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EXISTENCE AND UNIQUENESS RESULTS FOR BEST PROXIMITY POINTS

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Abstract. Let us consider a non-self mapping $T : A \to B$, where A and B are two nonempty subsets of a metric space (X,d). The aim of this paper is to solve the nonlinear programming problem that consists in minimizing the real valued function $x \longmapsto d(x,Tx)$, where T belongs to a new class of non-self mappings. In especial case, existence and uniqueness of fixed point for Kannan type self mappings are also obtained.

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1. INTRODUCTION

Let A and B be two nonempty subsets of a metric space X. A non-self mapping $T : A \to B$ is said to be a *contraction* if there exists a constant $r \in [0,1)$, such that $d(Tx,Ty) \le rd(x,y)$, for all $x,y \in A$. The well-known Banach contraction principle states that if A is a complete subset of X and T is a contraction self-mapping, then the fixed point equation $Tx = x$ has exactly one solution.

The Banach contraction principle is a very important tools in nonlinear analysis and there are many extensions of this principle; see, e.g., [13] and the references therein.

Let (X,d) be a metric space. A self-mapping $T : X \to X$ is called *Kannan mapping* if there exists $\alpha \in [0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le \alpha[d(x,Tx)+d(y,Ty)],$$

for all $x,y \in X$. We know that if X is complete metric space, every Kannan self-mapping defined on X has a unique fixed point ([12]). Note that, the notion of contraction mappings and Kannan mappings are independent. That is, there exists a contraction mapping, which is not Kannan and a Kannan mapping, which is not a contraction. Therefore, we cannot compare these two class of mappings directly.

Recently, Kikkawa and Suzuki in [14], established the following fixed point theorem, which is an extension of Kannan's fixed point theorem.

Theorem 1 ([14]). *Define a non-increasing function φ from $[0,1)$ into $(\frac{1}{2},1]$ by*

$$\varphi(r)=\begin{cases}1 & \text{if} \quad 0\le r<\frac{1}{\sqrt{2}},\\ \frac{1}{1+r} & \text{if} \quad \frac{1}{\sqrt{2}}\le r<1.\end{cases}$$

Let (X,d) be a complete metric space and let T be a self-mapping on X. Let $\alpha\in[0,\frac{1}{2})$ and put $r:=\frac{\alpha}{1-\alpha}\in[0,1)$. Assume that

$$\varphi(r)d(x,Tx)\le d(x,y) \quad \text{implies} \quad d(Tx,Ty)\le\alpha[d(x,Tx)+d(y,Ty)],$$

for all $x,y\in X$. Then T has a unique fixed point z and $\lim_n T^n x=z$ holds for every $x\in X$.

It is interesting to note that the function $\varphi(r)$ defined in Theorem 1 is the best constant for every r (see Theorem 2.4 of [14]).

2. Preliminaries

Consider the non-self mapping $T:A\to X$, in which A is a nonempty subset of a metric space (X,d). Clearly, the fixed point equation $Tx=x$ may not have solution. Hence, it is contemplated to find an approximate $x\in A$ such that the error $d(x,Tx)$ is minimum. Indeed, best approximation theory has been derived from this idea. Here, we state the following well-known best approximation theorem due to Kay Fan.

Theorem 2 ([8]). *Let A be a nonempty compact convex subset of a normed linear space X and $T:A\to X$ be a continuous mapping. Then there exists $x\in A$ such that $\|x-Tx\|=dist(Tx,A):=\inf\{\|Tx-a\|:a\in A\}$.*

A point $x\in A$ in the above theorem is called a *best approximant point* of T in A. The notion of best proximity point for non-self mappings is introduced in a similar fashion:

Definition 1. Let A and B be nonempty subsets of a metric space (X,d) and $T:A\to B$ be a non-self mapping. A point $p\in A$ is called a best proximity point of T if

$$d(p,Tp)=dist(A,B):=\{d(x,y):(x,y)\in A\times B\}.$$

In fact, best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{x\in A} d(x,Tx), \tag{2.1}$$

has at least one solution.

