations and briefly introduce the variational sequence, according to [2, 3, 5], and we recall known properties of the

Helmholtz morphism [6, 7]. In Section 3, we recall the basic concepts of the theory of symmetries of differential forms, the Noether theorem, and the Galilei group of transformations. We present a theorem on the structure of Helmholtz forms invariant with respect to the Galilei group.

2. THE HELMHOLTZ MORPHISM

Let $\pi: Y \to X$ be a smooth fibred manifold, $\dim X = 1$, $\dim Y = m + 1$, and $\pi_r : J^r Y \to X$, $r \geq 1$, its jet prolongations. Denote by $\pi_{r,s} : J^r Y \to J^s Y$, $r > s \geq 0$, canonical jet projections. A mapping $\gamma: W \to Y$, where W is an open subset of X, is called a *section* of the manifold $\pi: Y \to X$ if $\gamma \circ \pi = \mathrm{id}_W$.

Fibred coordinates on Y are denoted by (t, q^σ), $1 \leq \sigma \leq m$, associated coordinates on $J^r Y$ are denoted by (t, q^σ_i), $1 \leq \sigma \leq m$, $0 \leq i \leq r$. We usually use the notation $q^\sigma_0 = q^\sigma$, $q^\sigma_1 = \dot{q}^\sigma$, $q^\sigma_2 = \ddot{q}^\sigma$, $q^\sigma_3 = \dddot{q}^\sigma$.

A vector field ξ on $J^r Y$ is called π_r*-vertical* if $T\pi_r \cdot \xi = 0$, and π_r*-projectable* if there exists a vector field ξ_0 on X such that $T\pi_r \cdot \xi = \xi_0 \circ \pi_r$.

A differential q-form ($q > 1$) η on $J^r Y$ is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π, *horizontal* or 0*-contact* if $i_\xi \eta = 0$ for every vertical vector field ξ on $J^r Y$, and k*-contact*, $1 \leq k \leq q$, if for every vertical vector field ξ, $i_\xi \eta$ is $(k-1)$-contact. If lifted to $J^{r+1} Y$, every q-form η on $J^r Y$ can be canonically decomposed into a sum of k-contact components, η_k, where $k = 0, 1, \dots, q$. We write $\eta_k = p_k \eta$, and $p_0 = h$, then,

$$\pi^*_{r+1,r}\eta = h\eta + p_1\eta + \cdots + p_q\eta. \tag{2.1}$$

A contact q-form is called *strongly contact* if $\pi^*_{r+1,r}\eta = p_q\eta$.

A general framework for our exposition is the *variational sequence* [2, 3].

Let $\Omega^r_{0,c} = \{0\}$, and let $\Omega^r_{p,c}$ be the sheaf of contact p-forms, if $p \leq n$, or the sheaf of strongly contact p-forms, if $p > n$, on $J^r Y$. Set

$$\Theta^r_p = \Omega^r_{p,c} + \mathrm{d}\Omega^r_{p-1,c},$$

where $\mathrm{d}\Omega^r_{p-1,c}$ is the image sheaf of $\Omega^r_{p-1,c}$ by the exterior derivative d. We get an exact sequence of soft sheaves

$$0 \longrightarrow \Theta^r_1 \longrightarrow \Theta^r_2 \longrightarrow \Theta^r_3 \longrightarrow \cdots,$$

where the morphisms are the exterior derivative, i. e., a subsequence of the De Rham seguence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^r_0 \longrightarrow \Omega^r_1 \longrightarrow \Omega^r_2 \longrightarrow \Omega^r_3 \longrightarrow \cdots$$

The quotient sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^r_0 \longrightarrow \Omega^r_1/\Theta^r_1 \longrightarrow \Omega^r_2/\Theta^r_2 \longrightarrow \Omega^r_3/\Theta^r_3 \longrightarrow \cdots$$

is also exact. It is called the *variational sequence* of order r on π. The variational sequence is an acyclic resolution of the constant sheaf $\mathbb{R}$ over Y. We denote by

$$\mathcal{E}_p : \Omega^r_p/\Theta^r_p \to \Omega^r_{p+1}/\Theta^r_{p+1}$$

the quotient mapping. The class of a form $\rho \in \Omega_p^r$ is denoted by $[\rho]$. Hence, $\mathcal{E}_p([\rho]) = [\mathrm{d}\rho]$.

