Convergence to Stable Limits for Ratios of Trimmed Lévy Processes and their Jumps

Yuguang F. Ipsen, Péter Kevei* and Ross A. Maller†

Research School of Finance, Actuarial Studies & Statistics
Australian National University
Canberra, ACT, 0200, Australia
and
MTA-SZTE Analysis and Stochastics Research Group
Bolyai Institute, Aradi vértanúk tere 1
6720 Szeged, Hungary

February 22, 2018

Abstract

We derive characteristic function identities for conditional distributions of an r-trimmed Lévy process given its r largest jumps up to a designated time t. Assuming the underlying Lévy process is in the domain of attraction of a stable process as $t \downarrow 0$, these identities are applied to show joint convergence of the trimmed process divided by its large jumps to corresponding quantities constructed from a stable limiting process. This generalises related results in the 1-dimensional subordinator case developed in Kevei & Mason (2014) and produces new discrete distributions on the infinite simplex in the limit.

1 Introduction and Lévy Process Setup

Deleting the r largest jumps up to a designated time t from a Lévy process gives the "r-trimmed Lévy process". We derive useful characteristic function identities for conditional distributions of the process given some of its largest jumps. As

*Research supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, and by NKFIH grant FK124141.

†Research partially supported by ARC Grant DP1092502.

Email: Yuguang.Ipsen@anu.edu.au; kevei@math.u-szeged.hu; Ross.Maller@anu.edu.au

corollaries, representations for the characteristic functions of the trimmed process divided by its large jumps are found. Assuming X is in the domain of attraction of a stable process as $t \downarrow 0$, the representations are applied to show joint convergence of those ratios to corresponding quantities constructed from the stable limiting process.

In the case of subordinators, Kevei & Mason (2014) considered one-dimensional convergence to stable subordinators and derived the limit distribution of the ratio of an r-trimmed subordinator to its r^{th} largest jump occurring up till a specified time $t > 0$, as $t \downarrow 0$ or $t \to \infty$. Perman (1993), also considering subordinators, derived exact expressions for the joint density of the ratios of the first r largest jumps up till time $t = 1$ of a subordinator, taken as ratios of the value of the subordinator itself at time 1. In Perman's case the canonical measure of the subordinator was assumed to have a density with respect to Lebesgue measure. His results, when applied to a Gamma subordinator, produce formulae for the Poisson-Dirichlet process.

For our asymptotic results we allow a general Lévy measure, making no continuity assumptions on it. Our main result, Theorem 2.1, is a multivariate version of part of Theorem 1.1 of Kevei & Mason (2014), and, as a generalisation, we consider a trimmed Lévy process in the domain of attraction of a stable distribution with parameter α in $(0, 2)$, taken as a ratio of one of its large jumps at time t. We show the joint convergence of these ratios to corresponding quantities constructed from the stable limiting process, as time t tends to 0.

When $0 < \alpha < 1$, the limit distribution in Theorem 2.1 is related to the generalised Poisson-Dirichlet distribution $\mathrm{PD}_\alpha^{(r)}$ in Ipsen & Maller (2016) derived from the trimmed stable subordinator, which includes as a special case the $\mathrm{PD}(\alpha, 0)$ distribution in Pitman & Yor (1997). When $\alpha > 1$ the process is not a subordinator, and there is no direct connection with the Poisson-Dirichlet distribution. In this case the process has to be centered appropriately to get the required convergence. We note that (since the Lévy measure has infinite mass) there are always infinitely many "large" jumps of X_t, a.s., in any right neighbourhood of 0.

These considerations form the basis of further generalised versions of Poisson-Dirichlet distributions explored in Ipsen & Maller (2016). In the present paper we limit ourselves to proving Theorem 2.1 (in Section 2) and the foundational results needed for its proof (in Section 3). A second Theorem 2.2 proves a kind of "large trimming" result, showing that the trimmed process is of small order of the largest jump trimmed, uniformly in t, as the order tends to infinity. Section 4 contains the proofs of the results in Section 2. For the remainder of this section we give a brief introduction to the Lévy process ideas we will need.

1.1 Lévy Process Setup

We consider a real valued Lévy process $(X_t)_{t\geq 0}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$, with canonical triplet (γ, σ^2, Π); thus, having characteristic function $\mathrm{E}e^{\mathrm{i}\theta X_t} = e^{t\Psi(\theta)}$, $t \geq 0$, $\theta \in \mathbb{R}$, with exponent

$$\Psi(\theta) := \mathrm{i}\theta\gamma - \tfrac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\leq 1\}}\big)\Pi(\mathrm{d}x). \tag{1.1}$$

Here $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a Lévy measure on $\mathbb{R}$, i.e., a Borel measure on $\mathbb{R}$ with $\int_{\mathbb{R}\setminus\{0\}}(x^2 \wedge 1)\Pi(\mathrm{d}x) < \infty$. The positive, negative and two-sided tails of Π are defined for $x > 0$ by

$$\overline{\Pi}^+(x) := \Pi\{(x,\infty)\},\ \overline{\Pi}^-(x) := \Pi\{(-\infty,-x)\},\ \text{and}\ \overline{\Pi}(x) := \overline{\Pi}^+(x) + \overline{\Pi}^-(x). \tag{1.2}$$

