Ergodic properties of generalized Ornstein–Uhlenbeck processes

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Abstract

We investigate ergodic properties of the solution of the SDE $\mathrm{d}V_t = V_{t-}\mathrm{d}U_t + \mathrm{d}L_t$, where (U, L) is a bivariate Lévy process. This class of processes includes the generalized Ornstein–Uhlenbeck processes. We provide sufficient conditions for ergodicity, and for subexponential and exponential convergence to the invariant probability measure. We use the Foster–Lyapunov method. The drift conditions are obtained using the explicit form of the generator of the continuous process. In some special cases the optimality of our results can be shown.

MSC2010: 60J25; 60H10

Keywords: Generalized Ornstein–Uhlenbeck processes; Foster–Lyapunov technique; exponential / subexponential ergodicity; petite set.

1 Introduction

Let $(U_t, L_t)_{t\geq 0}$ be a bivariate Lévy process with characteristic triplet $((\gamma_U, \gamma_L), \Sigma, \nu_{UL})$. In the present paper we investigate ergodic properties of the unique solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}V_t &= V_{t-}\mathrm{d}U_t + \mathrm{d}L_t, \quad t \geq 0, \\ V_0 &= x_0, \end{aligned} \tag{1.1}$$

with deterministic initial value $x_0 \in \mathbb{R}$.

When U has no jumps smaller than, or equal to -1, then the unique solution of (1.1) is

$$V_t = e^{-\xi_t} \left[x_0 + \int_0^t e^{\xi_{s-}} \mathrm{d}\eta_s \right], \tag{1.2}$$

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where (ξ, η) is another bivariate Lévy process, defined in section 2 in details. In this case V is the *generalized Ornstein–Uhlenbeck process* (GOU) corresponding to the bivariate Lévy process (ξ, η). Thus the class of GOU processes is a subclass of the solutions to (1.1). For precise definitions and more detailed description see the next section. In the present paper we deal with the general case, therefore we use the description through (U, L). When $U_t = -\mu t$, $\mu > 0$, V is called Lévy-driven Ornstein–Uhlenbeck process, and if L_t is a Brownian motion, then we obtain the classical Ornstein–Uhlenbeck process.

Stationary GOU processes, or more generally stationary solutions to (1.1) have long attracted much attention in the probability community. De Haan and Karandikar [14] showed that GOU processes are the natural continuous time analogues of perpetuities. Carmona, Petit, and Yor [13] gave sufficient conditions in order that V in (1.2) converges in distribution to the stationary distribution for any nonstochastic $V_0 = x_0$. Necessary and sufficient conditions for the existence of a stationary solution were given by Lindner and Maller [23] in the GOU case, and by Behme, Lindner, and Maller [4] in case of solutions to (1.1). Tail behavior and moments of the stationary solution was investigated by Behme [5]. The stationary solution of (1.1) under appropriate conditions is $\int_0^\infty e^{-\xi_{s-}} \mathrm{d}L_s$, which is the exponential functional of the bivariate Lévy process (ξ, L). Continuity properties of these exponential functionals were investigated by Carmona, Petit, and Yor [12], Bertoin, Lindner, and Maller [7], Lindner and Sato [24], and Kuznetsov, Pardo, and Savov [21]. Wiener–Hopf factorization of exponential functionals of Lévy processes (when $L_t \equiv t$) was extensively studied by Patie and Savov [32, 33, 34]. As a result of their new analytical approach smoothness properties of the densities were also obtained. GOU processes have a wide range of applications, among others in mathematical physics, in finance, and in risk theory. For a more complete account on GOU processes and on exponential functionals of Lévy processes we refer to the survey paper by Bertoin and Yor [8], to Behme and Lindner [3], and to [21], and the references therein.

Here we deal with ergodic properties of GOU processes. Ergodicity of stochastic processes is important on its own right, and also in applications, such as estimation of certain parameters. Ergodic theory for general Markov process, both in the discrete and in the continuous case was developed by Meyn and Tweedie [28, 29, 30]. Using the so-called Foster–Lyapunov techniques, they worked out conditions for ergodicity and exponential ergodicity in terms of the generator of the underlying process. Recently, much attention is drawn to situations where the rate of convergence is only subexponential. Fort and Roberts [17], Douc, Fort, and Guillin [15] and Bakry, Cattiaux, and Guillin [2] proved general conditions for subexponential rates. See also the lecture notes by Hairer [18].

Ergodicity and mixing properties of diffusions with jumps were investigated by Masuda [27] and Kulik [20]. Sandrić [37] proved ergodicity for Lévy-type processes. Concerning OU processes, Sato and Yamazato [39] gave necessary and sufficient conditions for the convergence of a Lévy-driven OU process. Exponential ergodicity was investigated by Masuda [26] and Wang [41] in the Lévy-driven case, and by Fasen [16] and Lee [22] for GOU processes. Parameter estimation for GOU processes was treated by Belomestny and Panov [6].

The paper is organized as follows. In section 2 we fix the notations, and give some background on the process V, and on ergodicity. Section 3 contains the description of the Foster–Lyapunov technique. Using the explicit form of the generator of the process we give here the drift conditions corresponding to Theorems 1–4. The infinitesimal generator of the process V is determined in [36, 3]. The difficulty in our case is to show that the domain of the extended generator contains unbounded norm-like functions; this is done in Proposition 3 in section 6. The proof of the drift conditions relies on Lemma 1, which states that a two-dimensional integral with respect to the Lévy measure ν_{UL} asymptotically equals to a one-dimensional integral with respect to the Lévy measure ν_U. This is the reason why the drift conditions in the theorems depend only on the law of U. However, the integrability condition does depend on ν_{UL}. Finally, we investigate the petite sets. In Theorems 1–4 we assume that all compact sets are petite sets for some skeleton chain. In Proposition 2 and in the remarks afterwards we give a sufficient condition for this assumption. It turns out that under natural conditions the petiteness assumption is satisfied.

Section 4 contains the main results of the paper. In Theorem 1 under general integrability assumptions we prove ergodicity for V. In particular, the assumptions in Theorem 1 reduce to the necessary and sufficient condition by Sato and Yamazato [39] in the Lévy-driven OU case. In Theorems 2 and 3 we obtain two different subexponential rates: a polynomial and an 'almost exponential' one. Moreover, we point out in Proposition 4 in section 6 that under more complex moment assumptions more general subexponential rates can be obtained. These results are particularly interesting in view of the rare subexponential convergence rates. In fact, in his Remark 4.4 [26] Masuda claimed that in most cases stationary Lévy-driven OU processes are exponentially ergodic. Finally, Theorem 4 provides sufficient conditions for exponential ergodicity. In Theorems 1–4 we assume that $\nu_U(\{-1\}) = 0$. It turns out that the process behaves very differently if $\nu_U(\{-1\}) > 0$. In the latter case the process restarts itself in finite exponential times from a random initial value, therefore it cannot go to infinity regardless of the moment properties of the Lévy measure. Indeed, as a consequence of a general result by Avrachenkov, Piunovskiy, and Zhang [1] we show in Theorem 5 below that in this case the process is always exponentially ergodic. Thus, concerning ergodic properties the case $\nu_U(\{-1\}) = 0$ is more interesting, and we largely concentrate on it.

In section 5 we compare our results to earlier ones, and also spell out some statements in special cases.

The proofs are gathered in section 6. First we show that the domain of the extended generator is large enough. Then we deal with the drift conditions.

2 Preliminaries

Here we gather together the most important properties of the solution V to equation (1.1), and we give the basic definitions on ergodicity. We also fix the notation.

2.1 The SDE (1.1)

A bivariate Lévy process $(U_t, L_t)_{t\geq 0}$ with characteristic triplet $((\gamma_U, \gamma_L), \Sigma, \nu_{UL})$ has characteristic exponent

$$\begin{aligned} \log \mathbf{E} e^{\mathrm{i}(\theta_1 U_t + \theta_2 L_t)} &= \mathrm{i}t(\theta_1\gamma_U + \theta_2\gamma_L) - \frac{t}{2}\langle \boldsymbol{\theta}\Sigma, \boldsymbol{\theta}\rangle \\ &\quad + t \iint_{\mathbb{R}^2} \left(e^{\mathrm{i}(\theta_1 z_1 + \theta_2 z_2)} - 1 - \mathrm{i}(\theta_1 z_1 + \theta_2 z_2) I(|\mathbf{z}| \leq 1) \right) \nu_{UL}(\mathrm{d}\mathbf{z}), \end{aligned} \tag{2.1}$$

where $\gamma_U, \gamma_L \in \mathbb{R}$,

$$\Sigma = \begin{pmatrix} \sigma_U^2 & \sigma_{UL} \\ \sigma_{UL} & \sigma_L^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

is a nonnegative semidefinite matrix, ν_{UL} is a bivariate Lévy measure. Here and later on $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^2$, $|\cdot|$ is the Euclidean norm both in $\mathbb{R}$ and $\mathbb{R}^2$, and $I(\cdot)$ stands for the indicator function. To ease the notation we also write $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $\mathbf{z} = (z_1, z_2)$, i.e. vectors are always denoted by boldface letters. Let ν_U, ν_L denote the Lévy measure of U and L, respectively.

The unique solution to (1.1) was determined by Behme, Lindner, and Maller [4, Proposition 3.2]. Introduce the process η as

$$\eta_t = L_t - \sum_{s\le t, \Delta U_s \neq -1} \frac{\Delta U_s \Delta L_s}{1+\Delta U_s} - \sigma_{UL} t, \tag{2.2}$$

where for any càdlàg process Y its jump at t is $\Delta Y_t = Y_t - Y_{t-}$. If $\nu_U(\{-1\}) = 0$ then the solution to (1.1) can be written as

$$V_t = \mathcal{E}(U)_t \left[x_0 + \int_0^t \mathcal{E}(U)_{s-}^{-1} \mathrm{d}\eta_s \right],$$

where the stochastic exponential (see Protter [36, p. 84–85]; also called Doléans–Dade exponential) $\mathcal{E}(U)$ is

$$\mathcal{E}(U)_t = e^{U_t - \sigma_U^2 t/2} \prod_{s\le t} (1+\Delta U_s) e^{-\Delta U_s}.$$

While for $\nu_U(\{-1\}) > 0$

$$\begin{aligned} V_t &= \mathcal{E}(U)_t \left[x_0 + \int_0^t \mathcal{E}(U)_{s-}^{-1} \mathrm{d}\eta_s \right] I(K(t)=0) \\ &\quad + \mathcal{E}(U)_{(T(t),t]} \left[\Delta L_{T(t)} + \int_{(T(t),t]} \mathcal{E}(U)_{(T(t),s)}^{-1} \mathrm{d}\eta_s \right] I(K(t)\ge 1), \end{aligned} \tag{2.3}$$

where $K(t) = \#\{s \in (0,t] : \Delta U(s) = -1\}$, $T(t) = \sup\{s \in (0,t] : \Delta U(s) = -1\}$, and for $0 \le s < t$

$$\begin{aligned} \mathcal{E}(U)_{(s,t]} &= e^{U_t - U_s - \sigma_U^2 (t-s)/2} \prod_{s<u\le t} (1+\Delta U_u) e^{-\Delta U_u}, \\ \mathcal{E}(U)_{(s,t)} &= e^{U_{t-} - U_s - \sigma_U^2 (t-s)/2} \prod_{s<u<t} (1+\Delta U_u) e^{-\Delta U_u}, \end{aligned}$$

while $\mathcal{E}(U)_{(s,t]} = 1$ for $s \ge t$. From this form we see that the process restarts from $\Delta L_{T(t)}$ whenever a jump of size -1 occurs, so the cases $\nu_U(\{-1\}) = 0$ and $\nu_U(\{-1\}) > 0$ are significantly different. In fact the latter is much easier to handle.