The quotient mapping

$$\mathcal{E}_1 : \Omega_1^r/\Theta_1^r \to \Omega_2^r/\Theta_2^r$$

then identifies with the *Euler–Lagrange mapping*. The quotient mapping

$$\mathcal{E}_2 : \Omega_2^r/\Theta_2^r \to \Omega_3^r/\Theta_3^r$$

is called the *Helmholtz mapping*. The image of a class $[\rho] \in \Omega_2^r/\Theta_2^r$, i. e., the class $[\mathrm{d}\rho] \in \Omega_3^r/\Theta_3^r$ is called *Helmholtz class.*

Due to the exactness of the variational sequence, if $[\alpha] \in \Omega_2^r/\Theta_2^r$ is such that

$$\mathcal{E}_2([\alpha]) = [d\alpha] = 0, \tag{2.2}$$

there exists $[\rho] \in \Omega_1^r/\Theta_1^r$ such that $[\alpha] = [d\rho] = \mathcal{E}_1([\rho])$, i. e. $[\alpha]$ is the image by the Euler–Lagrange mapping of a class $[\rho]$. In other words, the class $[\alpha]$ is locally variational – it comes from a class $[\rho]$ that has the meaning of a *local Lagrangian*. If, moreover, $H^2 Y = \{0\}$, a *global* Lagrangian exists. Condition (2.2) for "local variationality" then provides Helmholtz conditions (of order r).

Classes in the variational sequence can be represented by differential forms [1, 4]. We shall use the representation by so-called *source forms*, $(q-1)$-contact q-forms belonging to the ideal generated by contact forms

$$\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma \mathrm{d}t, \quad \dot{\omega}^\sigma = \mathrm{d}\dot{q}^\sigma - \ddot{q}^\sigma \mathrm{d}t, \quad \cdots, \quad \omega_{r-1}^\sigma = \mathrm{d}q_{r-1}^\sigma - \dot{q}_r^\sigma \mathrm{d}t \tag{2.3}$$

where $1 \le i \le m$.

Source forms for classes $[\rho] \in \Omega_1^r/\Theta_1^r$ are *horizontal forms* $\lambda = L\mathrm{d}t$, called *Lagrangians*. Source forms for classes $[\alpha] \in \Omega_2^r/\Theta_2^r$ are two-forms $E = E_i \omega^i \wedge \mathrm{d}t$, called *dynamical forms* (corresponding to *differential equations*). Note that, in this representation, if $[\rho]$ is represented by λ, then $[\mathrm{d}\rho] = \mathcal{E}_1([\rho])$ is represented by the dynamical form E_λ, the Euler–Lagrange form of λ. If $[\alpha] \in \Omega_2^r/\Theta_2^r$ is represented by a dynamical form E, $[\mathrm{d}\alpha] = \mathcal{E}_2([\alpha])$ is represented by a source three-form H_E, the *Helmholtz-form* of E. As shown in [3], H_E is then given by (1.1).

Source forms representing elements in Ω_3^r/Θ_3^r are called *Helmholtz-like forms*. We shall be interested in Helmholtz-like forms of order 3 (in particular, they correspond to second order ordinary differential equations). In coordinates,

$$H = H^0_{\sigma\nu}\, \omega^\nu \wedge \omega^\sigma \wedge \mathrm{d}t + H^1_{\sigma\nu}\, \dot{\omega}^\nu \wedge \omega^\sigma \wedge \mathrm{d}t + H^2_{\sigma\nu}\, \ddot{\omega}^\nu \wedge \omega^\sigma \wedge \mathrm{d}t \tag{2.4}$$

where $H^0_{\sigma\nu} = -H^0_{\nu\sigma}$, $H^1_{\sigma\nu} = H^1_{\nu\sigma}$, $H^2_{\sigma\nu} = -H^2_{\nu\sigma}$.