Let $\overline{\Pi}^{+,\leftarrow}$ denote the inverse function of $\overline{\Pi}^+$, defined by

$$\overline{\Pi}^{+,\leftarrow}(x) = \inf\{y > 0 : \overline{\Pi}^+(y) \leq x\},\ x > 0, \tag{1.3}$$

and similarly for $\overline{\Pi}^{\leftarrow}$. Throughout, let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

Write $(\Delta X_t := X_t - X_{t-})_{t>0}$, with $\Delta X_0 = 0$, for the jump process of X, and $\Delta X_t^{(1)} \geq \Delta X_t^{(2)} \geq \cdots$ for the jumps ordered by their magnitudes at time $t > 0$. Assume throughout that $\Pi\{(0,\infty)\} = \infty$, so there are infinitely many positive jumps, a.s., in any right neighbourhood of 0. Thus the $\Delta X_t^{(i)}$ are positive a.s. for all $t > 0$ but $\lim_{t\downarrow 0} \Delta X_t^{(i)} = 0$ for all $i \in \mathbb{N}$. Our objective is to study the "one-sided trimmed process", by which we mean X_t minus its large positive jumps, at a given time t. Thus, the one-sided r-trimmed version of X_t is

$${}^{(r)}X_t := X_t - \sum_{i=1}^{r} \Delta X_t^{(i)},\ r \in \mathbb{N},\ t > 0 \tag{1.4}$$

(and we set ${}^{(0)}X_t \equiv X_t$). Detailed definitions and properties of this kind of ordering and trimming are given in Buchmann, Fan & Maller (2016), where we identify the positive ΔX_t with the points of a Poisson point process on $[0,\infty)$.

Our main result, in Theorem 2.1, is to show that ratios formed by dividing ${}^{(r)}X_t$, possibly after centering, by its ordered positive jumps, converge to the corresponding stable ratios when X is in the domain of attraction of a non-normal stable law.

2 Convergence of Lévy Ratios to Stable Limits

Throughout, X will be assumed to be in the domain of attraction of a non-normal stable random variable at 0 (or at ∞).[1] By this we mean that there are nonstochastic

[1]The convergences in this section can be worked out as $t \downarrow 0$ or as $t \to \infty$. For definiteness and in keeping with modern trends in the area we supply the versions for $t \downarrow 0$, but little modification is needed for the case $t \to \infty$.

functions $a_t \in \mathbb{R}$ and $b_t > 0$ such that $(X_t - a_t)/b_t \xrightarrow{\mathrm{D}} Y$, for an a.s. finite random variable Y, not degenerate at a constant, and not normally distributed, as $t \downarrow 0$. The Lévy tail $\overline{\Pi}(x)$ is then regularly varying of index $-\alpha$ at 0, and the balance conditions

$$\lim_{x \downarrow 0} \frac{\overline{\Pi}^{\pm}(x)}{\overline{\Pi}(x)} = a_{\pm}, \tag{2.1}$$

where $a_+ + a_- = 1$, are satisfied. If this is the case then the limit random variable Y must be a stable random variable with index α in $(0,2)$. We consider one-sided (positive) trimming, so we always assume $a_+ > 0$, and then also $\overline{\Pi}^+(x)$ is regularly varying at 0 with index $-\alpha$, $\alpha \in (0,2)$.

Denote by $RV_0(\alpha)$ ($RV_\infty(\alpha)$) the regularly varying functions of index $\alpha \in \mathbb{R}$ at 0 (or ∞). When $\overline{\Pi}^+(\cdot) \in RV_0(-\alpha)$ with $0 < \alpha < \infty$ or, equivalently, the inverse function $\overline{\Pi}^{+,\leftarrow}(\cdot) \in RV_\infty(-1/\alpha)$ (e.g. Bingham, Goldie and Teugels (1987, Sect. 7, pp.28-29)), we have the easily verified convergence

$$t\overline{\Pi}^+(u\overline{\Pi}^{+,\leftarrow}(y/t)) \sim \frac{\overline{\Pi}^+(uy^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t))}{\overline{\Pi}^+(\overline{\Pi}^{+,\leftarrow}(1/t))} \to u^{-\alpha}y \text{ as } t \downarrow 0, \text{ for all } u, y > 0. \tag{2.2}$$

For $r > 0$ write

$$\mathrm{P}(\Gamma_r \in \mathrm{d}x) = \frac{x^{r-1}e^{-x}\mathrm{d}x}{\Gamma(r)}\mathbf{1}_{\{x>0\}},$$

for the density of Γ_r, a Gamma$(r,1)$ random variable, which should not be confused with the Gamma function, $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}\mathrm{d}x$. Denote the Beta random variable on $(0,1)$ with parameters $a, b > 0$ by $\mathrm{B}_{a,b}$, having density function

$$f_B(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1} = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \ 0 < x < 1.$$

Denote by $(S_t)_{t \geq 0}$ a stable process of index $\alpha \in (0,2)$ having Lévy measure

$$\Lambda(\mathrm{d}x) = \Lambda_S(\mathrm{d}x) = -\mathrm{d}(x^{-\alpha})\mathbf{1}_{\{x>0\}} + (a_-/a_+)\mathrm{d}(-x)^{-\alpha}\mathbf{1}_{\{x<0\}}, \ x \in \mathbb{R}, \tag{2.3}$$

with characteristic exponent

$$\Psi_S(\theta) := \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x| \leq 1\}}\right)\Lambda(\mathrm{d}x), \tag{2.4}$$

and by $(\Delta S_t := S_t - S_{t-})_{t>0}$ the jump process of S. Let

$$\Delta S_t^{(1)} \geq \Delta S_t^{(2)} \geq \cdots \geq \Delta S_t^{(n)} \geq \cdots$$

be the ordered stable jumps at time $t > 0$. These are uniquely defined a.s. (no tied values a.s.) since the Lévy measure of S has no atoms. The positive and negative tails of Λ are $\overline{\Lambda}^+(x) := \Lambda\{(x,\infty)\} = x^{-\alpha}$ and $\overline{\Lambda}^-(x) := \Lambda\{(-\infty,-x)\} =$

$(a_-/a_+)x^{-\alpha}$, for $x>0$. Since $\overline{\Lambda}^+(0+)=\infty$, the $\Delta S_t^{(i)}$ are positive a.s., $i=1,2,\ldots$, but tend to 0 a.s. as $t\downarrow 0$.