Here we always consider deterministic initial value $V_0 = x_0 \in \mathbb{R}$, however, we note that the existence and uniqueness of the solution to (1.1) holds under more general conditions on the initial value V_0; see Proposition 3.2 in [4].

The processes U and L are semimartingales with respect to the smallest filtration, which satisfies the usual hypotheses and contains the filtration generated by (U, L). Stochastic integrals are always meant with respect to this filtration. The integral $\int_s^t$ for $s \le t$ stands for the integral on the closed interval $[s,t]$. Since we do not directly use stochastic integration theory, we prefer to suppress unnecessary notation.

If $\nu_U((-\infty,-1]) = 0$ then we may introduce the process ξ as

$$\xi_t = -\log \mathcal{E}(U)_t = -U_t + \frac{\sigma_U^2}{2} t + \sum_{s \le t} [\Delta U_s - \log(1+\Delta U_s)]. \tag{2.4}$$

Then, it is easy to see that (ξ,η) has independent stationary increments, i.e. it is a bivariate Lévy process. In fact there is a one-to-one correspondence between bivariate Lévy processes (ξ,η) and (U,L), where $\nu_U((-\infty,-1]) = 0$. To see this, for a given bivariate Lévy process (ξ,η) define

$$\begin{aligned} U_t &= -\xi_t + \sum_{0<s\le t} \left(e^{-\Delta\xi_s} - 1 + \Delta\xi_s\right) + t\frac{\sigma_\xi^2}{2}, \\ L_t &= \eta_t + \sum_{0<s\le t} \left(e^{-\Delta\xi_s} - 1\right) \Delta\eta_s - t\sigma_{\xi\eta}, \end{aligned}$$

where σ_ξ^2 is the variance of the Gaussian part of ξ, and $\sigma_{\xi\eta}$ is the covariance of the Gaussian part of ξ and η. Note that $\eta \equiv L$ if η and ξ are independent, or equivalently from (2.2), if U and L are independent. For more details and verification of these statements see the discussions on p. 428 by Maller, Müller, and Szimayer [25], and [3, p.4]. Thus, without the restriction $\nu_U((-\infty,-1]) = 0$ the class of solutions to (1.1) is larger than the class of GOU processes.

2.2 Ergodicity

We use the methods developed by Meyn and Tweedie [28, 29, 30], and we also use their terminology. First, we recall some basic notions about Markov processes, which we need later. The definitions are from [29, 30].

As usual for a Markov process $(X_t)_{t\ge0}$ for any $x \in \mathbb{R}$, $\mathbf{P}_x$ and $\mathbf{E}_x$ stands for the probability and expectation conditioned on $X_0 = x$. In the following X is always a Markov process.

The process $(X_t)_{t\ge0}$ is a *Feller process*, if $T_t f(x) := \mathbf{E}_x f(X_t) \in C_0$ for any $f \in C_0$, $t \ge 0$, and $\lim_{t\downarrow0} T_t f(x) = f(x)$ for any $f \in C_0$, where $C_0 = \{f : f \text{ is continuous}, \lim_{|x|\to\infty} f(x) = 0\}$. If $T_t f$, $t > 0$, is only continuous, but does not necessarily tend to 0 at infinity, then the process is a *weak Feller process*.

Let $T_n = \inf\{t \ge 0 : |X_t| \ge n\}$, $n \ge 1$. If $\mathbf{P}_x\{\lim_{n\to\infty} T_n = \infty\} = 1$ for all $x \in \mathbb{R}$, then X is *nonexplosive*. A time-homogeneous Markov process $(X_t)_{t\ge0}$ on $\mathbb{R}$ is *ϕ-irreducible* (or simply irreducible), if for some σ-finite measure ϕ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mathcal{B}(\mathbb{R})$ being the Borel sets, $\phi(B) > 0$ implies $\int_0^\infty \mathbf{P}_x\{X_t \in B\}\mathrm{d}t > 0$, for all $x \in \mathbb{R}$. The notion of petite sets is a technical tool for linking stability properties of Markov processes with the different drift conditions. Stochastic stability is closely related to the return time behavior of the process on petite sets. A nonempty set $C \in \mathcal{B}(\mathbb{R})$ is *petite set* (respect to the process X), if there is a probability distribution a on $(0,\infty)$, and a nontrivial measure ψ such that

$$\int_0^\infty \mathbf{P}_x\{X_t \in A\} a(\mathrm{d}t) \ge \psi(A), \ \forall x \in C, \ \forall A \in \mathcal{B}(\mathbb{R}).$$

A probability measure π is *invariant* (for the process X), if

$$\pi(A) = \int_{\mathbb{R}} \mathbf{P}_x\{X_t \in A\}\pi(\mathrm{d}x), \quad \forall t \geq 0,\ \forall A \in \mathcal{B}(\mathbb{R}).$$

The process X is *positive Harris recurrent*, if there is an invariant probability measure π such that

$$\mathbf{P}_x\{X_t \in A \text{ for some } t \geq 0\} = 1$$

whenever $\pi(A) > 0$.

For a continuous time Markov process $(X_t)_{t\geq 0}$ the discretely sampled process $(X_{n\delta})_{n\in\mathbb{N}}$, $\delta > 0$, which is a Markov chain, called *skeleton chain.* Irreducibility and petiteness are defined analogously for Markov chains. A Markov chain $(X_n)_{n\in\mathbb{N}}$ is *weak Feller chain* if $Tf(x) = \mathbf{E}_x f(X_1)$ is continuous and bounded for any continuous and bounded f.

For a measurable function $g \geq 1$ and a signed measure μ introduce the notation

$$\|\mu\|_g = \sup\left\{ \int h\mathrm{d}\mu : |h| \leq g,\ h \text{ measurable} \right\}.$$

When $g \equiv 1$ we obtain the total variation norm, which is simply denoted by $\|\cdot\|$. The process X is *ergodic*, if there exists an invariant probability measure π such that

$$\lim_{t\to\infty} \|\mathbf{P}_x\{X_t \in \cdot\} - \pi\| = 0 \quad \text{for all } x \in \mathbb{R}. \tag{2.5}$$

For a measurable function $f \geq 1$ the process X is *f-ergodic*, if it is positive Harris recurrent with invariant probability measure π, $\int f\mathrm{d}\pi < \infty$, and

$$\lim_{t\to\infty} \|\mathbf{P}_x\{X_t \in \cdot\} - \pi\|_f = 0 \quad \text{for all } x \in \mathbb{R}. \tag{2.6}$$

If the convergence in (2.5), (2.6) is exponentially fast, i.e. there exists a finite valued function g, and $c > 0$ such that

$$\|\mathbf{P}_x\{X_t \in \cdot\} - \pi\|_f \leq g(x)e^{-ct} \quad \text{for all } x \in \mathbb{R}, \tag{2.7}$$

then X is *f-exponentially ergodic* (or simply exponentially ergodic when $f \equiv 1$).

We mention that the terminology is not completely unified. Sometimes 'geometrically' refers to a discrete time process, and 'exponentially' to a continuous time process, see [28, 30]. However, some authors ([15, 17]) use the term f-geometrically ergodic, instead of f-exponentially ergodic for continuous time processes. More importantly, when $f \equiv 1$, and (2.7) holds only for π-almost every x, then X is called geometrically ergodic; see Bradley [11, p. 121], Nummelin, Tuominen [31, Definition 1.1] (also [16, 26]). Here we follow Meyn and Tweedie.

2.3 Infinitesimal and extended generators

The *infinitesimal generator* $\mathcal{A}$ of a Markov process X is defined as

$$\mathcal{A}f(x) = \lim_{t\downarrow 0} t^{-1}\mathbf{E}_x[f(X_t) - f(x)]$$

whenever it exists. Its domain is denoted by $\mathcal{DI}(X)$. The *extended generator* $\mathcal{A}$ of the Markov process X is defined as $\mathcal{A}f = g$ whenever $f(X_t) - f(X_0) - \int_0^t g(X_s)\mathrm{d}s$ is a local martingale with respect to the natural filtration. Its domain is denoted by $\mathcal{DE}(X)$. The same notation should not cause confusion, since the two operators are the same, only the domains are different.

Let us define the operator $\mathcal{A}$ as

$$\begin{aligned}\mathcal{A}f(x) &= (x\gamma_U + \gamma_L)f'(x) + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{UL} + \sigma_L^2)f''(x) \\ &\quad + \iint_{\mathbb{R}^2} \left[f(x + xz_1 + z_2) - f(x) - f'(x)(xz_1 + z_2)I(|z| \le 1)\right] \nu_{UL}(\mathrm{d}z),\end{aligned} \tag{2.8}$$

where $f \in C^2$ (the set of twice continuously differentiable functions) is such that the integral in the definition exists. In Exercise V.7 in Protter [36] and in Theorem 3.1 in Behme and Lindner [3] it is shown that the infinitesimal generator of V is $\mathcal{A}$, and $C_c^\infty \subset \mathcal{DI}(V)$, where C_c^∞ is the set of infinitely many times differentiable compactly supported functions. Moreover, if $\nu_U(\{-1\}) = 0$ then V is a Feller process, and $\mathcal{DI}(V) \supset \{f \in C_0^2 : \lim_{|x|\to\infty}(|xf'(x)| + x^2|f''(x)|) = 0\}$, which is a core, see [3, Theorem 3.1]. Here $C_0^2 = \{f : f$ twice continuously differentiable, and $\lim_{|x|\to\infty} f''(x) = 0\}$. It is clear from the regenerative property of the process in (2.3) that if $\nu_U(\{-1\}) > 0$ then it is *not* Feller process, only weak Feller. A slightly different form of the generator in terms of (ξ, η), for independent ξ and η is given in [21, Proposition 2.3], see also [3, Remark 3.4].

3 The Foster–Lyapunov method

The Foster–Lyapunov method is a well-established technique for proving ergodicity (recurrence, ergodicity with rate, etc.) of Markov-processes both in discrete and in continuous time. The method was worked out by Meyn and Tweedie in the series of papers [28, 29, 30]. There are two basic components: (i) to prove drift conditions (or Foster–Lyapunov inequalities) for the extended generator of the Markov process; (ii) to show that the topological properties of the process are not too pathological (e.g. certain sets are petite sets, or some skeleton chain is irreducible). In this section we describe the Foster–Lyapunov method.

3.1 Drift conditions

In order to apply Foster–Lyapunov techniques we have to truncate the process V, defined in (1.1). For $n \in \mathbb{N}$ let

$$V_t^n = V_{t\wedge T_n} \tag{3.1}$$

where $T_n = \inf\{t \ge 0 : |V_t| \ge n\}$. Note that this is not exactly the process defined in [30, p.521], but the results in [30] are valid for our process; see the comment after formula (2) in [30, p.521]. We also emphasize that the stopped process is not necessarily bounded, which causes some difficulties by proving that the domain of the extended generator is large enough.