In [6, 7] was studied and solved the question when a three-form corresponds to a system of differential equations (such form is called *locally Helmholtz*). This problem is closely related to the question of existence of a closed counterpart of a three-form. This problem is solved for second-order Helmholtz-like forms in [6] and for third-order Helmholtz-like forms in [7].

3. Galilei invariance of the Helmholtz form

Let ξ be a projectable vector field on Y, η a differential form on $J^r Y$. ξ is called *point symmetry* of η, if

$$\partial_{J^r\xi}\eta = 0 \tag{3.1}$$

where ∂ denotes the Lie derivative. If η is a Helmholtz-like form of order three (2.4) the symmetry conditions (3.1) reads as follows:

$$\begin{aligned}
0 = {} & H^0_{\sigma\rho}\frac{\partial\xi^\rho}{\partial q^\nu} + H^0_{\rho\nu}\frac{\partial\xi^\rho}{\partial q^\sigma} + H^0_{\sigma\nu}\frac{\partial\xi^0}{\partial t} + H^1_{\sigma\rho}\frac{\partial}{\partial q^\nu}\left(\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} - \dot q^\rho\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) \\
& + H^2_{\sigma\rho}\frac{\partial}{\partial q^\nu}\left(\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\dot q^\rho - 2\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\ddot q^\rho\right) \\
& + \frac{\partial H^0_{\sigma\nu}}{\partial t}\xi^0 + \frac{\partial H^0_{\sigma\nu}}{\partial q^\rho}\xi^\rho + \frac{\partial H^0_{\sigma\nu}}{\partial \dot q^\rho}\left(\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} - \dot q^\rho\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) \\
& + \frac{\partial H^0_{\sigma\nu}}{\partial \ddot q^\rho}\left(\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\dot q^\rho - 2\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\ddot q^\rho\right) \\
& + \frac{\partial H^0_{\sigma\nu}}{\partial \dddot q^\rho}\left(\frac{\mathrm{d}^3\xi^\rho}{\mathrm{d}t^3} - \frac{\mathrm{d}^3\xi^0}{\mathrm{d}t^3}\dot q^\rho - 3\frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\ddot q^\rho - 3\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\dddot q^\rho\right)
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
0 = {} & H^0_{\sigma\rho}\frac{\partial\xi^\rho}{\partial \dot q^\nu} - H^0_{\rho\sigma}\frac{\partial\xi^\rho}{\partial \dot q^\nu} \\
& + H^1_{\sigma\nu}\frac{\partial\xi^0}{\partial t} + H^1_{\rho\nu}\frac{\partial\xi^\rho}{\partial q^\sigma} + H^1_{\sigma\rho}\frac{\partial}{\partial \dot q^\nu}\left(\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} - \dot q^\rho\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) \\
& + H^2_{\sigma\rho}\frac{\partial}{\partial \dot q^\nu}\left(\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\dot q^\rho - 2\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\ddot q^\rho\right) \\
& + \frac{\partial H^1_{\sigma\nu}}{\partial t}\xi^0 + \frac{\partial H^1_{\sigma\nu}}{\partial q^\rho}\xi^\rho + \frac{\partial H^1_{\sigma\nu}}{\partial \dot q^\rho}\left(\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} - \dot q^\rho\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) \\
& + \frac{\partial H^1_{\sigma\nu}}{\partial \ddot q^\rho}\left(\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\dot q^\rho - 2\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\ddot q^\rho\right) \\
& + \frac{\partial H^1_{\sigma\nu}}{\partial \dddot q^\rho}\left(\frac{\mathrm{d}^3\xi^\rho}{\mathrm{d}t^3} - \frac{\mathrm{d}^3\xi^0}{\mathrm{d}t^3}\dot q^\rho - 3\frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\ddot q^\rho - 3\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\dddot q^\rho\right)
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
0 = {} & H^0_{\sigma\rho}\frac{\partial\xi^\rho}{\partial \ddot q^\nu} - H^0_{\rho\sigma}\frac{\partial\xi^\rho}{\partial \ddot q^\nu} + H^1_{\sigma\rho}\frac{\partial}{\partial \ddot q^\nu}\left(\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} - \dot q^\rho\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) \\
& + H^2_{\sigma\nu}\frac{\partial\xi^0}{\partial t} + H^2_{\rho\nu}\frac{\partial\xi^\rho}{\partial q^\sigma} + H^2_{\sigma\rho}\frac{\partial}{\partial \ddot q^\nu}\left(\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\dot q^\rho - 2\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\ddot q^\rho\right) \\
& + \frac{\partial H^2_{\sigma\nu}}{\partial t}\xi^0 + \frac{\partial H^2_{\sigma\nu}}{\partial q^\rho}\xi^\rho + \frac{\partial H^2_{\sigma\nu}}{\partial \dot q^\rho}\left(\frac{\mathrm{d}\xi^\rho}{\mathrm{d}t} - \dot q^\rho\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right)
\end{aligned} \tag{3.4}$$