Define a centering function $\rho_X(\cdot)$ for X by

$$\rho_X(w) := \begin{cases} \gamma - \displaystyle\int_{[-1,-w)\cup[w,1]} x\Pi(\mathrm{d}x), & 0<w\le 1, \\ \\ \gamma + \int_{[-w,-1)\cup(1,w)} x\Pi(\mathrm{d}x), & w>1, \end{cases} \tag{2.5}$$

and similarly for $\rho_S(w)$, but with γ taken as 0 and Λ replacing Π in that case.

To state Theorem 2.1, we need some further notation. For each $n=2,3,\ldots$ and $0<u<1$, suppose random variables $J_{n-1}^{(1)}(u)\ge J_{n-1}^{(2)}(u)\ge\cdots\ge J_{n-1}^{(n-1)}(u)$ are distributed like the decreasing order statistics of $n-1$ independent and identically distributed (i.i.d.) random variables $(J_i(u))_{1\le i\le n-1}$, each having the distribution

$$\mathrm{P}(J_1(u)\in \mathrm{d}x) = \frac{\Lambda(\mathrm{d}x)\mathbf{1}_{\{1\le x\le 1/u\}}}{1-u^\alpha},\ x>0. \tag{2.6}$$

Also let $L_{n-1}^{(1)}\ge L_{n-1}^{(2)}\ge\cdots\ge L_{n-1}^{(n-1)}$ be distributed like the decreasing order statistics of $n-1$ i.i.d. random variables $(L_i)_{1\le i\le n-1}$, each having the distribution

$$\mathrm{P}(L_1\in\mathrm{d}x) = \Lambda(\mathrm{d}x)\mathbf{1}_{\{x>1\}}. \tag{2.7}$$

Define

$$\psi(\theta) = \int_{(-\infty,1)} \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x\mathbf{1}_{\{|x|\le 1\}}\right)\Lambda(\mathrm{d}x),\ \theta\in\mathbb{R}, \tag{2.8}$$

and choose $\theta_0>0$ such that $|\psi(\theta)|<1$ for $|\theta|\le\theta_0$ (as is possible since $\psi(0)=0$). Also define $\phi(\theta,u)=\mathrm{E}e^{\mathrm{i}\theta J_1(u)}$, $\theta\in\mathbb{R}$, with $J_1(u)$ having the distribution in (2.6):

$$\phi(\theta,u) = (1-u^\alpha)^{-1}\int_1^{1/u} e^{\mathrm{i}\theta x}\Lambda(\mathrm{d}x),\ 0<u<1. \tag{2.9}$$

Let $W=(W_v)_{v\ge 0}$ be a Lévy process on $\mathbb{R}$ with triplet $(0,0,\Lambda(\mathrm{d}x)\mathbf{1}_{(-\infty,1)})$, and Γ_{r+n} a Gamma $(r+n,1)$ random variable independent of W.

When $n=2,3,\ldots$, $x_k>0$, $1\le k\le n-1$, $x_n=1$, and $\theta_k\in\mathbb{R}$, $1\le k\le n$, write for shorthand

$$x_{n+} = \sum_{k=1}^n x_k \quad\text{and}\quad \widetilde{\theta}_{n+} = \widetilde{\theta}_{n+}(x_1,\ldots,x_n) := \sum_{k=1}^n \frac{\theta_k}{x_k}, \tag{2.10}$$

and let $\int_{\mathbf{x}^\uparrow\ge 1}$ denote integration over the region $\{x_1\ge x_2\ge\cdots\ge x_{n-1}\ge 1\}\subseteq\mathbb{R}^{n-1}$. Recall that ${}^{(0)}X\equiv X$.

Theorem 2.1. *Assume $\overline{\Pi} \in RV_0(-\alpha)$ for some $0<\alpha<2$ and* (2.1).

(i) *Then for each $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, as $t \downarrow 0$, we have the joint convergence*

$$\Big(\frac{{}^{(r)}X_t - t\rho_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+1)}}, \ldots, \frac{{}^{(r)}X_t - t\rho_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+n)}}\Big)$$
$$\stackrel{\mathrm{D}}{\longrightarrow} \Big(\frac{{}^{(r)}S_1 - \rho_S(\Delta S_1^{(r+n)})}{\Delta S_1^{(r+1)}}, \ldots, \frac{{}^{(r)}S_1 - \rho_S(\Delta S_1^{(r+n)})}{\Delta S_1^{(r+n)}}\Big). \tag{2.11}$$