Best proximity point theory is an interesting topic in optimization theory which recently attracted the attention of many authors (see for instance [1–9, 16]).

Let A and B be two nonempty subsets of a metric space (X,d). Let us fix the following notations which will be needed throughout this article:

$$A_0:=\{x\in A: d(x,y)=dist(A,B) \quad \text{for some} \quad y\in B\},$$

$$B_0 := \{y \in B : d(x,y) = dist(A,B) \quad \text{for some} \quad x \in A\}.$$

It is easy to see that if (A,B) is a nonempty and weakly compact pair of subsets of a Banach space X, then (A_0, B_0) is also nonempty pair X.

The notion of *proximal contractions* was defined by Sadiq Basha, as follows.

Definition 2 ([15])**.** Let (A,B) be a pair of nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a proximal contraction if there exists a non-negative real number $\alpha < 1$ such that, for all $u_1, u_2, x_1, x_2 \in A$,

$$\begin{cases} d(u_1, Tx_1) = dist(A,B) \\ d(u_2, Tx_2) = dist(A,B) \end{cases} \Rightarrow d(u_1,u_2) \le \alpha d(x_1,x_2).$$

Definition 3 ([15])**.** Let A, B be two nonempty subsets of a metric space (X,d). A is said to be approximatively compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y,x_n) \to D(y,A)$ for some $y \in B$ has a convergent subsequence.

Next theorem is the main result of [15].

Theorem 3. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a proximal contraction such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

We mention that in [10], the current author extended Theorem 3 and established a best proximity point theorem under weaker conditions with respect to Theorem 3, due to Sadiq Basha (see Theorem 2.1 and Corollary 2.1 of [10]).

In this article, we introduce a new class of mappings called *weak proximal Kannan non-self mappings* and obtain a similar result of Theorem 1 for this new class of non-self mappings.

3. Main results

To establish our main results, we introduce the following new class of non-self mappings.

Definition 4. Define a strictly decreasing function θ from $[0, \frac{1}{2})$ onto $(\frac{1}{2}, 1]$ by

$$\theta(r) = 1 - r.$$

Let (A,B) be a nonempty pair of subsets of a metric space (X,d). Let $\alpha \in [0, \frac{1}{2})$ and put $r := \frac{\alpha}{1-\alpha}$. Then $T : A \to B$ is said to be a weak proximal Kannan non-self mapping if for all $u, v, x, y \in A$ with

$$d(u,Tx) = dist(A,B) \ \ \& \ \ d(v,Ty) = dist(A,B),$$

we have

$$\theta(r) d^*(x,Tx) \le d(x,y) \ \text{ implies } \ d(u,v) \le \alpha[d^*(x,Tx) + d^*(y,Ty)]. \tag{3.1}$$

The notion of *proximal Kannan non-self mapping* can be defined as below.

Definition 5. Let (A, B) be a nonempty pair of subsets of a metric space (X, d). Then $T : A \to B$ is said to be a proximal Kannan non-self mapping if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $u, v, x, y \in A$ with

$$d(u, Tx) = dist(A, B) \ \& \ d(v, Ty) = dist(A, B),$$

we have

$$d(u, v) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)].$$

It is clear that the class of weak proximal Kannan non-self mappings contains the class of proximal Kannan non-self mappings as a subclass. Also, the class of proximal Kannan non-self mappings contains the class of Kannan non-self mappings.

We now state our main result of this article.