$$+\frac{\partial H^2_{\sigma\nu}}{\partial \ddot{q}^\rho}\left(\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}t^2}-\frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\dot{q}^\rho-2\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\ddot{q}^\rho\right)$$
$$+\frac{\partial H^2_{\sigma\nu}}{\partial \dddot{q}^\rho}\left(\frac{\mathrm{d}^3\xi^\rho}{\mathrm{d}t^3}-\frac{\mathrm{d}^3\xi^0}{\mathrm{d}t^3}\dot{q}^\rho-3\frac{\mathrm{d}^2\xi^0}{\mathrm{d}t^2}\ddot{q}^\rho-3\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\dddot{q}^\rho\right)$$

The Galilei group on $\mathbb{R}^4$ is the 10-parametric transformation group, generated by the vector fields

$$\frac{\partial}{\partial t},\quad \frac{\partial}{\partial q^\sigma},\quad -\varepsilon_{\sigma\nu\rho}q^\nu\frac{\partial}{\partial q^\rho},\quad t\frac{\partial}{\partial q^\sigma},\quad \sigma=1,2,3 \tag{3.5}$$

for the time translation, the space translations, the space rotations, and the Galilei transformations, respectively.

Theorem 1. *A Helmholtz-like form H* (2.4) *is invariant under the Galilei group of transformations if and only if the component $H^0_{\sigma\nu}$ of H takes the form:*

$$H^0_{\sigma\nu}=\varepsilon_{\sigma\nu\rho}f(\eta)\ddot{q}^\rho+\varepsilon_{\sigma\nu\rho}g(\bar{\eta})\dddot{q}^\rho \tag{3.6}$$

where $\eta=\sum_{\kappa=1}^3(\ddot{q}^\kappa)^2$, $\bar{\eta}=\sum_{\kappa=1}^3(\dddot{q}^\kappa)^2$, and components $H^1_{\sigma\nu}$, $H^2_{\sigma\nu}$ of H vanish.

Proof. Substituing the generators of the Galilei group (3.5) into (3.2)–(3.4) we obtain

$$\begin{array}{lll}\dfrac{\partial H^0_{\sigma\nu}}{\partial t}=0 & \dfrac{\partial H^0_{\sigma\nu}}{\partial q^\rho}=0 & \dfrac{\partial H^0_{\sigma\nu}}{\partial \dot{q}^\rho}=0\\ \dfrac{\partial H^1_{\sigma\nu}}{\partial t}=0 & \dfrac{\partial H^1_{\sigma\nu}}{\partial q^\rho}=0 & \dfrac{\partial H^1_{\sigma\nu}}{\partial \dot{q}^\rho}=0\\ \dfrac{\partial H^2_{\sigma\nu}}{\partial t}=0 & \dfrac{\partial H^2_{\sigma\nu}}{\partial q^\rho}=0 & \dfrac{\partial H^2_{\sigma\nu}}{\partial \dot{q}^\rho}=0\end{array} \tag{3.7}$$