(ii) *When $r \in \mathbb{N}$, $n = 2, 3, \ldots$, the random vector on the RHS of* (2.11) *has characteristic function which can be represented, for $\theta_k \in \mathbb{R}$, $1 \le k \le n$, as*

$$\mathrm{E}\exp\Big(\mathrm{i}\sum_{k=1}^{n} \frac{\theta_k\big({}^{(r)}S_1 - \rho_S(\Delta S_1^{(r+n)})\big)}{\Delta S_1^{(r+k)}}\Big) =$$
$$\int_{\mathbf{x}^{\uparrow}\ge 1} e^{\mathrm{i}\widetilde{\theta}_{n+}x_{n+}}\mathrm{E}\big(e^{\mathrm{i}\widetilde{\theta}_{n+}W_{\Gamma_{r+n}}}\big)\mathrm{P}\big(J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha}) \in \mathrm{d}x_k, 1 \le k \le n-1\big), \tag{2.12}$$

where $B_{r,n}$ is a Beta(r,n) random variable independent of the $(J_i(u))$. Alternatively, recalling (2.8), *when $\max_{1\le k\le n}|\theta_k| \le \theta_0$ the RHS of* (2.12) *can be written as*

$$\int_{\mathbf{x}^{\uparrow}\ge 1} \frac{e^{\mathrm{i}\widetilde{\theta}_{n+}x_{n+}}}{\big(1-\psi(\widetilde{\theta}_{n+})\big)^{r+n}}\mathrm{P}\big(J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha}) \in \mathrm{d}x_k, 1 \le k \le n-1\big). \tag{2.13}$$

When $r = 0$, (2.12) *and* (2.13) *remain true as stated if the rvs $J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha})$ are replaced respectively by $L_{n-1}^{(k)}$, being the order statistics associated with the distribution in* (2.7).

(iii) *When $r \in \mathbb{N}$, $n \in \mathbb{N}$ we have*

$$\frac{{}^{(r)}X_t - t\rho_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+n)}} \stackrel{\mathrm{D}}{\longrightarrow} \frac{{}^{(r)}S_1 - \rho_S(\Delta S_1^{(r+n)})}{\Delta S_1^{(r+n)}}, \text{ as } t \downarrow 0, \tag{2.14}$$

where, recalling (2.9), *the random variable on the RHS of* (2.14) *has characteristic function*

$$\frac{e^{\mathrm{i}\theta}}{(1-\psi(\theta))^{r+n}}\,\mathrm{E}\big(\phi^{n-1}(\theta, \mathrm{B}_{r,n}^{1/\alpha})\big),\ \theta \in \mathbb{R}, |\theta| \le \theta_0. \tag{2.15}$$

When $r = 0$, (2.14) *remains true, as does* (2.15), *if $\phi(\theta, \mathrm{B}_{r,n}^{1/\alpha})$ in* (2.15) *is replaced by $\phi(\theta, 0) := \int_1^\infty e^{\mathrm{i}\theta x}\Lambda(\mathrm{d}x)$.*

Setting $n = 1$ in (2.14), and (since ${}^{(r)}X_t/\Delta X_t^{(r+1)} = 1 + {}^{(r+1)}X_t/\Delta X_t^{(r+1)}$) replacing $r+1$ by r gives

Corollary 2.1. *For each $r \in \mathbb{N}$, $\theta \in \mathbb{R}$, $|\theta| \le \theta_0$,*

$$\frac{{}^{(r)}X_t - t\rho_X(\Delta X_t^{(r)})}{\Delta X_t^{(r)}} \stackrel{\mathrm{D}}{\longrightarrow} \frac{{}^{(r)}S_1 - \rho_S(\Delta S_1^{(r)})}{\Delta S_1^{(r)}}, \text{ as } t \downarrow 0, \tag{2.16}$$

where

$$\mathrm{E}\big(e^{\mathrm{i}\theta({}^{(r)}S_1-\rho_S(\Delta S_1^{(r)}))/\Delta S_1^{(r)}}\big) = \mathrm{E}\big(e^{\mathrm{i}\theta W_{\Gamma_r}}\big) = \frac{1}{\big(1-\psi(\theta)\big)^r}. \tag{2.17}$$

Further, $\big({}^{(r)}S_1 - \rho_S(\Delta S_1^{(r)})\big)/\Delta S_1^{(r)} \stackrel{\mathrm{D}}{=} W_{\Gamma_r}$*, being a Gamma-subordinated Lévy process, is infinitely divisible for each* $r \in \mathbb{N}$*.*

The unwieldy centering functions ρ_X and ρ_S in (2.11)–(2.17) can be simplified in many cases. Especially, when X is a subordinator with drift d_X, ρ_X can be replaced by d_X, and without loss of generality we can assume $\mathrm{d}_X = 0$. The convergences in (2.11)–(2.16) can then be written in terms of Laplace transforms. This case recovers a result proved in Theorem 1.1 of Kevei & Mason (2014): *assume* X *is a driftless subordinator in the domain of attraction (at 0) of a stable random variable with index* $\alpha \in (0,1)$*. Then for* $r \in \mathbb{N}$

$$\frac{{}^{(r)}X_t}{\Delta X_t^{(r)}} \xrightarrow{\mathrm{D}} {}^{(r)}Y, \text{ as } t \downarrow 0, \tag{2.18}$$