Theorem 4. *Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a weak proximal Kannan non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$. Moreover, if $\{x_n\}$ is a sequence in A such that $d(x_{n+1}, Tx_n) = dist(A, B)$, then $x_n \to x^*$.*

Proof. Assume $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$. Again, since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = dist(A, B)$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = dist(A, B), \quad \text{for all } \ n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Thus,

$$d(x_0, Tx_0) \leq d(x_0, x_1) + d(x_1, Tx_0) = d(x_0, x_1) + dist(A, B),$$

and so,

$$\theta(r) d^*(x_0, Tx_0) \leq d^*(x_0, Tx_0) \leq d(x_0, x_1) \quad \& \quad \begin{cases} d(x_1, Tx_0) = dist(A, B), \\ d(x_2, Tx_1) = dist(A, B). \end{cases}$$

Since T is a weak proximal Kannan non-self mapping, we conclude that

$$\begin{aligned} d(x_1, x_2) &\leq \alpha[d^*(x_0, Tx_0) + d^*(x_1, Tx_1)] \\ &\leq \alpha[d(x_0, x_1) + d^*(x_1, Tx_0) + d(x_1, x_2) + d^*(x_2, Tx_1)] \\ &= \alpha[d(x_0, x_1) + d(x_1, x_2)]. \end{aligned}$$

Therefore,

$$d(x_1, x_2) \leq \frac{\alpha}{1-\alpha} d(x_0, x_1) = r d(x_0, x_1).$$

Similarly, we can see that

$$\theta(r)d^*(x_1,Tx_1) \le d(x_1,x_2) \quad \& \quad \begin{cases} d(x_2,Tx_1) = dist(A,B), \\ d(x_3,Tx_2) = dist(A,B). \end{cases}$$

This implies that

$$\begin{aligned} d(x_2,x_3) &\le \alpha[d^*(x_1,Tx_1) + d^*(x_2,Tx_2)] \\ &\le \alpha[d(x_1,x_2) + d^*(x_2,Tx_1) + d(x_2,x_3) + d^*(x_3,Tx_2)] \\ &= \alpha[d(x_1,x_2) + d(x_2,x_3)]. \end{aligned}$$

So,

$$d(x_2,x_3) \le \frac{\alpha}{1-\alpha} d(x_1,x_2) = rd(x_1,x_2) \le r^2 d(x_0,x_1).$$

Hence, by induction, we conclude that

$$d(x_n,x_{n+1}) \le r^n d(x_0,x_1),$$

which implies that

$$\Sigma_{n=1}^{\infty} d(x_n,x_{n+1}) \le \Sigma_{n=1}^{\infty} r^n d(x_0,x_1) < \infty.$$

That is, $\{x_n\}$ is a Cauchy sequence in A_0. Since A_0 is closed and X is complete metric space, we deduce that $\{x_n\}$ is a convergent sequence. Let $x^* \in A_0$ be such that $x_n \to x^*$. We claim that x^* is a unique best proximity point of T. At first, we prove that

$$d^*(x^*,Tx) \le \alpha d(x^*,x), \quad \forall x \in A_0 \quad \text{with} \quad x \ne x^*. \tag{3.3}$$

Let $x \in A_0$ and $x \ne x^*$. Since $T(A_0) \subseteq B_0$, there exists $y \in A_0$ such that $d(y,Tx) = dist(A,B)$. By the fact that $x_n \to x^*$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n,x^*) \le \frac{1}{3} d(x,x^*), \quad \forall n \ge N_1.$$

We now have

$$\begin{aligned} \theta(r)d^*(x_n,Tx_n) &\le d^*(x_n,Tx_n) = d(x_n,Tx_n) - dist(A,B) \\ &\le d(x_n,x^*) + d(x^*,x_{n+1}) + d(x_{n+1},Tx_n) - dist(A,B) \\ &= d(x_n,x^*) + d(x^*,x_{n+1}) \le \frac{2}{3} d(x,x^*) \\ &= d(x,x^*) - \frac{1}{3} d(x,x^*) \le d(x,x^*) - d(x_n,x^*) \\ &\le d(x_n,x). \end{aligned}$$

Thus,

$$\theta(r)d^*(x_n,Tx_n) \le d(x_n,x) \quad \& \quad \begin{cases} d(x_{n+1},Tx_n) = dist(A,B), \\ d(y,Tx) = dist(A,B). \end{cases}$$