Let us define the generator of V^n as

$$\mathcal{A}_n f(x) = \begin{cases} \mathcal{A}f(x), & |x| < n, \\ 0, & |x| \ge n. \end{cases}$$

In Proposition 3 we show that $\mathcal{A}_n$ is indeed the extended generator of the process V^n, and

$$\mathcal{DE}(V^n) \supset \left\{ f \in C^2 : \iint_{|\mathbf{z}|>1} |f(|\mathbf{z}|)| \nu_{UL}(\mathrm{d}\mathbf{z}) < \infty \right\}.$$

From this result it also follows that

$$\mathcal{DE}(V) \supset \left\{ f \in C^2 : \iint_{|\mathbf{z}|>1} |f(|\mathbf{z}|)| \nu_{UL}(\mathrm{d}\mathbf{z}) < \infty \right\}.$$

Following [30] we introduce the various ergodicity conditions for the generator $\mathcal{A}_n$. A function $f : \mathbb{R} \to [0, \infty)$ is *norm-like* if $f(x) \to \infty$ as $|x| \to \infty$. In the conditions below f is always a norm-like function. The recurrence condition is

$$\exists f,\, \exists d > 0,\, \exists C \text{ compact such that } \mathcal{A}_n f(x) \le d I_C(x),\ \forall |x| < n, \forall n \in \mathbb{N}. \tag{3.2}$$

The ergodicity condition is

$$\begin{gathered} \exists f,\, \exists c, d > 0,\, \exists g \ge 1 \text{ measurable, } \exists C \text{ compact such that} \\ \mathcal{A}_n f(x) \le -c g(x) + d I_C(x),\ \forall |x| < n, \forall n \in \mathbb{N}. \end{gathered} \tag{3.3}$$

The exponential ergodicity condition is

$$\exists f,\, \exists c, d > 0 \text{ such that } \mathcal{A}_n f(x) \le -c f(x) + d,\ \forall |x| < n, \forall n \in \mathbb{N}. \tag{3.4}$$

For subexponential rates of convergence we use more recent results due to Douc, Fort and Guillin [15], Bakry, Cattiaux and Guillin [2]. For a survey see also Hairer's notes [18]. The subexponential ergodicity condition ([15, Theorems 3.4 and 3.2]) is

$$\begin{gathered} \exists f \ge 1, d > 0,\, C \text{ compact}, \varphi \text{ positive concave} \\ \text{such that } \mathcal{A} f(x) \le -\varphi(f(x)) + d I_C(x),\ \forall x \in \mathbb{R}. \end{gathered} \tag{3.5}$$

In the following we state the drift conditions corresponding to Theorems 1–4. Define the finite measure ν' on $[-1, 1]$ by

$$\nu'(A) = \nu_{UL}((A \times \mathbb{R}) \cap \{|\mathbf{z}| > 1\}), \tag{3.6}$$

where $A \subset [-1, 1]$ is Borel measurable. With this notation, from (2.1) we obtain

$$\mathbf{E} e^{\mathrm{i}\theta U_1} = \exp\left\{ \mathrm{i}\theta \left(\gamma_U + \int_{-1}^{1} z \nu'(\mathrm{d}z) \right) - \frac{\sigma_U^2}{2} \theta^2 + \int_{\mathbb{R}} \left(e^{\mathrm{i}\theta z} - 1 - \mathrm{i}\theta z I(|z| \le 1) \right) \nu_U(\mathrm{d}z) \right\}. \tag{3.7}$$

Proposition 1. *Assume that* $\nu_U(\{-1\}) = 0$. *Assuming the drift condition*

$$\gamma_U + \int_{-1}^{1} z \nu'(\mathrm{d}z) - \frac{\sigma_U^2}{2} + \int_{\mathbb{R}} [\log|1 + z| - z I(|z| \le 1)]\, \nu_U(\mathrm{d}z) < 0 \tag{3.8}$$

(i) (3.3) holds with $f(x) = \log|x|$, $|x| \geq e$, *and* $g \equiv 1$ *whenever the integrability condition*

$$\iint_{|\mathbf{z}|\geq 1} \log|\mathbf{z}|\, \nu_{UL}(\mathrm{d}\mathbf{z}) < \infty, \quad \int_{-3/2}^{-1/2} |\log|1+z||\, \nu_U(\mathrm{d}z) < \infty \tag{3.9}$$

is satisfied;

(ii) (3.5) holds with $f(x) = (\log|x|)^\alpha$, $|x| \geq 3$, *and* $\varphi(x) = x^{1-1/\alpha}$ *for some* $\alpha > 1$, *whenever*

$$\iint_{|\mathbf{z}|\geq 1} (\log|\mathbf{z}|)^\alpha \nu_{UL}(\mathrm{d}\mathbf{z}) < \infty, \quad \int_{-3/2}^{-1/2} |\log|1+z||\, \nu_U(\mathrm{d}z) < \infty; \tag{3.10}$$

(iii) (3.5) holds with $f(x) = \exp\{\gamma(\log|x|)^\alpha\}$, $|x| \geq e$, *and* $\varphi(x) = x\,(\log x)^{1-1/\alpha}$ *for some* $\alpha \in (0,1)$, $\gamma > 0$, *whenever*

$$\iint_{|\mathbf{z}|\geq 1} \exp\{\gamma(\log|\mathbf{z}|)^\alpha\}\nu_{UL}(\mathrm{d}\mathbf{z}) < \infty, \quad \int_{-3/2}^{-1/2} |\log|1+z||\, \nu_U(\mathrm{d}z) < \infty. \tag{3.11}$$

For some $\beta \in (0,1]$, *assuming*

$$\gamma_U + \int_{-1}^{1} z\nu'(\mathrm{d}z) - \frac{\sigma_U^2(1-\beta)}{2} + \int_{\mathbb{R}} \frac{|1+z|^\beta - 1 - z\beta I(|z|\leq 1)}{\beta}\nu_U(\mathrm{d}z) < 0 \tag{3.12}$$

(iv) (3.4) holds with $f(|x|) = |x|^\beta$, $|x| \geq 1$, *if*

$$\iint_{|\mathbf{z}|\geq 1} |\mathbf{z}|^\beta \nu_{UL}(\mathrm{d}\mathbf{z}) < \infty. \tag{3.13}$$

Condition (3.8) is the limiting condition of condition (3.12) as β tends to 0.

Note that in the drift conditions (3.8), (3.12) depend only on the law of U. However, the integral conditions do depend on the joint law of (U, L).

By Theorem 25.3 in Sato [38], noting that the corresponding functions are submultiplicative, the first part of conditions (3.9), (3.10), (3.11), and condition (3.13) are equivalent to the finiteness of $\mathbf{E}\log(|(U_1, L_1)| \vee e)$, $\mathbf{E}[\log(|(U_1, L_1)| \vee e)]^\alpha$, $\mathbf{E}\exp\{\gamma[\log(|(U_1, L_1)| \vee e)]^\alpha\}$, and $\mathbf{E}|(U_1, L_1)|^\beta$, respectively. However, the other conditions cannot be rewritten in terms of (U, L).

Assume that any of the integrability conditions of Proposition 1 is satisfied. If U has a large negative drift, then the corresponding drift condition holds, while it fails for large positive drift. In this way it is easy to construct examples, when the conditions hold true, and when do not.

3.2 Petite sets

In this subsection we give sufficient condition for all compact sets to be petite sets for some skeleton chain. Under natural assumptions this condition holds.

By investigating ergodicity rates a minimal necessary assumption is that the process converges in distribution. If V_t converges in distribution for any initial value $x_0 \in \mathbb{R}$ then V is π-irreducible, where π is the law of the limit distribution. Certain properties of the limit distribution imply that compact sets are petite sets. Recall the relation (U, L) and (η, ξ) from (2.2), (2.4).

Proposition 2. *Assume that* $\lim_{t\to\infty} \xi_t = \infty$ *a.s.,* $\int_0^\infty e^{-\xi_{s-}} \mathrm{d}L_s$ *exists a.s., and its distribution* π *is such that the interior of its support is not empty. Then all compact sets are petite sets for the skeleton chain* $(V_n)_{n\in\mathbb{N}}$.

The assumptions above imply the existence of a stationary causal solution to (1.1). Moreover, for any initial value x_0 the solution converges in distribution to the stationary solution; see [4, Theorem 2.1 (a)]. Necessary and sufficient conditions for $\lim_{t\to\infty} \xi_t = \infty$ a.s., and for the existence of $\int_0^\infty e^{\xi_{s-}} \mathrm{d}L_s$ are given in [4, Theorem 3.5 and 3.6].

In the Lévy-driven OU case, when $U_t = -\mu t$, $\mu > 0$, the distributional convergence of V_t holds if and only if $\int_{|z|>1} \log |z| \nu_L(\mathrm{d}z) < \infty$, in which case the limit distribution is self-decomposable; see [39, Theorem 4.1]. A nondegenerate selfdecomposable distribution has absolute continuous density with respect to the Lebesgue measure, in particular the interior of its support is not empty; see [38, Theorem 28.4].

In the general case, there is much less known about the properties of the integral $J = \int_0^\infty e^{-\xi_{s-}} \mathrm{d}L_s$. Continuity properties of these integrals were investigated in [7]. It was shown in [7, Theorem 2.2] that if π has an atom then it is necessarily degenerate. If ξ is spectrally negative (does not have positive jumps), then π is still self-decomposable; see [7] Theorem 2.2, and the remark after it. When $L_t = t$ sufficient conditions for the existence of the density of $\int_0^\infty e^{-\xi_s} \mathrm{d}s$ were given in [12, Proposition 2.1]. When L and U are independent, $\mathbf{E}|\xi_1| < \infty$, $\mathbf{E}\xi_1 > 0$, $\mathbf{E}|L_1| < \infty$, and $\sigma_U^2 + \sigma_L^2 > 0$ then J has continuously differentiable density; see [21, Corollary 2.5]. The case, when $(\xi_t, L_t)_{t\geq 0} = ((\log c) N_t, Y_t)$, $c > 1$, where $(N_t)_{t\geq 0}, (Y_t)_{t\geq 0}$ are Poisson processes, and $(N_t, Y_t)_{t\geq 0}$ is a bivariate Lévy process was treated in [24]. Whether the distribution of J is absolute continuous or continuous singular depends on algebraic properties of the constant c, see Theorems 3.1 and 3.2 [24]. The problem of absolute continuity in this case is closely related to infinite Bernoulli convolutions; see Peres, Schlag, and Solomyak [35]. For further results in this direction we refer to [7, 24, 21] and the references therein.

4 Results

4.1 The case $\nu_U(\{-1\}) = 0$

In the theorems below we need that all compact sets are petite sets for some skeleton chain. As we have seen in Proposition 2, this assumption is satisfied under mild conditions.

First we give a sufficient condition for the ergodicity of the process.

Theorem 1. *Assume that* $\nu_U(\{-1\}) = 0$*, all compact sets are petite for some skeleton chain, (3.8) and (3.9) hold. Then* V *is ergodic, i.e. there is an invariant probability measure* π *such that for any* $x \in \mathbb{R}$

$$\lim_{t\to\infty} \|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| = 0.$$

Proof of Theorem 1. The process is clearly nonexplosive, and Proposition 1 (i) shows that the ergodicity condition holds. Thus [30, Theorem 5.1] proves the statement. □

In the Lévy-driven OU case our assumptions reduce to the the necessary and sufficient condition for convergence to an invariant measure, given by Sato and Yamazato [39] (in any dimension); see Corollary 1 below.

Assuming stronger moment assumptions we obtain polynomial rate of convergence. However, note that the drift condition is the same as in the previous result.