where ${}^{(r)}Y$ is an a.s. finite non-degenerate random variable. From Theorem 2.1 we can identify ${}^{(r)}Y$ as having the distribution of ${}^{(r)}S_1/\Delta S_1^{(r)}$, in our notation. Kevei and Mason show, conversely, in this subordinator case, that when (2.18) holds with ${}^{(r)}Y$ a finite non-degenerate random variable, then X is in the domain of attraction (at 0) of a stable random variable with index $\alpha \in (0,1)$. They also give in their Theorem 1.1 a formula for the Laplace transform of ${}^{(r)}Y$. We can state an equivalent version as: *suppose* (2.18) *holds. Then* (2.17) *becomes*

$$\mathrm{E}\big(e^{-\lambda {}^{(r)}S_1/\Delta S_1^{(r)}}\big) = \mathrm{E}\big(e^{-\lambda W_{\Gamma_r}}\big) = \frac{1}{(1+\Psi(\lambda))^r}, \ r \in \mathbb{N}, \tag{2.19}$$

where now $W = (W_v)_{v \geq 0}$ *is a driftless subordinator with measure* $\Lambda(\mathrm{d}x)\mathbf{1}_{(0,1)}$*, and*

$$\Psi(\lambda) = \int_{(0,1)} \big(1 - e^{-\lambda x}\big)\Lambda(\mathrm{d}x), \ \lambda > 0.$$

Remark 2.1 (Negative Binomial Point Process)**.** The form of the Laplace transform in (2.19) suggests a connection with the negative binomial point process of Gregoire (1984). That connection is developed in detail in Ipsen & Maller (2018), and also forms the basis for a general point measure treatment when $0 \leq \alpha \leq \infty$ in Ipsen et al. (2017), which contains a converse proof generalising that of Kevei & Mason (2014). Those results motivate further investigation of the "large trimming" properties of general Lévy processes in the spirit of Buchmann, Maller & Resnick (2016).

Remark 2.2 (Modulus Trimming)**.** Rather than removing large jumps from X as we do in (1.4), we can remove jumps large in modulus and obtain analogous formulae and results, with appropriate modifications. The centering function ρ_X in (2.5) should then be changed to $\gamma - \int_{[-1,-w]\cup[w,1]} x\Pi(\mathrm{d}x)$ when $0 < w \le 1$, and to $\gamma + \int_{(-w,-1)\cup(1,w)} x\Pi(\mathrm{d}x)$ when $w > 1$, and similarly for ρ_S. The norming in Theorem 2.1 would then be by jumps large in modulus rather than by large (positive) jumps, and the convergence would be to the analogous modulus trimmed stable process. The identities in Section 3 required for the modified proofs can be obtained from analogous formulae for modulus trimming in Buchmann, Fan & Maller (2016).

Remark 2.3 (Connection with $\mathrm{PD}_\alpha^{(r)}$)**.** When X is a driftless subordinator, we obtain from (2.11) with $n \in \mathbb{N}$ that

$$\left(\frac{\Delta X_t^{(r+1)}}{{}^{(r)}X_t}, \ldots, \frac{\Delta X_t^{(r+n)}}{{}^{(r)}X_t} \right) \xrightarrow{\mathrm{D}} \left(\frac{\Delta S_1^{(r+1)}}{{}^{(r)}S_1}, \ldots, \frac{\Delta S_1^{(r+n)}}{{}^{(r)}S_1} \right), \text{ as } t \downarrow 0. \tag{2.20}$$

When $n \to \infty$, the n-vector on the RHS tends to a vector $(V_1^{(r)}, V_2^{(r)}, \ldots)$ on the infinite simplex with the generalised Poisson-Dirichlet distribution $\mathrm{PD}_\alpha^{(r)}$ defined in Ipsen & Maller (2016). When $r = 0$, this reduces to the Poisson-Dirichlet distribution generated from the stable subordinator, denoted by $\mathrm{PD}(\alpha, 0)$ in Pitman & Yor (1997), which was first noted by Kingman (1975).

To complete this section we continue to consider the case when X is a driftless subordinator. Our final result shows that ratios of the form ${}^{(r+n)}X_t/\Delta X_t^{(r)}$ as in (2.20) have strong stability properties. In the next theorem the interesting aspect is the uniformity of convergence in neighbourhoods of 0; although $\Delta X_t^{(r)} \downarrow 0$ a.s. as $t \downarrow 0$, the remainder after removing an increasing number of jumps, $r+n$, from X is shown to be small order $\Delta X_t^{(r)}$, in probability as $n \to \infty$, uniformly on compacts.

Theorem 2.2. *Suppose X is a driftless subordinator with $\overline{\Pi} \equiv \overline{\Pi}^+ \in RV_0(-\alpha)$ for some $0 < \alpha < 1$. Then for each $r \in \mathbb{N}$*

$$\frac{{}^{(r+n)}X_t}{\Delta X_t^{(r)}} \xrightarrow{P} 0, \text{ as } n \to \infty, \tag{2.21}$$

uniformly in $t \in (0, t_0]$, for any $t_0 > 0$.

Remark 2.4. By the uniform in probability convergence in Theorem 2.2 we mean

$$\lim_{n\to\infty} \mathrm{P}({}^{(r+n)}X_t > \varepsilon \Delta X_t^{(r)}) = 0, \text{ uniformly in } 0 < t \le t_0, \text{ for all } \varepsilon > 0. \tag{2.22}$$

Since ${}^{(r+n)}X_t$ is monotone in n, this is equivalent to a kind of "uniform almost sure" convergence, as follows. With "i.o." standing for "infinitely often", and $\varepsilon > 0$, $t > 0$,

$$\mathrm{P}({}^{(r+n)}X_t > \varepsilon \Delta X_t^{(r)} \text{ i.o., } n \to \infty) = \lim_{m\to\infty} \mathrm{P}({}^{(r+n)}X_t > \varepsilon \Delta X_t^{(r)} \text{ for some } n > m)$$

$$\leq \lim_{m\to\infty} \mathrm{P}({}^{(r+m)}X_t > \varepsilon \Delta X_t^{(r)}) = 0,$$

where the convergence is uniform in $0 < t \leq t_0$, by (2.22).