and

$$\begin{aligned}\varepsilon_{\theta\rho\nu}H^0_{\sigma\rho}+\varepsilon_{\theta\rho\sigma}H^0_{\rho\nu}+\varepsilon_{\theta\rho\kappa}\frac{\partial H^0_{\sigma\nu}}{\partial\ddot{q}^\rho}\ddot{q}^\kappa+\varepsilon_{\theta\rho\kappa}\frac{\partial H^0_{\sigma\nu}}{\partial\dddot{q}^\rho}\dddot{q}^\kappa&=0\\ \varepsilon_{\theta\rho\sigma}H^1_{\rho\nu}+\varepsilon_{\theta\rho\kappa}\frac{\partial H^1_{\sigma\nu}}{\partial\ddot{q}^\rho}\ddot{q}^\kappa+\varepsilon_{\theta\rho\kappa}\frac{\partial H^1_{\sigma\nu}}{\partial\dddot{q}^\rho}\dddot{q}^\kappa&=0\\ \varepsilon_{\theta\rho\sigma}H^2_{\rho\nu}+\varepsilon_{\theta\rho\kappa}\frac{\partial H^2_{\sigma\nu}}{\partial\ddot{q}^\rho}\ddot{q}^\kappa+\varepsilon_{\theta\rho\kappa}\frac{\partial H^2_{\sigma\nu}}{\partial\dddot{q}^\rho}\dddot{q}^\kappa&=0\end{aligned} \tag{3.8}$$

From equalities (3.7), we get that $H^0_{\sigma\nu}$, $H^1_{\sigma\nu}$ and $H^2_{\sigma\nu}$ do not depend on $t, q^\rho, \dot{q}^\rho$. From (3.8), we get formula (3.6) and the vanishing of the components $H^1_{\sigma\nu}$, $H^2_{\sigma\nu}$. □

Theorem 2. *Let H be a Helmholtz-like form* (2.4) *which is locally Helmholtz. H is invariant under the Galilei group if and only if the component $H^0_{\sigma\nu}$ of H takes the form*

$$H^0_{\sigma\nu}=\varepsilon_{\sigma\nu\rho}\,c\,\ddot{q}^\rho \tag{3.9}$$

where c is constant and the components $H^1_{\sigma\nu}$, $H^2_{\sigma\nu}$ of H vanish. The corresponding dynamical form takes the form

$$E = -\frac{2}{3}\varepsilon_{\sigma\nu\rho}\, c\, q^\nu \ddot{q}^\rho \omega^\sigma \wedge \mathrm{d}t \tag{3.10}$$

Proof. H is locally Helmholtz, i. e., it corresponds to a system of differential equations, if and only if its components satisfy the following identities (see [7]):