Theorem 2. *Assume that $\nu_U(\{-1\}) = 0$, all compact sets are petite for some skeleton chain, (3.8) and (3.10) hold. Then there is an invariant probability measure π such that for some $C > 0$ for any $x \in \mathbb{R}$, $t > 0$,*

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \leq C\,(\log|x|)^\alpha\, t^{1-\alpha}.$$

Proof of Theorem 2. Here we use the notation in [15]. A petite set for the skeleton chain is petite set for the continuous process. In particular, the compact set in condition (3.5) is petite. By Theorem 1 the process V is ergodic, therefore some (any) skeleton chain is irreducible. Proposition 1 (ii) and [15, Theorem 3.4] imply that the further assumptions of [15, Theorem 3.2] are satisfied with $f(x) = (\log|x|)^\alpha$ and $\varphi(x) = x^{1-1/\alpha}$. From the discussion after [15, Theorem 3.2] we see that for the rate of convergence corresponding to the total variation distance we have

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \leq C(\log|x|)^\alpha r_*(t)^{-1},$$

with $r_*(t) = \varphi(H_\varphi^{\leftarrow}(t))$, where $H_\varphi(t) = \int_1^t \varphi(s)^{-1}\mathrm{d}s$, and $H_\varphi^{\leftarrow}$ is the inverse function of H_φ. After a short calculation we see that this is exactly the statement. □

We can show not only polynomial but more general convergence rates. However, in the general case the assumptions are more complicated. We spell out one more example. A general result on the drift condition is given in Proposition 4. From its proof it will be clear why the drift condition is the same in Theorems 1, 2, and 3; see the comment after (6.4).

Theorem 3. *Assume that $\nu_U(\{-1\}) = 0$, all compact sets are petite for some skeleton chain, (3.8) and (3.11) hold. Then there is an invariant probability measure π such that for some $C > 0$ for any $x \in \mathbb{R}$, $t > 0$,*

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \leq C \exp\{\gamma(\log|x|)^\alpha\}\, e^{-(t/\alpha)^\alpha} t^{1-\alpha}.$$

Proof of Theorem 3. The proof is the same as the previous one. Now $f(x) = \exp\{\gamma(\log|x|)^\alpha\}$ and $\varphi(x) = x(\log x)^{1-1/\alpha}$. Short calculation gives that $H_\varphi(t) = \alpha(\log t)^{1/\alpha}$, and the statement follows. □

As in [15, Theorem 3.2], under the same assumptions as in Theorems 2 and 3 above it is possible to prove convergence rates in other norms, i.e. for $\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\|_g$ with specific g. There is a trade-off between the convergence rate and the norm function g: larger g corresponds to weaker rate, and vice versa. See [15, Theorem 3.2] and the remark after it.

Last we deal with exponential ergodicity.

Theorem 4. *Assume that $\nu_U(\{-1\}) = 0$, all compact sets are petite for some skeleton chain, (3.12) and (3.13) hold. Then V is exponentially ergodic, that is there is an invariant probability measure π such that for some $c, C > 0$,*

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\|_g \leq C(1 + |x|^\beta)e^{-ct},$$

for any $x \in \mathbb{R}$, $t > 0$, with $g(x) = 1 + |x|^\beta$.

Proof of Theorem 4. Proposition 1 (iv) shows that the exponential ergodicity condition holds, thus [30, Theorem 6.1] implies the statement. □

4.2 The case $\nu_U(\{-1\}) > 0$

In this subsection we consider the significantly different case $\nu_U(\{-1\}) > 0$. From (2.3) we see that in this case the process returns to $\Delta L_{T(t)}$ in exponential times and restarts. It is natural to expect that exponential ergodicity holds without further moment conditions. This is exactly the situation treated by Avrachenkov, Piunovskiy and Zhang [1]. Put $\lambda = \nu_U(\{-1\}) > 0$. Then, if $T_1 = \min\{t : \Delta U_t = -1\}$, then the restart distribution is

$$m(A) := \mathbf{P}(\Delta L_{T_1} \in A) = \frac{\nu_{UL}(\{-1\} \times A)}{\nu_U(\{-1\})} = \frac{\nu_{UL}(\{-1\} \times A)}{\lambda},$$

where A is a Borel set of $\mathbb{R}$. Let $\widetilde{V}$ be the process with the same characteristics as V, except $\nu_{\widetilde{U}}(\{-1\}) = 0$. Then the process V can be seen as the process $\widetilde{V}$ which restarts from a random initial value with distribution m after independent exponential random times with parameter λ. In Corollary 2.1 [1] it is shown that

$$\pi(A) = \int_{\mathbb{R}} \int_0^\infty \mathbf{P}_y\{\widetilde{V}_s \in A\} \lambda e^{-\lambda s} \mathrm{d}s \, m(\mathrm{d}y)$$

is the unique invariant probability measure for V. From Theorem 2.2 in [1] we obtain the following.

Theorem 5. *Assume that $\lambda = \nu_U(\{-1\}) > 0$. Then the process V is exponentially ergodic, that is*

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \le 2\, e^{-\lambda t},$$

where the invariant measure π is defined above.

In this case, by [3, Theorem 2.2] a strictly stationary causal solutions always exists, which has marginal distribution π.

5 Previous results and special cases

Ergodic properties of Lévy-driven stochastic differential equations (in d-dimension) were investigated by Masuda [27] and Kulik [20]. They obtained very general conditions for ergodicity and exponential ergodicity. Due to the generality of their setting the drift conditions (3.3) and (3.4) explicitly appear in their results, therefore these theorems are difficult to apply to our specific process. Sandrić treated Lévy-type processes, which class is still too large to obtain explicit conditions, as we will see below. We are not aware of any previous result on the ergodicity properties of the solution of (1.1). However, there are various results concerning ergodicity of GOU processes, and Lévy driven OU processes, which we spell out below, and compare them to our theorems.

5.1 Lévy-type processes

In a recent paper Sandrić [37] analyzed ergodicity of Lévy-type processes. (Sandrić considered d-dimensional processes. Here we spell out everything in one dimension.) These processes are such

Feller processes $(X_t)_{t\geq 0}$, whose generator is an integro-differential operator of the form

$$\begin{aligned} \mathcal{L}f(x) &= -a(x)f(x) + b(x)f'(x) + \frac{c(x)}{2}f''(x) \\ &\quad + \int_{\mathbb{R}} \big(f(x+y) - f(x) - yf'(x)I(|y|\leq 1)\big)\,\mu_x(\mathrm{d}y), \end{aligned} \tag{5.1}$$

and $C_c^\infty \subset \mathcal{DI}(X)$. For precise definition and more details see [37], and the monograph on Lévy-type processes by Böttcher, Schilling and Wang [10]. In terms of the coefficients (a, b, c, μ_x) in Theorem 3.3 [37] Sandrić gave sufficient conditions for the transience, recurrence, ergodicity, polynomial/exponential ergodicity of the process. Due to the generality of the setup, these conditions are necessarily complicated.

Schilling and Schnurr [40, Theorem 3.1] showed that the solution of a Lévy-driven SDE is a Feller process if the coefficients are bounded and locally Lipschitz. The coefficients in (1.1) are not bounded, however if $\nu_U(\{-1\} = 0$, then the solution is still a Feller process, and $C_c^\infty \subset \mathcal{DI}(V)$; see [3, Theorem 3.1]. Thus, the solution of (1.1) is a Lévy-type process, so the generator $\mathcal{A}$ in (2.8) can be written in the form in (5.1). Indeed, after some calculation one has

$$\begin{aligned} &\iint_{\mathbb{R}^2} \big[f(x + xz_1 + z_2) - f(x) - f'(x)(xz_1 + z_2)I(|\mathbf{z}| \leq 1)\big]\,\nu_{UL}(\mathrm{d}z) \\ &= \int_{\mathbb{R}} \big[f(x+y) - f(x) - f'(x)yI(|y|\leq 1)\big]\,\mu_x(\mathrm{d}y) \\ &\quad - f'(x)\iint_{\mathbb{R}^2} (xz_1 + z_2)(I(|\mathbf{z}|\leq 1) - I(|xz_1 + z_2| \leq 1))\nu_{UL}(\mathrm{d}\mathbf{z}), \end{aligned}$$

where $\mu_x(A) = \nu_{UL}(\{(z_1, z_2) : xz_1 + z_2 \in A\})$, $A \in \mathcal{B}(\mathbb{R})$. Thus, we obtain that $\mathcal{A}$ in (2.8) has the representation (5.1) with

$$\begin{aligned} a(x) &\equiv 0, \\ b(x) &= x\gamma_U + \gamma_L - \iint_{\mathbb{R}^2} (xz_1 + z_2)[I(|\mathbf{z}|\leq 1) - I(|xz_1 + z_2|\leq 1)]\nu_{UL}(\mathrm{d}\mathbf{z}), \\ c(x) &= x^2\sigma_U^2 + 2x\sigma_{UL} + \sigma_L^2, \\ \mu_x(A) &= \nu_{UL}(\{(z_1, z_2) : xz_1 + z_2 \in A\}), \quad A \in \mathcal{B}(\mathbb{R}). \end{aligned} \tag{5.2}$$

From this representation, we see that it is very difficult to obtain reasonable conditions in terms of the Lévy-triplet $((\gamma_U, \gamma_L), \Sigma, \nu_{UL})$ for the ergodicity (with or without rate) of the process V using Theorem 3.3 in [37]. However, in special cases the representation (5.2) simplifies. If U is continuous, then $\mathcal{A}$ in (2.8) has the form (5.1) with $a(x) \equiv 0$, $b(x) = x\gamma_U + \gamma_L$, $c(x) = x^2\sigma_U^2 + 2x\sigma_{UL} + \sigma_L^2$, $\mu_x \equiv \nu_L$. In this case, after some calculation we obtain from Theorem 3.3 (iii) [37] that if $\int_{|z|\geq 1} \log|z|\nu_L(\mathrm{d}z) < \infty$ and $\gamma_U < \sigma_U^2/2$ then V is ergodic. This is exactly the condition in our Theorem 1; see also subsection 5.3. Theorem 3.3 [37] also provides conditions for transience and recurrence. We note that our Theorems 2 and 3 have no counterpart in [37], as there the integrability of $\log|z|$ or of $|z|^\alpha$, $\alpha > 0$, is assumed.

5.2 GOU case

Recall the definition of (ξ, η) from (2.2), (2.4).

Fasen [16] investigated ergodic and mixing properties of strictly stationary GOU processes under the following conditions: η is a subordinator, the initial value V_0 is independent of (ξ, η), there is a positive stationary version V such that $\mathbf{P}\{V_0 > x\} \sim Cx^{-\alpha}$ as $x \to \infty$ for some $C > 0$, $\alpha > 0$, $\mathbf{E}e^{-\alpha\xi_1} = 1$, $\mathbf{E}e^{-d\xi_1} < \infty$ for some $d > \alpha$, and for some $h > 0$

$$\mathbf{E}\left| e^{-\xi_h} \int_0^h e^{\xi_{s-}} \mathrm{d}\eta_s \right|^d < \infty. \tag{5.3}$$

In [16, Proposition 3.4], along the lines of Masuda's proof of [26, Theorem 4.3], it is shown that if the conditions above hold, and any δ-skeleton chain of V is ϕ-irreducible (with the same measure ϕ), then there is a $g : \mathbb{R} \to (0, \infty)$, and $c > 0$ such that

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \leq g(x)e^{-ct} \quad \text{for } \pi\text{-a.e. } x,$$

where π is the unique invariant probability measure of V. (Indeed, the term geometrically ergodic is used in the sense of [31] both in [16] and [26].) The function g is not specified.