3 Representations for Trimmed Lévy Processes

In the present section we revert to considering an arbitrary real valued Lévy process $(X_t)_{t\geq 0}$, set up as in Section 1 (see (1.1) and (1.2)), and derive the identities required for the proofs of the results in Section 2. Fundamental to these identities is a general representation for the joint distribution of ${}^{(r)}X_t$ and its large jumps, given in Buchmann et al. (2016), which allows for possible tied values in the jumps.[2] Our main theorem in this section, Theorem 3.1, applies it to derive formulae for the conditional distributions of the trimmed Lévy given some of its large jumps. We expect these formulae will have useful applications in other areas too.

To state the Buchmann et al. (2016) representation, recall the definition of the right-continuous inverse $\overline{\Pi}^{+,\leftarrow}(x)$ of $\overline{\Pi}^{+}$ in (1.3), and for each $v > 0$ introduce a Lévy process $(X_t^v)_{t\geq 0}$ having the canonical triplet

$$\begin{aligned}&\big(\gamma^v, \sigma^2, \Pi^v(\mathrm{d}x)\big) := \\ &\bigg(\gamma - \mathbf{1}_{\{\overline{\Pi}^{+,\leftarrow}(v)\leq 1\}} \int_{\overline{\Pi}^{+,\leftarrow}(v)\leq x\leq 1} x\Pi(\mathrm{d}x),\, \sigma^2,\, \Pi(\mathrm{d}x)\mathbf{1}_{\{x<\overline{\Pi}^{+,\leftarrow}(v)\}}\bigg). \end{aligned} \tag{3.1}$$

Further, let $G_t^v = \overline{\Pi}^{+,\leftarrow}(v)Y_{t\kappa(v)}$ for $v > 0$, $t > 0$, with $\kappa(v) := \overline{\Pi}^{+}(\overline{\Pi}^{+,\leftarrow}(v)-) - v$ and $(Y_t)_{t\geq 0}$ a homogeneous Poisson process with $\mathrm{E}(Y_1) = 1$, independent of $(X_t^v)_{t\geq 0}$. Let $r \in \mathbb{N}$ and recall that (Γ_i) are Gamma$(i, 1)$ random variables, $i \in \mathbb{N}$. Assume that $(X_t^v)_{t\geq 0}$, $(G_t^v)_{t\geq 0}$ and (Γ_i) are independent as random elements for each $v > 0$. Then Theorem 2.1, p.2329, together with Lemma 1, p.2333, of Buchmann et al. (2016) give, for each $t > 0$, $r, m \in \mathbb{N}$, $1 \leq m \leq r$,

$$\big({}^{(r)}X_t,\, \Delta X_t^{(m)}, \ldots, \Delta X_t^{(r)}\big) \stackrel{\mathrm{D}}{=} \big(X_t^{\Gamma_r/t} + G_t^{\Gamma_r/t},\, \overline{\Pi}^{+,\leftarrow}(\Gamma_m/t), \ldots, \overline{\Pi}^{+,\leftarrow}(\Gamma_r/t)\big). \tag{3.2}$$

We need some further notions. For each $y > 0$ introduce another Lévy process $(X_t^{(y)})_{t\geq 0}$ having the canonical triplet

$$\Big(\gamma^{(y)}, \sigma^2, \Pi^{(y)}(\mathrm{d}x)\Big) := \Big(\gamma - \mathbf{1}_{\{y\leq 1\}} \int_{y\leq x\leq 1} x\Pi(\mathrm{d}x),\, \sigma^2,\, \Pi(\mathrm{d}x)\mathbf{1}_{\{x<y\}}\Big), \tag{3.3}$$

and another process $(G_t^{(y,v)})$ defined such that $G_t^{(y,v)} = yY_{t\kappa(y,v)}$ for $y, v, t > 0$, where again $(Y_t)_{t\geq 0}$ is a homogeneous Poisson process with $\mathrm{E}(Y_1) = 1$, now independent of $(X_t^{(y)})_{t\geq 0}$, and $\kappa(y, v) := \overline{\Pi}^{+}(y-) - v$.

[2] A different but equivalent distributional representation when X is a subordinator is in Proposition 1 of Kevei & Mason (2013).