Again, since T is weak proximal Kannan non-self mapping we conclude that

$$d(x_{n+1},y) \le \alpha[d^*(x_n,Tx_n)+d^*(x,Tx)] \le \alpha[d(x_n,x_{n+1})+d^*(x,Tx)].$$

Thereby,

$$\begin{aligned} d(x^*,Tx) &= \lim_{n\to\infty} d(x_n,Tx) \\ &\le \lim_{n\to\infty}[d(x_n,x_{n+1})+d(x_{n+1},y)+d(y,Tx)] \\ &\le \lim_{n\to\infty}[(1+\alpha)d(x_n,x_{n+1})+\alpha d^*(x,Tx)+d(y,Tx)] \\ &\le \lim_{n\to\infty}[(1+\alpha)r^n d(x_0,x_1)+\alpha d^*(x,Tx)+dist(A,B)] \\ &= \alpha d^*(x,Tx)+dist(A,B). \end{aligned}$$

Then,

$$d^*(x^*,Tx) \le \alpha d^*(x,Tx), \quad \forall x\in A_0, \quad \text{with} \quad x\ne x^*.$$

That is, (3.3) holds. It now follows from (3.3) that

$$\begin{aligned} d^*(x_n,Tx_n) &\le d(x_n,x^*)+d^*(x^*,Tx_n) \\ &\le d(x_n,x^*)+\alpha d^*(x_n,Tx_n). \end{aligned}$$

Thus,

$$\theta(r)d^*(x_n,Tx_n) = (1-r)d^*(x_n,Tx_n) \le (1-\alpha)d^*(x_n,Tx_n) \le d(x_n,x^*). \tag{3.4}$$

On the other hand, since $x^*\in A_0$ and $T(A_0)\subseteq B_0$, there exists $y^*\in B_0$ such that $d(y^*,Tx^*)=dist(A,B)$. Therefore,

$$\theta(r)d^*(x_n,Tx_n) \le d(x_n,x^*) \quad \& \quad \begin{cases} d(x_{n+1},Tx_n)=dist(A,B), \\ d(y^*,Tx^*)=dist(A,B), \end{cases}$$

which implies that

$$\begin{aligned} d(x_{n+1},y^*) &\le \alpha[d^*(x_n,Tx_n)+d^*(x^*,Tx^*)] \\ &\le \alpha[d(x_n,x_{n+1})+d^*(x_{n+1},Tx_n)+d^*(x^*,Tx^*)]. \end{aligned}$$

If in above relation $n\to\infty$, we obtain

$$\begin{aligned} d(y^*,x^*) &\le \alpha d^*(x^*,Tx^*) \\ &= \alpha[d(x^*,y^*)+d^*(y^*,Tx^*)] = \alpha d(x^*,y^*). \end{aligned}$$

This deduces that $d(x^*,y^*)=0$ or $x^*=y^*$. Hence x^* is a best proximity point of the mapping T. The uniqueness of best proximity point follows from the condition that T is weak proximal Kannan non-self mapping. That is, suppose that x_1^*,x_2^* are two distinct points in A such that $d(x_i^*,Tx_i^*)=dist(A,B)$, for $i=1,2$. So,

$$\theta(r)d^*(x_1^*,Tx_1^*) \le d(x_1^*,x_2^*) \quad \& \quad \begin{cases} d(x_1^*,Tx_1^*)=dist(A,B), \\ d(x_2^*,Tx_2^*)=dist(A,B), \end{cases}$$

Then,

$$0 < d(x_1^*, x_2^*) \leq \alpha[d^*(x_1^*, Tx_1^*) + d^*(x_2^*, Tx_2^*)] = 0,$$

which is a contradiction. Hence, the best proximity point is unique.

□

The following corollaries are obtained from Theorem 4.