It is clear from the formulation that this type of ergodicity result is weaker than the one in Theorem 4. Indeed, in Theorem 4 the function g is explicitly given, and the result holds for all $x \in \mathbb{R}$. The conditions in [16, Proposition 3.4] are also more demanding. For example, in Theorem 4 L is not necessarily a subordinator. (Recall that if ξ and η are independent, then $\eta \equiv L$.)

Exponential ergodicity and β-mixing for more general GOU processes was investigated by Lee [22]. In Theorem 2.1 [22] it is shown that the distribution of $(V_{nh})_{n\in\mathbb{N}}$ converges to a probability measure π, which is the unique invariant distribution for the process, if $0 < \mathbf{E}\xi_h \leq \mathbf{E}|\xi_h| < \infty$ and $\mathbf{E}\log^+ |\eta_h| < \infty$. The condition $\mathbf{E}|\xi_h| < \infty$ is much stronger than our condition in Theorem 1. However, when U_t is continuous, one sees easily that $\mathbf{E}\xi_h = -h(\gamma_U - \sigma_U^2/2)$, and so conditions $\mathbf{E}\xi_h > 0$, $\mathbf{E}\log^+ |\eta_h| < \infty$ are the same as our conditions in Theorem 1. Moreover, Lee showed in her Theorem 2.6 that there is a $g : \mathbb{R} \to (0, \infty)$, $c > 0$ such that

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \leq g(x)e^{-ct}, \quad \text{for } \pi\text{-a.e. } x,$$

whenever for some $r > 0$

$$\mathbf{E}e^{-r\xi_h} < \infty, \text{ and } \mathbf{E}\left| e^{-\xi_h} \int_0^h e^{\xi_{s-}} \mathrm{d}\eta_s \right|^r < \infty, \tag{5.4}$$

the transition density functions exist, and they are uniformly bounded on compact sets. Again, g remains unspecified. Thus, as above in some cases Theorem 4 states more.

By Proposition 3.1 in [5] for $r \geq 1$ condition $\mathbf{E}e^{-r\xi_h} < \infty$ holds if and only if $\mathbf{E}|U_1|^r < \infty$. For the other condition note that by Proposition 2.3 in [23]

$$e^{-\xi_t} \int_0^t e^{\xi_{s-}} \mathrm{d}\eta_s \overset{\mathcal{D}}{=} \int_0^t e^{-\xi_{s-}} \mathrm{d}L_s.$$

Combining [23, Proposition 4.1] (or rather its proof) and [5, Proposition 3.1] we have that the latter has finite rth moment, $r \geq 1$, if $\mathbf{E}|U_1|^{\max\{1,r\}p} < \infty$, $\mathbf{E}(\mathcal{E}(U)_1)^r < 1$, and $\mathbf{E}|\eta_1|^{\max\{1,r\}q} < \infty$, for some $p, q > 1$, $p^{-1} + q^{-1} = 1$.

For $r = 1$, by [5, Proposition 3.1] $\mathbf{E}\mathcal{E}(U)_1 = e^{\mathbf{E}U_1}$, so the condition $\mathbf{E}\mathcal{E}(U)_1 < 1$ is equivalent to $\mathbf{E}U_1 < 0$. From (3.7) we see that (whenever it exists)

$$\mathbf{E}U_1 = \gamma_U + \int_{-1}^{1} z\nu'(\mathrm{d}z) + \int_{|z|>1} z\nu_U(\mathrm{d}z). \tag{5.5}$$

If $\nu_U((-\infty, -1]) = 0$ (which is the case for GOU process), then $\mathbf{E}U_1 < 0$ is exactly the drift condition (3.12) with $\beta = 1$.

We do not claim that Theorem 4 implies the results in [16] or in [22]. However, on the one hand, the statement of our theorem is stronger (g is explicitly given, and the inequality holds for all x, not only for π-a.e.). On the other hand, the moment conditions (5.3), (5.4) involve complicated stochastic integrals, so these are not easy to check. Our conditions (3.13), (3.12) are simpler, and seem to be less restrictive. Theorems 1 – 3 are completely new.

5.3 Lévy-driven OU

Here we consider the Lévy-driven OU processes, i.e. when $U_t = -\mu t$, $\mu > 0$. We spell out Theorems 1–4 in this case.

Corollary 1. *Assume that $U_t = -\mu t$, $\mu > 0$, and all compact sets are petite sets for some skeleton chain.*

(i) If $\int_{|z|\geq 1} \log|z| \nu_L(\mathrm{d}z) < \infty$, then V is ergodic; i.e. there is an invariant probability measure π such that $\lim_{t\to\infty} \|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| = 0$ for any $x \in \mathbb{R}$.

(ii) If $\int_{|z|\geq 1} (\log|z|)^\alpha \nu_L(\mathrm{d}z) < \infty$ for some $\alpha > 1$, then there is an invariant probability measure π, $C > 0$ such that $\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \leq C(\log|x|)^\alpha t^{1-\alpha}$ for any $x \in \mathbb{R}$, $t > 0$.

(iii) If $\int_{|z|\geq 1} e^{\gamma(\log|z|)^\alpha} \nu_L(\mathrm{d}z) < \infty$ for some $\gamma > 0, \alpha \in (0,1)$, then there is an invariant probability measure π, $C > 0$ such that $\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \leq Ce^{\gamma(\log|x|)^\alpha} e^{-(t/\alpha)^\alpha} t^{1-\alpha}$ for any $x \in \mathbb{R}$, $t > 0$.

(iv) If $\int_{\mathbb{R}\setminus[-1,1]} |x|^\beta \nu_L(\mathrm{d}x) < \infty$ for some $\beta \in (0,1]$, then there is an invariant probability measure π such that for some $c, C > 0$

$$\|\mathbf{P}_x\{V_t \in \cdot\} - \pi\|_g \leq C(1 + |x|^\beta)e^{-ct}, \tag{5.6}$$

for any $x \in \mathbb{R}$, $t > 0$, with $g(x) = 1 + |x|^\beta$.

Parts (i) and (iv) in Corollary 1 were given by Sandrić [37, Example 3.7] (in any dimension); see also Masuda [27, Theorem 2.6].

In the Lévy-driven OU case the necessary and sufficient condition for convergence to an invariant measure was given by Sato and Yamazato [39] (in any dimension). They showed that V_t converges in distribution if and only if $\int_{|z|\geq 1} \log|z| \nu_L(\mathrm{d}z) < \infty$, which is exactly our assumption.

Otherwise $|V_t|$ tends to infinity in probability. This suggests that the conditions in Theorem 1 are optimal.

Our results concerning the rate are also optimal in the following sense. Fort and Roberts gave examples for a compound Poisson-driven OU-process, which fails to be exponentially ergodic, or even is not positive recurrent [17, Example 3.3]. Assume that V is a Lévy-driven OU process such that $L_t = S_{N_t}$, where N_t is a standard Poisson process, and $S_n = X_1 + \ldots + X_n$, where $X, X_1, \ldots$ are i.i.d. nonnegative random variables. In [17, Lemma 17] it was shown that if $\mathbf{E}X^\alpha = \infty$ for any $\alpha > 0$, then the process is not exponentially ergodic. Moreover, if $\mathbf{E}\log X = \infty$ then the process is not positive recurrent.

For d-dimensional Lévy-driven OU processes Masuda [26] and Wang [41] proved exponential ergodicity. We spell out their results in one dimension. Using Foster–Lyapunov techniques, in Theorem 4.3 [26] Masuda proved (see also [27, Theorem 2.6]) that if $\int_{\mathbb{R}} |x|^\alpha \pi(\mathrm{d}x) < \infty$, $\alpha > 0$, where π is the stationary distribution, then V_t is exponential β-mixing, i.e. for some $c, C > 0$

$$\int_{\mathbb{R}} \|\mathbf{P}_x\{V_t \in \cdot\} - \pi\| \pi(\mathrm{d}x) \le C e^{-ct}, \quad t > 0.$$

Using coupling methods, Wang [41, Theorem 1] showed that (5.6) holds with $\beta = 1$, i.e. the process is exponentially ergodic, if $\int_{\mathbb{R}\setminus[-1,1]} |x| \nu_L(\mathrm{d}x) < \infty$, and the Lévy measure satisfies a smoothness condition. In Theorem 2 [41] it was proved that (5.6) holds if $\int_{\mathbb{R}\setminus[-1,1]} |x|^\beta \nu_L(\mathrm{d}x) < \infty$, $\beta \in (0,1]$, and the Lévy measure satisfies a growth condition at 0. The latter condition implies $\nu_L(\mathbb{R}) = \infty$, and it is satisfied for stable processes.

In this special case Theorems 1 and 4 are roughly the same as [27, Theorem 2.6] and [37, Example 3.7], while Theorems 2 and 3 are new.

5.4 Exponential functionals of Lévy processes

Another important special case is when $L_t \equiv t$. The law of the stationary solution is the integral $\int_0^\infty e^{-\xi_s} \mathrm{d}s$, which is called the exponential functional of the Lévy process ξ, and has been attracted much attention; see [8, 7, 21, 32, 33, 34].

In this particular case our condition does not simplify too much. Recalling (3.6), note that $\nu' \equiv 0$. Moreover, the double integrals with respect to ν_{UL} simplifies to an integral with respect to ν_U. We are not aware of any ergodicity results in this special case.

5.5 Subexponential rates

Subexponential rates are rare in the literature. For compound Poisson driven Ornstein–Uhlenbeck processes with nonnegative step size Fort and Roberts [17, Lemma 18] proved polynomial rate of convergence, while in the same setup (under stronger moment conditions) Douc, Fort, and Guillin [15, Proposition 5.7] showed more general subexponential convergence rates. Theorems 2 and 3 are generalizations of their results.

In Theorem 3.3 in [37] Sandrić gave polynomial convergence rate for Lévy-type processes. However, this result cannot be applied to our process V; see at the end of subsection 5.1.

5.6 Diffusion

Finally, we spell out some of the results in the continuous case, i.e. when the process V is a diffusion, which is much easier to handle. More importantly, in this case there is a necessary and sufficient condition for recurrence.

Let (U, L) be a bivariate Brownian motion with drift γ_Z and covariance matrix Σ_Z. The infinitesimal generator of the process V is

$$\mathcal{A}f(x) = (\gamma_U x + \gamma_L) f'(x) + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{UL} + \sigma_L^2) f''(x).$$

The main difference compared to the general case is that this operator is a local operator, therefore the domain of the extended generator automatically contains all C^2 functions. (See (2.8) and Proposition 3.)

The following result follows easily from Example 3.10 in Khasminskii [19, p. 95].

Corollary 2. *The process V is recurrent if and only if $\gamma_U \leq \sigma_U^2/2$.*

For terminology and more results in this direction we refer to [19].

6 Proofs

First we show that the domain of the extended generator is large enough, and contains usual norm-like functions, which are not bounded. In subsection 6.2 after some preliminary technical lemmas we prove that the various drift conditions hold. Subsection 6.3 contains the proof of the sufficient condition for the petiteness assumption.

6.1 Extended generator and infinitesimal generator

In the following $(\mathcal{F}_t)_{t\geq 0}$ stands for the natural filtration induced by the bivariate Lévy process (U, L). Martingales are meant to be martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Proposition 3. *Assume that $f \in C^2$, and for each fixed $n \in \mathbb{N}$,*

$$\sup_{|x|\leq n} \iint_{|x+xz_1+z_2|>m} (1 + |f(x + xz_1 + z_2)|)\, \nu_{UL}(\mathrm{d}\mathbf{z}) =: \eta_m^n < \infty, \tag{6.1}$$

and $\lim_{m\to\infty} \eta_m^n = 0$. Then $f \in \mathcal{DE}(V^n)$, $n \in \mathbb{N}$, and $f \in \mathcal{DE}(V)$.