We need to distinguish situations when $\overline{\Pi}^+$ is or is not continuous at a point. Let A_Π denote the points of discontinuity of $\overline{\Pi}^+$ in $(0,\infty)$. When $y_i \in A_\Pi$, set

$$a_i = a_i(y_i) = \overline{\Pi}^+(y_i) < b_i = b_i(y_i) = \overline{\Pi}^+(y_i-). \tag{3.4}$$

Note that $\overline{\Pi}^{+,\leftarrow}(v)$ takes the same value, namely, y_i, for any $v \in [a_i, b_i)$. When $r, m \in \mathbb{N}$ with $1 \le m \le r$ and $y_r \in A_\Pi$, define conditional expectations

$$K_{m,r}(\theta, t, y_m, \ldots, y_r) = \mathrm{E}\Big(e^{\mathrm{i}\theta G_t^{(y_r,\Gamma_r/t)}} \,\big|\, \Gamma_i/t \in [a_i, b_i),\, m \le i \le r\Big), \tag{3.5}$$

for $t > 0$, $\theta \in \mathbb{R}$. When $y_r \notin A_\Pi$, i.e., $\overline{\Pi}^+(y_r) = \overline{\Pi}^+(y_r-)$, we set $G_t^{(y_r,\cdot)} = 0$ and in this case we understand $K_{m,r}(\theta, t, y_m, \ldots, y_r) = 1$. When $y_r \in A_\Pi$ but $y_i \notin A_\Pi$ for one or more i, $m \le i < r$, we understand the corresponding events $\{\Gamma_i/t \in [a_i, b_i)\}$ are omitted from the conditioning in (3.5).

With this notation in place we can now state Theorem 3.1, the main result of this section, which provides in characteristic function form the conditional distribution of ${}^{(r)}X_t$, given $\Delta X_t^{(r)}$, or given $\Delta X_t^{(m)}, \ldots, \Delta X_t^{(r)}$.

Theorem 3.1. *Take integers $r, m \in \mathbb{N}$ with $1 \le m \le r$, and real numbers $y_m \ge \cdots \ge y_r > 0$, $\theta \in \mathbb{R}$, $t > 0$. Then we have the identities*

$$\mathrm{E}\big(e^{\mathrm{i}\theta{}^{(r)}X_t} \,\big|\, \Delta X_t^{(r)} = y_r\big) = \mathrm{E}\big(e^{\mathrm{i}\theta X_t^{(y_r)}}\big) K_{r,r}(\theta, t, y_r) \tag{3.6}$$

and

$$\mathrm{E}\big(e^{\mathrm{i}\theta{}^{(r)}X_t} \,\big|\, \Delta X_t^{(i)} = y_i,\, m \le i \le r\big) = \mathrm{E}\big(e^{\mathrm{i}\theta X_t^{(y_r)}}\big) K_{m,r}(\theta, t, y_m, \ldots, y_r). \tag{3.7}$$

Proof of Theorem 3.1: We prove (3.6), then show how it can be extended to (3.7). First suppose $y_r \in A_\Pi$. From (3.2) we have

$$\begin{aligned} & \mathrm{P}(\Delta X_t^{(r)} = y_r) = \mathrm{P}\big(\overline{\Pi}^{+,\leftarrow}(\Gamma_r/t) = y_r\big) \\ = \; & \mathrm{P}(\Gamma_r/t \in [\overline{\Pi}^+(y_r), \overline{\Pi}^+(y_r-))) = \mathrm{P}(\Gamma_r/t \in [a_r, b_r)) > 0. \end{aligned} \tag{3.8}$$

(In the last equality, recall (3.4).) Since the probability in (3.8) is positive, we can compute, by elementary means, using (3.2) again,

$$\begin{aligned} & \mathrm{P}({}^{(r)}X_t \le x \,|\, \Delta X_t^{(r)} = y_r) = \frac{\mathrm{P}({}^{(r)}X_t \le x,\, \Delta X_t^{(r)} = y_r)}{\mathrm{P}(\Delta X_t^{(r)} = y_r)} \\ & = \frac{\mathrm{P}\Big(X_t^{\Gamma_r/t} + G_t^{\Gamma_r/t} \le x, \Gamma_r/t \in [a_r, b_r)\Big)}{\mathrm{P}(\Gamma_r/t \in [a_r, b_r))} \\ & = \mathrm{P}\Big(X_t^{\Gamma_r/t} + G_t^{\Gamma_r/t} \le x \Big| \Gamma_r/t \in R(y_r)\Big), \end{aligned} \tag{3.9}$$

where $R(y_r) := [a_r, b_r)$. Going over to characteristic functions, we find, since X_t^v, G_t^v and Γ_r are independent for each $v > 0$,

$$\mathrm{E}\big(e^{\mathrm{i}\theta^{(r)}X_t}\,\big|\,\Delta X_t^{(r)} = y_r\big) = \int_{v\in R(y_r)} \mathrm{E}(e^{\mathrm{i}\theta X_t^v})\,\mathrm{E}(e^{\mathrm{i}\theta G_t^v})\frac{\mathrm{P}\left(\Gamma_r/t \in \mathrm{d}v\right)}{\mathrm{P}(\Gamma_r/t \in R(y_r))}. \tag{3.10}$$

Whenever $v \in R(y_r)$, then $\overline{\Pi}^{+,\leftarrow}(v) = y_r$ and $X_t^v = X_t^{(y_r)}$ (recall (3.3)), while $\kappa(v) = \overline{\Pi}^{+}(y_r-) - v = \kappa(y_r, v)$ and $G_t^v = y_r Y_{t\kappa(y_r,v)} = G_t^{(y_r,v)}$. Consequently the RHS of (3.10) is

$$\mathrm{E}(e^{\mathrm{i}\theta X_t^{(y_r)}})\mathrm{E}\big(e^{\mathrm{i}\theta G_t^{(y_r,\Gamma_r/t)}}\big|\Gamma_r/t \in R(y_r)\big) = \mathrm{E}\big(e^{\mathrm{i}\theta X_t^{(y_r)}}\big)K_{r,r}(\theta, t, y_r),$$

as required for (3.6).