Corollary 1. *Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a proximal Kannan non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$. Moreover, if $\{x_n\}$ is a sequence in A such that $d(x_{n+1}, Tx_n) = dist(A, B)$ then, $x_n \to x^*$.*

Corollary 2. *Let (A, B) be a nonempty pair of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a Kannan non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$. Moreover, if $\{x_n\}$ is a sequence in A such that $d(x_{n+1}, Tx_n) = dist(A, B)$ then, $x_n \to x^*$.*

Corollary 3. *Let A be a nonempty and closed subset of a complete metric space (X, d). Assume that $T : A \to A$ is a self mapping such that*

$$\theta(r) d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in A$, where $\theta(r)$ is defined as in the Definition 4. Then T has a unique fixed point $x^* \in A$. Moreover, if $x_0 \in A$ and we define $x_{n+1} = Tx_n$, then $x_n \to x^*$.

Corollary 4 (Kannan fixed point theorem)**.** *Let A be a nonempty and closed subset of a complete metric space (X, d). Assume that $T : A \to A$ is a Kannan mapping. Then T has a unique fixed point. Moreover, for each $x_0 \in A$ if we define $x_{n+1} = Tx_n$ then the sequence $\{x_n\}$ converges to the fixed point of T.*

Example 1. Suppose that $X = \mathbb{R}$ with the usual metric. Suppose that

$$A := [0, 2] \cup \{5\} \quad \& \quad B := [3, 4].$$

Then A and B are nonempty closed subsets of X and $A_0 = \{2, 5\}$ and $B_0 = \{3, 4\}$. Note that $dist(A, B) = 1$. Let $T : A \to B$ be a mapping defined as

$$T(x) = \begin{cases} \frac{7}{2} & \text{if} \quad x = 0, \\ 4 & \text{if} \quad x \neq 0. \end{cases}$$

It is easy to see that T is weak proximal Kannan non-self mapping for each $\alpha \in [0, \frac{1}{2})$. Indeed, it is sufficient to note that $d(u, Tx) = dist(A, B)$, holds for $u = 5$ and $x \in A - \{0\}$. Therefore, Theorem 4 guaranties the existence and uniqueness of a best proximity point for T and this point is $x^* = 5$.

Example 2. Suppose that $X=\mathbb{R}$ with the usual metric. Suppose that

$$A:=[0,\frac{1}{100}]\cup\{1\}\quad \&\quad B:=[2,3].$$

Then A and B are nonempty closed subsets of X and $dist(A,B)=1$. Define a non-self mapping $T:A\to B$ as follows

$$T(x)=\begin{cases}2 & \text{if}\quad x\in\mathbb{Q}\cap A,\\ \frac{101}{50} & \text{if}\quad x\in\mathbb{Q}^c\cap A.\end{cases}$$

Note that T is not continuous. We claim that T is Kannan nons-elf mapping with $\alpha=\frac{1}{3}$. For this purpose, it is sufficient to consider two following cases.

Case 1. If $x\in\mathbb{Q}\cap A-\{1\}$ and $y\in\mathbb{Q}^c\cap A$, then

$$\alpha[d^*(x,Tx)+d^*(y,Ty)]=\frac{1}{3}[\frac{101}{50}-(x+y)]\geq\frac{2}{3}>\frac{1}{50}=d(Tx,Ty).$$

Case 2. If $x=1$ and $y\in\mathbb{Q}^c\cap A$, then

$$\alpha[d^*(x,Tx)+d^*(y,Ty)]=\frac{1}{3}[\frac{51}{50}-y]\geq\frac{1}{3}\times\frac{101}{100}>\frac{1}{50}=d(Tx,Ty).$$

It now follows from Corollary 2 that T has a unique best proximity point and this point is $x^*=1$.

Remark 1. We mention that in [11] the author studied the existence of best proximity points in metric spaces with a partial order, where weak proximal Kannan non-self mappings are satisfied only for comparable elements.

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