We mention that the same method was used earlier by Masuda [27]. He defined the truncation $\widehat{V}_t^n = V_t$ for $t < T_n$, and $\widehat{V}_t^n = \Delta_n$ for $t \geq T_n$, where $|\Delta_n| = n$ arbitrary. Lemma 3.7 in [27] wrongly states that $\mathcal{A}_n$ is the extended generator of $\widehat{V}^n$, as it can be seen in a simple Poisson process example. In order to get a process, which has extended generator $\mathcal{A}_n$, one has to consider the stopped process in (3.1). However, some technical difficulties arise, since this process is not bounded in general. We also note that Masuda's Lemma 3.7 can be amended along the lines of Proposition 3.

We use this proposition for norm-like functions f, which for $|x|$ large enough equals to $(\log|x|)^\alpha$, $\alpha \geq 1$, $\exp\{\gamma(\log x)^\alpha\}$, $\gamma > 0, \alpha \in (0, 1)$, or $|x|^\beta$, $\beta \in (0, 1]$. For these 'nice' functions (6.1) is satisfied when $\iint_{|\mathbf{z}|\geq 1} f(|\mathbf{z}|)\nu_{UL}(\mathrm{d}\mathbf{z}) < \infty$.

Proof. First let $g \in \mathcal{DI}(V)$. It is well-known that

$$M(t) = g(V_t) - g(x_0) - \int_0^t \mathcal{A}g(V_s)\mathrm{d}s, \quad t \ge 0,$$

is martingale. Consider the stopping time $T_n = \inf\{t \ge 0 : |V_t| \ge n\}$, then by (3.1)

$$\begin{aligned} M(t \wedge T_n) &= g(V_{t\wedge T_n}) - g(x_0) - \int_0^{t\wedge T_n} \mathcal{A}g(V_s)\mathrm{d}s \\ &= g(V_t^n) - g(x_0) - \int_0^{t\wedge T_n} \mathcal{A}_n g(V_s)\mathrm{d}s \\ &= g(V_t^n) - g(x_0) - \int_0^{t} \mathcal{A}_n g(V_s^n)\mathrm{d}s, \end{aligned} \tag{6.2}$$

where we used that $|V_{s-}| < n$ if $s \le T_n$. Since $M(t \wedge T_n)$ is a martingale, we have proved that $\mathcal{DI}(V) \subset \mathcal{DE}(V^n)$.

Now we handle the general case. We may and do assume that f is nonnegative. Consider a sequence of nonnegative functions $\{g_m\} \subset \mathcal{DI}(V)$ with the following properties: $g_m(x) \equiv f(x)$ for $|x| \le m$, and $\equiv 0$ for $|x| \ge m+1$, $\max_{x\in\mathbb{R}} g_m(x) \le \sup_{x\in[-m,m]} f(x) + 1$, and $g_m \le g_{m+1}$. Define the martingales

$$M_m(t) = g_m(V_t^n) - g_m(x_0) - \int_0^t \mathcal{A}_n g_m(V_s^n)\mathrm{d}s.$$

Let $\ell \ge m$ and to ease the notation put $h(x) = g_\ell(x) - g_m(x)$. Since $h(x) \equiv 0$ for $|x| \le m$ and for $|x| \ge \ell + 1$ we have for $|x| < n < m$

$$\begin{aligned} \mathcal{A}_n h(x) &= (x\gamma_U + \gamma_L)h'(x) + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{UL} + \sigma_L^2)h''(x) \\ &\quad + \iint_{\mathbb{R}^2} \left[h(x + xz_1 + z_2) - h(x) - h'(x)(xz_1 + z_2)I(|\mathbf{z}| \le 1)\right] \nu_{UL}(\mathrm{d}\mathbf{z}) \\ &= \iint_{\mathbb{R}^2} h(x + xz_1 + z_2)\nu_{UL}(\mathrm{d}\mathbf{z}). \end{aligned}$$

Thus for all $|x| < n$

$$|\mathcal{A}_n h(x)| \le \eta_m = \eta_m^n,$$

therefore we have

$$\left| \int_0^t \mathcal{A}_n[g_\ell(V_s^n) - g_m(V_s^n)]\mathrm{d}s \right| \le t\eta_m. \tag{6.3}$$

Using $g_\ell \ge g_m$ and the martingale property for $m \ge |x_0|$

$$\begin{aligned} \mathbf{E}|g_\ell(V_t^n) - g_m(V_t^n)| &= \mathbf{E}[g_\ell(V_t^n) - g_m(V_t^n)] \\ &= \mathbf{E}\left(M_\ell(t) - M_m(t) + g_\ell(x_0) - g_m(x_0) + \int_0^t \mathcal{A}_n[g_\ell(V_s^n) - g_m(V_s^n)]\mathrm{d}s \right) \\ &= \mathbf{E}\int_0^t \mathcal{A}_n[g_\ell(V_s^n) - g_m(V_s^n)]\mathrm{d}s, \end{aligned}$$

thus

$$\mathbf{E}|g_\ell(V_t^n) - g_m(V_t^n)| \le t\eta_m.$$

Letting $\ell \to \infty$ Fatou's lemma gives

$$\mathbf{E}[f(V_t^n) - g_m(V_t^n)] \le t\eta_m.$$

Moreover, as in (6.3)

$$\left| \int_0^t \mathcal{A}_n[f(V_s^n) - g_m(V_s^n)]\mathrm{d}s \right| \le t\eta_m.$$

Thus we obtain for each $t \ge 0$ as $m \to \infty$

$$M_m(t) \to f(V_t^n) - f(x_0) - \int_0^t \mathcal{A}_n f(V_s^n)\mathrm{d}s =: M(t) \quad \text{in } L^1.$$

Since M_m is martingale, we have for each $0 \le u < t$

$$\begin{aligned}
&\mathbf{E}\,|M(u) - \mathbf{E}[M(t)|\mathcal{F}_u]| \\
&\le \mathbf{E}|M(u) - M_m(u)| + \mathbf{E}\,|M_m(u) - \mathbf{E}[M_m(t)|\mathcal{F}_u]| + \mathbf{E}\,|\mathbf{E}[M_m(t) - M(t)|\mathcal{F}_u]| \\
&\le 2u\eta_m + \mathbf{E}|M_m(t) - M(t)| \le 4t\eta_m \to 0,
\end{aligned}$$

as $m \to \infty$. Thus $\mathbf{E}[M(t)|\mathcal{F}_u] = M(u)$ a.s., i.e. M is a martingale, and $f \in \mathcal{DE}(V^n)$.

Finally, (6.2) shows that $f(V_t) - f(x_0) - \int_0^t \mathcal{A}f(V_s)\mathrm{d}s$ is a local martingale with localizing sequence T_n. □

6.2 Drift conditions

We frequently use the following technical lemma. Recall the definition ν' from (3.6).

Lemma 1. *Let f be an even norm-like C^2 function, for which there exists $k_f > 0$ such that $f(x + y) \le k_f + f(x) + f(y)$, $x, y \in \mathbb{R}$, $\lim_{x\to\infty} x \sup_{|y|\ge x} |f''(y)| = 0$, $\lim_{x\to\infty} f'(x) = 0$, and $\iint_{\mathbb{R}^2} f(|\mathbf{z}|)\nu_{UL}(\mathrm{d}\mathbf{z}) < \infty$. Assume that $\nu_U(\{-1\}) = 0$. Then*

$$\begin{aligned}
&\iint_{\mathbb{R}^2} [f(x + xz_1 + z_2) - f(x) - f'(x)(xz_1 + z_2)I(|\mathbf{z}| \le 1)]\nu_{UL}(\mathrm{d}\mathbf{z}) \\
&= \int_{\mathbb{R}} [f(x + xz) - f(x) - f'(x)xzI(|z| \le 1)]\nu_U(\mathrm{d}z) + f'(x)x \int_{-1}^1 z\nu'(\mathrm{d}z) + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $|x| \to \infty$. Moreover, the same holds with $O(1)$ when $f'(x)$ is bounded.

Proof. We may write

$$\begin{aligned}
&\iint_{\mathbb{R}^2} [f(x + xz_1 + z_2) - f(x) - f'(x)(xz_1 + z_2)I(|\mathbf{z}| \le 1)]\nu_{UL}(\mathrm{d}\mathbf{z}) \\
&= \iint_{\mathbb{R}^2} [f(x + xz_1 + z_2) - f(x + xz_1) - f'(x)z_2 I(|\mathbf{z}| \le 1)]\nu_{UL}(\mathrm{d}\mathbf{z}) \\
&\quad + \iint_{\mathbb{R}^2} [f(x + xz_1) - f(x) - f'(x)xz_1 I(|\mathbf{z}| \le 1)]\nu_{UL}(\mathrm{d}\mathbf{z}) \\
&=: I_1 + I_2,
\end{aligned}$$

since an application of the mean value theorem implies that both integral above is finite. Indeed, for the integrand in I_1 we have for x large enough

$$\left|f(x+xz_1+z_2)-f(x+xz_1)-f'(x)z_2\right| \le |z_2|(|xz_1|+|z_2|)\max_y |f''(y)|,$$

which implies the integrability. Let $I_1 = I_{11} + I_{12}$, where I_{11} stands for the integral on $\{|\mathbf{z}| \le 1\}$, and I_{12} on $\{|\mathbf{z}| > 1\}$. Let $\delta > 0$ be arbitrary. For $z_1 > -1+\delta$

$$\begin{aligned}&\iint_{|\mathbf{z}|\le 1, z_1>-1+\delta} [f(x+xz_1+z_2)-f(x+xz_1)-f'(x)z_2 I(|\mathbf{z}|\le 1)]\nu_{UL}(\mathrm{d}\mathbf{z})\\ &\le \sup_{|y|\ge\delta|x|-1} |f''(y)| \iint_{|\mathbf{z}|\le 1, z_1>-1+\delta} |z_2|(|xz_1|+|z_2|)\nu_{UL}(\mathrm{d}\mathbf{z}),\end{aligned}$$

which goes to 0, since $x\sup_{|y|\ge x} f''(y) \to 0$. Cutting further the remaining set

$$\begin{aligned}&\left|\iint_{|\mathbf{z}|\le 1, z_1<-1+\delta} [f(x+xz_1+z_2)-f(x+xz_1)-f'(x)z_2 I(|\mathbf{z}|\le 1)]\nu_{UL}(\mathrm{d}\mathbf{z})\right|\\ &\le |f'(x)| \iint_{|\mathbf{z}|\le 1, z_1<-1+\delta} |z_2|\nu_{UL}(\mathrm{d}\mathbf{z}) + \max_{y\in\mathbb{R}} |f'(y)| \iint_{|\mathbf{z}|\le 1, z_1<-1+\delta} |z_2|\nu_{UL}(\mathrm{d}\mathbf{z}).\end{aligned}$$

Since $\delta > 0$ can be arbitrarily small we have that $I_{11} = o(1)$.