$$\begin{gathered}
\frac{\partial H^2_{\sigma\nu}}{\partial \dddot{q}^\rho} = 0 \qquad \frac{\partial H^0_{\sigma\nu}}{\partial \dddot{q}^\rho} + \frac{1}{2}\frac{\partial H^2_{\sigma\rho}}{\partial \dot{q}^\nu} = 0 \\
\left(\frac{\partial H^1_{\sigma\nu}}{\partial \dddot{q}^\rho} - \frac{\partial H^2_{\sigma\nu}}{\partial \ddot{q}^\rho}\right)_{[\nu\rho]} = 0 \qquad \left(\frac{\partial H^1_{\sigma\nu}}{\partial \ddot{q}^\rho} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^1_{\sigma\nu}}{\partial \dddot{q}^\rho}\right)_{(\nu\rho)} = 0 \\
\left(\frac{\partial H^0_{\sigma\nu}}{\partial \ddot{q}^\rho} + \frac{1}{2}\frac{\partial H^2_{\sigma\nu}}{\partial q^\rho} + \frac{1}{2}\frac{\partial H^2_{\nu\rho}}{\partial q^\sigma} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(3\frac{\partial H^0_{\sigma\nu}}{\partial \dddot{q}^\rho} + \frac{\partial H^2_{\sigma\nu}}{\partial \dot{q}^\rho}\right)\right)_{[\nu\rho]} = 0, \\
\left(\frac{\partial H^1_{\sigma\nu}}{\partial \dot{q}^\rho} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^1_{\sigma\nu}}{\partial \ddot{q}^\rho} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial H^1_{\sigma\nu}}{\partial \dddot{q}^\rho}\right)_{[\nu\rho]} = 0, \\
\left(\frac{\partial H^0_{\sigma\nu}}{\partial \dot{q}^\rho} - \frac{1}{2}\left(\frac{\partial H^1_{\sigma\rho}}{\partial q^\nu} - \frac{\partial H^1_{\nu\rho}}{\partial q^\sigma}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H^0_{\sigma\nu}}{\partial \ddot{q}^\rho} - \frac{1}{2}\frac{\partial H^2_{\sigma\nu}}{\partial q^\rho}\right) + \right. \\
\left. + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial H^0_{\sigma\nu}}{\partial \dddot{q}^\rho}\right)_{(\nu\rho)} = 0, \\
\left(\frac{\partial H^0_{\sigma\nu}}{\partial q^\rho} - \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^0_{\sigma\nu}}{\partial \dot{q}^\rho} + \frac{1}{3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial H^0_{\sigma\nu}}{\partial \ddot{q}^\rho} + \frac{\partial H^2_{\sigma\nu}}{\partial q^\rho}\right) - \right. \\
\left. - \frac{1}{3}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\frac{\partial H^0_{\sigma\nu}}{\partial \dddot{q}^\rho}\right)_{[\nu\rho]} = 0.
\end{gathered} \tag{3.11}$$

The corresponding dynamical form is then in coordinates given by the formula (see [6]):

$$E = \left(\tilde{E}_\sigma - \frac{\mathrm{d}\tilde{E}^1_\sigma}{\mathrm{d}t} + \frac{\mathrm{d}^2 \tilde{E}^2_\sigma}{\mathrm{d}t^2}\right)\omega^\sigma \wedge \mathrm{d}t \tag{3.12}$$

where

$$\begin{aligned}
\tilde{E}_\sigma &= 2q^\nu \int_0^1 H^0_{\nu\sigma}(t, uq^\rho, u\dot{q}^\rho, u\ddot{q}^\rho) u \, \mathrm{d}u \\
&- \dot{q}^\nu \int_0^1 H^1_{\sigma\nu}(t, uq^\rho, u\dot{q}^\rho, u\ddot{q}^\rho) u \, \mathrm{d}u \\
&- \ddot{q}^\nu \int_0^1 H^2_{\sigma\nu}(t, uq^\rho, u\dot{q}^\rho, u\ddot{q}^\rho) u \, \mathrm{d}u
\end{aligned}$$

and

$$\tilde{E}^1_\sigma = q^\nu \int_0^1 H^1_{\nu\sigma}(t, uq^\rho, u\dot{q}^\rho, u\ddot{q}^\rho) u \, \mathrm{d}u,$$
$$\tilde{E}^2_\sigma = q^\nu \int_0^1 H^2_{\nu\sigma}(t, uq^\rho, u\dot{q}^\rho, u\ddot{q}^\rho) u \, \mathrm{d}u.$$

From equalities (3.7) and (3.11), we get that $H^0_{\sigma\nu}$, $H^1_{\sigma\nu}$ and $H^2_{\sigma\nu}$ do not depend on $t, q^\rho, \dot{q}^\rho, \dddot{q}^\rho$. From (3.8), we get (3.9). Substituting (3.9) into the (3.12), we get (3.10). □

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