To handle I_{12} fix $\varepsilon > 0$. There is an $R > 0$ such that $\iint_{|\mathbf{z}|>R}[k_f + f(z_2)]\nu_{UL}(\mathrm{d}\mathbf{z}) < \varepsilon$, and so

$$\left|\iint_{|\mathbf{z}|>R} [f(x+xz_1+z_2)-f(x+xz_1)]\nu_{UL}(\mathrm{d}\mathbf{z})\right| \le \iint_{|\mathbf{z}|>R} [k_f + f(z_2)]\nu_{UL}(\mathrm{d}\mathbf{z}) < \varepsilon.$$

Let us choose $\delta = \delta(R)$ so small that

$$\iint_{1<|\mathbf{z}|\le R, |z_1+1|\le\delta} |z_2|\nu_{UL}(\mathrm{d}\mathbf{z}) < \varepsilon.$$

This is possible, since $\nu_U(\{-1\}) = 0$. Then we have

$$\begin{aligned}|I_{12}| &\le \max_{|y|>\delta|x|-R} |f'(y)| \iint_{1<|\mathbf{z}|\le R, |z_1+1|>\delta} |z_2|\nu_{UL}(\mathrm{d}\mathbf{z}) + \max_{y\in\mathbb{R}} |f'(y)| \iint_{1<|\mathbf{z}|\le R, |z_1+1|\le\delta} |z_2|\nu_{UL}(\mathrm{d}\mathbf{z})\\ &\quad + \iint_{|\mathbf{z}|>R} [k_f + f(z_2)]\nu_{UL}(\mathrm{d}\mathbf{z})\\ &\le \max_{|y|>\delta|x|-R} |f'(y)| \iint_{1<|\mathbf{z}|\le R, |z_1+1|>\delta} |z_2|\nu_{UL}(\mathrm{d}\mathbf{z}) + \varepsilon(1 + \max_{y\in\mathbb{R}} |f'(y)|),\end{aligned}$$

which proves that $I_1 = o(1)$, if $f'(x) \to 0$. We also see that if f' is only bounded then $I_1 = O(1)$.

We turn to I_2. Note that in the integrand in I_2 only the indicator depends on z_2. Put $\nu^1(A) = \nu_{UL}((A \times \mathbb{R}) \cap \{|\mathbf{z}| \le 1\})$, $\nu^2(A) = \nu_{UL}((A \times \mathbb{R}) \cap \{|\mathbf{z}| > 1\})$. Then ν^2 is a finite measure on $\mathbb{R}$ and $\nu^2|_{\{|z|>1\}} \equiv \nu_U|_{\{|z|>1\}}$, $\nu^2 = \nu' + \nu_U|_{\{|z|>1\}}$, and $\nu_1 + \nu' = \nu_U|_{\{|z|\leq 1\}}$. Thus

$$\begin{aligned} I_2 &= \int_{-1}^{1} [f(x+xz) - f(x) - f'(x)xz]\nu^1(\mathrm{d}z) + \int_{\mathbb{R}} [f(x+xz) - f(x)]\nu^2(\mathrm{d}z) \\ &= \int_{\mathbb{R}} [f(x+xz) - f(x) - f'(x)xzI(|z| \leq 1)]\nu_U(\mathrm{d}z) + f'(x)x \int_{-1}^{1} z\nu'(\mathrm{d}z), \end{aligned}$$

and the statement is proven. □

The following simple statement shows that if $f(x)$ is concave for large x, then $f(x+y) \leq k_f + f(x) + f(y)$, for some $k_f > 0$. That is, whenever the integrability condition holds, our norm-like functions ($(\log|x|)^\alpha$, $\alpha \geq 1$; $\exp\{\gamma(\log x)^\alpha\}$, $\gamma > 0$, $\alpha \in (0,1)$; $|x|^\beta$, $\beta \in (0,1]$) satisfy the conditions of Lemma 1 and Proposition 3. The lemma follows from simple properties of concave functions. We omit the proof.

Lemma 2. *Assume that $f : [0,\infty) \to [0,\infty)$ is concave on the interval $[x_0,\infty)$ for some $x_0 > 0$. Then there is a $k_f > 0$ such that $f(x+y) \leq k_f + f(x) + f(y)$ for all $x, y \geq 0$.*

Proof of Proposition 1 (i). Let $f(x) = \log|x|$ for $|x| \geq e$, and consider a smooth nonnegative extension of it to $[-e,e]$. Since $\log x$ is concave f satisfies the assumptions of Lemma 1.

We have

$$\int_{|z+1|>e/|x|} [f(x+xz) - f(x) - f'(x)xzI(|z| \leq 1)]\nu_U(\mathrm{d}z) = \int_{|z+1|>e/|x|} [\log|1+z| - zI(|z| \leq 1)]\nu_U(\mathrm{d}z),$$

and

$$\left| \int_{-1-e/|x|}^{-1+e/|x|} [f(x+xz) - f(x) - f'(x)xzI(|z| \leq 1)]\nu_U(\mathrm{d}z) \right| \leq (\log x + 2)\nu_U([-1-e/|x|, -1+e/|x|]).$$

The integrability condition $\int_{-3/2}^{-1/2} |\log|1+z||\,\nu_U(\mathrm{d}z) < \infty$ implies that the latter bound tends to 0 as $|x| \to \infty$. Therefore

$$\int_{\mathbb{R}} [f(x+xz) - f(x) - f'(x)xzI(|z| \leq 1)]\nu_U(\mathrm{d}z) = \int_{\mathbb{R}} [\log|1+z| - zI(|z| \leq 1)]\nu_U(\mathrm{d}z) + o(1).$$

Combining this with Lemma 1 we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^2} [f(x+xz_1+z_2) - f(x) - f'(x)(xz_1+z_2)I(|\mathbf{z}| \leq 1)]\nu_{UL}(\mathrm{d}\mathbf{z}) \\ &= \int_{\mathbb{R}} [\log|1+z| - zI(|z| \leq 1)]\nu_U(\mathrm{d}z) + \int_{-1}^{1} z\nu'(\mathrm{d}z) + o(1). \end{aligned}$$

Therefore

$$\mathcal{A}_n f(x) = \gamma_U - \frac{\sigma_U^2}{2} + \int_{\mathbb{R}} [\log|1+z| - zI(|z| \leq 1)]\nu_U(\mathrm{d}z) + \int_{-1}^{1} z\nu'(\mathrm{d}z) + o(1),$$

so (3.3) holds. □

Before the proofs of (ii)-(iii) in Proposition 1 we show a more general statement.

Proposition 4. *Assume that f satisfies the conditions in Lemma 1, $g(x) = xf'(x)$ is positive, slowly varying at infinity, increases for $x > x_0 > 0$, there is a $k_g > 0$ such that $g(xu) \leq k_g g(x)g(u)$ for all $u, x \geq 1$, $xf''(x) \sim -f'(x)$, and there is a concave function φ such that $\varphi(f(x)) = g(x)$ for x large enough. Furthermore, assume that $\int_{|z|>1} g(z) \log|z| \, \nu_U(\mathrm{d}z) < \infty$, $\int_{-3/2}^{-1/2} |1+z|^{-\varepsilon} \, \nu_U(\mathrm{d}z) < \infty$ for some $\varepsilon > 0$, and (3.8) holds. Then (3.5) holds.*

We note that whenever $g(x) = xf'(x)$ is slowly varying, $f(x)$ is also slowly varying, and $f(x)/g(x) \to \infty$; see [9, Proposition 1.5.9a]. This is the reason why we have to assume some integrability conditions around -1. In particular, when $\nu_U(\{-1\}) > 0$ the situation is completely different. Moreover, the same argument shows that if $xg'(x)$ is slowly varying, then $g(x)$ is slowly varying, and $g(x)/(xg'(x)) \to \infty$, which implies that $xf''(x) \sim -f'(x)$.

Proof. Since g is slowly varying, for any $u > 0$ we have after a change of variables

$$\frac{f(xu) - f(x)}{xf'(x)} = \int_1^u \frac{g(xy)}{g(x)} y^{-1} \mathrm{d}y \to \log u, \tag{6.4}$$

where we used the uniform convergence theorem [9, Theorem 1.2.1]. That is, under some general conditions on f the limit above does not depend on f. This convergence is the reason why the drift condition in Theorems 1, 2, and 3 is the same.

Assuming for a moment that the interchangeability of the limit and the integral is justified, we have

$$\int_{\mathbb{R}} [f(x + xz) - f(x) - f'(x)xzI(|z| \leq 1)]\nu_U(\mathrm{d}z) \sim xf'(x) \int_{\mathbb{R}} [\log|1+z| - zI(|z| \leq 1)]\nu_U(\mathrm{d}z).$$

Thus, using also that $x^2 f''(x) \sim -xf'(x)$,

$$\mathcal{A}f(x) \sim xf'(x) \left[\gamma_U - \frac{\sigma_U^2}{2} + \int_{\mathbb{R}} [\log|1+z| - zI(|z| \leq 1)] \, \nu_U(\mathrm{d}z) + \int_{-1}^1 z\nu'(\mathrm{d}z) \right].$$

Since $\varphi(f(x)) = xf'(x)$, the statement follows.

So we only have to find a integrable majorant around infinity, around 0, where the measure may be infinity, and around -1, where $\log|1+z|$ has a singularity.

At infinity: Using the monotonicity of g, and $g(xu) \leq k_g g(x)g(u)$, $u \geq 1$, from (6.4) we obtain

$$\frac{f(xu) - f(x)}{xf'(x)} \leq k_g g(u) \log u,$$

which is integrable.

At 0: By the mean value theorem we have $f(x(1+z)) - f(x) = xzf'(\xi)$ with ξ between x and $x(1+z)$, and $|f'(\xi) - f'(x)| = |(\xi - x)f''(\xi')| \leq |xzf''(\xi')|$, with ξ' between x and $x(1+z)$. Therefore

$$\left| \frac{f(x(1+z)) - f(x)}{xf'(x)} - z \right| = \left| z \left(\frac{f'(\xi)}{f'(x)} - 1 \right) \right| \leq z^2 \frac{|xf''(\xi')|}{f'(x)},$$

and since $xf''(x) \sim -f'(x)$, and $xf'(x)$ is slowly varying $|xf''(\xi')|/f'(x)$ is uniformly bounded for $z \in [-1/2, 1/2]$.

At -1: Using the Potter bounds [9, Theorem 1.5.6], for any $\varepsilon > 0$ there is a $c = c(\varepsilon) > 0$ such that $g(xy)/g(x) \leq 2y^{-\varepsilon}$, $c/|x| \leq y \leq 1$. Thus for $c/|x| \leq |1+z| \leq 1$ by (6.4)

$$\left| \frac{f(x(1+z)) - f(x)}{xf'(x)} \right| = \left| \int_{|1+z|}^{1} \frac{g(xy)}{g(x)} y^{-1} \mathrm{d}y \right| \leq 2 \int_{|1+z|}^{1} y^{-1-\varepsilon} \mathrm{d}y \leq \frac{2}{\varepsilon} |1+z|^{-\varepsilon},$$

which is integrable for some ε with respect to ν_U, according to the assumptions. Finally,

$$\left| \int_{-1-c/|x|}^{-1+c/|x|} \left[\frac{f(x(1+z)) - f(x)}{g(x)} - zI(|z| \leq 1) \right] \nu_U(\mathrm{d}z) \right| \leq \left(\frac{f(x)}{g(x)} + 1 \right) \nu_U(-1 - c/|x|, -1 + c/|x|),$$

and since $f(x)/g(x)$ is slowly varying the latter bound tends to 0 due to the integrability assumption. □

Proof of Proposition 1 (ii). Let $f(x) = (\log|x|)^\alpha$ for $|x| \geq 3$, and consider a smooth extension of it to $[-3, 3]$, which is greater than, or equal to 1. We show that f satisfies the conditions of Proposition 4, except the integrability condition at -1. Since f is concave for $|x| \geq 3$ so it satisfies the assumptions of Lemma 1. Moreover,

$$g(x) = xf'(x) = \alpha(\log|x|)^{\alpha-1}$$

is increasing and slowly varying, and $\log g(e^x)$ is concave, which, combined with Lemma 2 implies that $g(ux) \leq k_g g(u) g(x)$, $u, x \geq 1$, for some $k_g > 0$. Simple calculation shows that $xf''(x) \sim -f'(x)$. The function $\varphi(y) = \alpha y^{1-1/\alpha}$ is concave, and $\varphi(f(x)) = g(x)$. Finally, as $g(x) \log x = \alpha(\log x)^\alpha$ the integrability condition at infinity is also satisfied.

Therefore, we only have to show that the integrability condition at -1 can be relaxed. Note that $(1-y)^\alpha \geq 1 - \alpha y$ for $y \in [0, 1]$. Thus, with $u = |1+z| \in [3/|x|, 1]$

$$\left| \frac{f(xu) - f(x)}{xf'(x)} \right| = \alpha^{-1} \log|x| \left[1 - \left(1 - \frac{\log u^{-1}}{\log|x|} \right)^\alpha \right] \leq \log u^{-1}.$$

While, for $u \leq 3/|x|$

$$\begin{aligned} &\int_{|1+z| \leq 3/|x|} \left| \frac{f(x(1+z)) - f(x)}{xf'(x)} - zI(|z| \leq 1) \right| \nu_U(\mathrm{d}z) \\ &\leq (\alpha^{-1} \log|x| + 1)\, \nu_U((-1 - 3/|x|, -1 + 3/|x|)) \to 0. \end{aligned}$$

At the last step we used that the integrability condition $\int_{-3/2}^{-1/2} |\log|1+z|| \nu_U(\mathrm{d}z) < \infty$ implies $\log|x|\, \nu_U((-1 - 1/|x|, -1 + 1/|x|)) \to 0$. □

Proof of Proposition 1 (iii). Let $f(x) = \exp\{\gamma(\log|x|)^\alpha\}$, $\alpha \in (0, 1)$, $\gamma > 0$, for $|x| \geq e$, and consider a smooth extension of it to $[-e, e]$, which is greater than, or equal to 1. First, we show that the function f satisfies the conditions of Proposition 4, except the integrability condition at infinity, and at -1.

Simply,

$$g(x) = xf'(x) = \gamma\alpha(\log|x|)^{\alpha-1}\exp\{\gamma(\log|x|)^{\alpha}\},$$

which is an increasing, slowly varying function on (e,∞). Moreover, $\log g(e^x)$ is concave, for x large enough, therefore Lemma 2 implies that $g(ux) \le k_g g(u)g(x)$ for some $k_g > 0$. Straightforward calculation shows that $xf''(x) \sim -f'(x)$. Finally, $\varphi(f(x)) = g(x)$ for the concave function $\varphi(x) = \alpha\gamma^{1/\alpha}x\,(\log x)^{1-1/\alpha}$.

Now we prove that the integrability condition at infinity and at -1 can be relaxed. We start with the condition at infinity. Let $u = |z+1| \ge 1$, and write

$$I(u) = \frac{f(xu)-f(x)}{xf'(x)} = \frac{1}{\gamma\alpha}(\log|x|)^{1-\alpha}\left[\exp\left\{\gamma(\log|x|u)^{\alpha} - \gamma(\log|x|)^{\alpha}\right\} - 1\right].$$

For $\log u \le \log|x|$, using $(1+y)^{\alpha} \le 1+\alpha y$, $y \ge 0$, we have

$$(\log|x|u)^{\alpha} - (\log|x|)^{\alpha} = (\log|x|)^{\alpha}\left[\left(1+\frac{\log u}{\log|x|}\right)^{\alpha} - 1\right] \le \alpha(\log|x|)^{\alpha-1}\log u.$$

If $\gamma\alpha\log u\,(\log|x|)^{\alpha-1} \le 1$, then using $e^y - 1 \le 2y$ for $y \in [0,1]$ we have

$$I(u) \le 2\log u,$$

which is integrable. While, for $\gamma\alpha\log u\,(\log|x|)^{\alpha-1} \ge 1$, noting that $(\log|x|)^{\alpha-1}\log u \le (\log u)^{\alpha}$, we obtain

$$I(u) \le \frac{1}{\gamma\alpha}(\log|x|)^{1-\alpha}e^{\gamma\alpha(\log u)^{\alpha}} \le \log u\, e^{\gamma\alpha(\log u)^{\alpha}} \le c_1 e^{\gamma(\log u)^{\alpha}},$$

with $c_1 > 0$, which is again integrable according to our assumptions. Here, $c_1, c_2, \ldots$ are strictly positive constants, whose value are not important.

For $\log u \ge \log|x|$ we use the inequality $(1+y)^{\alpha} - y^{\alpha} \le 1 - y^{\alpha}/2$, which holds for $y \in (0,\delta)$, for some $\delta > 0$ ($\delta(\alpha) = 1/(2^{1/(1-\alpha)}-1)$ works). If $\log|x| \le \delta\log u$, then

$$I(u) \le \frac{1}{\gamma\alpha}(\log|x|)^{1-\alpha}\exp\left\{\gamma(\log u)^{\alpha} - \frac{\gamma}{2}(\log|x|)^{\alpha}\right\} \le c_2\exp\{\gamma(\log u)^{\alpha}\},$$

with some $c_2 > 0$, which is integrable. Otherwise, for $\log|x| \ge \delta\log u$

$$\begin{aligned} I(u) &= \frac{1}{\gamma\alpha}(\log|x|)^{1-\alpha}\exp\left\{\gamma(\log u)^{\alpha}\left[\left(1+\frac{\log|x|}{\log u}\right)^{\alpha} - \left(\frac{\log|x|}{\log u}\right)^{\alpha}\right]\right\} \\ &\le \frac{1}{\gamma\alpha}(\log|x|)^{1-\alpha}\exp\left\{\gamma(\log u)^{\alpha}\eta\right\} \\ &\le c_3\exp\{\gamma(\log u)^{\alpha}\}, \end{aligned}$$

with $\eta = \sup_{y\in[\delta,1]}[(1+y)^{\alpha} - y^{\alpha}] < 1$, and some $c_3 > 0$.

We turn to the integrability condition at -1. Now $u = |1+z| \in [0,1]$. The integral condition $\int_{-3/2}^{-1/2}|\log|1+z||\,\nu_U(\mathrm{d}z) < \infty$ implies $(\log|x|)^{1-\alpha}\nu_U((-1-1/|x|, -1+1/|x|)) \to 0$, and so for any $\varepsilon > 0$ it also holds that $(\log|x|)^{1-\alpha}\nu_U(-1-|x|^{-\varepsilon}, -1+|x|^{-\varepsilon}) \to 0$. Therefore, we may and do assume that $|x|^{-\varepsilon} \leq u \leq 1$. Then $\log u / \log |x| \in [-\varepsilon, 0]$. Let $\varepsilon > 0$ be small enough such that $(1-y)^\alpha \geq 1 - 2\alpha y$, for $y \in [0, \varepsilon]$ ($\varepsilon(\alpha) = 1 - 2^{-1/(1-\alpha)}$ works). Then

$$\begin{aligned} I(u) &= \frac{1}{\gamma\alpha}(\log |x|)^{1-\alpha} \left[1 - \exp\left\{ -\gamma(\log|x|)^\alpha \left[1 - \left(1 + \frac{\log u}{\log |x|} \right)^\alpha \right] \right\} \right] \\ &\leq \frac{1}{\gamma\alpha}(\log |x|)^{1-\alpha} \left[1 - \exp\left\{ 2\gamma\alpha(\log|x|)^{\alpha-1} \log u \right\} \right] \\ &\leq -2 \log u, \end{aligned}$$

where at the last step we used the simple inequality $1 - e^u \leq -u$. Since $\log |1+z|$ is integrable around -1, the statement is proved. □

Proof of Proposition 1 (iv). Let $f(x) = |x|^\beta$ for $|x| \geq 1$, and consider an even, smooth nonnegative extension of it to $[-1, 1]$. Since x^β is concave f satisfies the assumptions of Lemma 1. For $\beta < 1$ we have $\lim_{|x|\to\infty} f'(x) = 0$, while $f'(x)$ is bounded for $\beta = 1$.

We have

$$\int_{-1+x^{-1}}^1 [f(x+xz) - f(x) - f'(x)xz] \nu_U(\mathrm{d}z) = x^\beta \int_{-1+x^{-1}}^1 [(1+z)^\beta - 1 - \beta z] \nu_U(\mathrm{d}z),$$

and

$$\left| \int_{-1}^{-1+x^{-1}} [f(x+xz) - f(x) - f'(x)xz] \nu_U(\mathrm{d}z) \right| \leq 3x^\beta \nu_U([-1, -1+x^{-1}]) = o(x^\beta),$$

since $\nu_U(\{-1\}) = 0$. Therefore

$$\int_{-1}^1 [f(x+xz) - f(x) - f'(x)xz] \nu_U(\mathrm{d}z) = x^\beta \int_{-1}^1 [(1+z)^\beta - 1 - \beta z] \nu_U(\mathrm{d}z) + o(x^\beta).$$

Using the same argument as above (now the exceptional set is $[-1-x^{-1}, -1]$), we obtain

$$\int_{|z|>1} [f(x+xz) - f(x)] \nu_U(\mathrm{d}z) = x^\beta \int_{|z|>1} [|1+z|^\beta - 1] \nu_U(\mathrm{d}z) + o(x^\beta).$$

Summarizing, we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^2} [f(x + xz_1 + z_2) - f(x) - f'(x)(xz_1 + z_2) I(|z| \leq 1)] \nu_{UL}(\mathrm{d}z) \\ &= x^\beta \left[\int_{\mathbb{R}} [|1+z|^\beta - 1 - z\beta I(|z| \leq 1)] \nu_U(\mathrm{d}z) + \beta \int_{-1}^1 z\nu'(\mathrm{d}z) \right] + o(|x|^\beta). \end{aligned} \tag{6.5}$$

Now the statement follows. According to (6.5), as $x \to \infty$

$$\mathcal{A}_n f(x) \sim \beta x^\beta \left[\gamma_U - \frac{\sigma_U^2(1-\beta)}{2} + \int_{\mathbb{R}} \frac{|1+z|^\beta - 1 - z\beta I(|z| \leq 1)}{\beta} \nu_U(\mathrm{d}z) + \int_{-1}^1 z\nu'(\mathrm{d}z) \right].$$

Since the expression in the square bracket is negative, we obtain (3.4). □

6.3 Petite sets

Proof of Proposition 2. Since $\lim_{n\to\infty} \mathbf{P}_x\{V_n \in A\} = \pi(A)$ for any $x \in \mathbb{R}$, we see that the skeleton process $(V_n)_{n\in\mathbb{N}}$ is irreducible with respect to the invariant measure π. According to our assumptions the interior of the support is nonempty, therefore Theorem 3.4 in [28] implies that the compact sets are petite sets. $\square$

Acknowledgement.
I thank Mátyás Barczy for useful comments on the manuscript, Anita Behme for suggesting me the problem, and the anonymous referee, whose comments helped to improve the presentation of the paper. This research was funded by a postdoctoral fellowship of the Alexander von Humboldt Foundation.

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