The conditional probability in (3.9) is in fact the Radon-Nikodym derivative of the measure $\mathrm{P}\big({}^{(r)}X_t \le x,\, \Delta X_t^{(r)} \le \cdot\big)$ with respect to the measure $\mathrm{P}\big(\Delta X_t^{(r)} \le \cdot\big)$ on $(0, \infty)$ when y_r is an atom of Π. Alternatively, suppose Π is continuous at y_r. Then we write, from (3.2), for $t > 0$, $y_r > 0$,

$$\mathrm{P}\big({}^{(r)}X_t \le x,\, \Delta X_t^{(r)} \le y_r\big) = \int_{\{v>0:\, \overline{\Pi}^{+,\leftarrow}(v)\le y_r\}} \mathrm{P}\left(X_t^v + G_t^v \le x\right)\mathrm{P}\left(\Gamma_r/t \in \mathrm{d}v\right) \tag{3.11}$$

and

$$\mathrm{P}\big(\Delta X_t^{(r)} \le y_r\big) = \int_{\{v>0:\, \overline{\Pi}^{+,\leftarrow}(v)\le y_r\}} \mathrm{P}\left(\Gamma_r/t \in \mathrm{d}v\right). \tag{3.12}$$

Since $\mathrm{P}\left(\Gamma_r/t \in \cdot\right)$ is absolutely continuous with respect to Lebesgue measure we can use the differentiation formula in Thm.2, p.156 of Zaanen (1958) to calculate the Radon-Nikodym derivative. Thus we evaluate (3.11) and (3.12) over intervals $(y_r - \varepsilon^-, y_r + \varepsilon^+)$ and take the limit of the ratio as $\varepsilon^\pm \downarrow 0$. This produces

$$\mathrm{P}({}^{(r)}X_t \le x\,\big|\,\Delta X_t^{(r)} = y_r) = \mathrm{P}(X_t^{(y_r)} \le x), \tag{3.13}$$

and since $K_{r,r}(\theta, t, y_r) = 1$ in this case, we get (3.6) again.

This completes the proof of (3.6). Next we extend it to (3.7). Assume $y_m \ge \cdots \ge y_r > 0$ are in A_Π. Then (3.8) generalises straightforwardly to

$$\mathrm{P}(\Delta X_t^{(i)} = y_i, m \le i \le r) = \mathrm{P}(\Gamma_i/t \in [a_i, b_i),\, m \le i \le r) > 0, \tag{3.14}$$

and (3.9) becomes

$$\begin{aligned}&\mathrm{P}({}^{(r)}X_t \le x\,\big|\,\Delta X_t^{(i)} = y_i,\, m \le i \le r)\\ &= \mathrm{P}\Big(X_t^{\Gamma_r/t} + G_t^{\Gamma_r/t} \le x\Big|\Gamma_i/t \in [a_i, b_i),\, m \le i \le r\Big).\end{aligned} \tag{3.15}$$

Going over to characteristic functions and recalling $K_{m,r}(\theta, t, y_m, \ldots, y_r)$ in (3.5) we get (3.7).

The cases when some or all of the y_i are continuity points of Π can be analysed as for (3.6). Since we do not need these formulae for the proofs we omit details. □

Remark 3.1. (i) When calculating conditional probabilities, we should check that they have the requisite measurability and integrability properties. The expressions in (3.9) and (3.15) are clearly measurable with respect to their variables, and that they integrate to give the respective joint distributions of ${}^{(r)}X_t$ and the relevant $\Delta X_t^{(i)}$ is easily checked by decomposing integrals into discrete and absolutely continuous components. Effectively, since all of our calculations ultimately involve integration with respect to the absolutely continuous gamma distributions, the needed properties follow automatically.

(ii) (3.5), (3.6) and (3.7) show that in general the Markov property for the ordered large jumps does not hold, as $K_{1,r}(\theta, t, y_1, \dots, y_r) \neq K_{r,r}(\theta, t, y_r)$ in general. But when y_r is a continuity point of $\overline{\Pi}^+$, then equality does hold here and we do have the Markov property. This parallels the similar situation for order statistics of i.i.d. random variables.

Using Theorem 3.1 and (3.3), conditional characteristic functions of ${}^{(r)}X_t$ can be written as in the next corollary. For (3.16) and (3.17), set $m = 1 \leq r$ in (3.7).

Corollary 3.1. *For $r \in \mathbb{N}$, $y_1 \geq y_2 \geq \cdots \geq y_r > 0$, $\theta \in \mathbb{R}$, $t > 0$,*

$$\begin{aligned}&\mathrm{E}\big(e^{\mathrm{i}\theta {}^{(r)}X_t} \,\big|\, \Delta X_t^{(i)} = y_i,\, 1 \leq i \leq r\big)\\ &= \exp\left(\mathrm{i}\theta t\gamma^{(y_r)} - \tfrac{1}{2}t\sigma^2\theta^2 + t\int_{(-\infty,y_r)} \big(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x\mathbf{1}_{\{|x|\leq 1\}}\big)\Pi(\mathrm{d}x)\right)\\ &\qquad\qquad \times K_{1,r}(\theta, t, y_1, y_2, \dots, y_r).\end{aligned} \tag{3.16}$$

Suppose X is a subordinator (so $\sigma^2 = 0$) with drift $\mathrm{d}_X := \gamma - \int_{0<x\leq 1} x\Pi(\mathrm{d}x)$. Then the RHS of (3.16) *can be replaced by*

$$\exp\left(\mathrm{i}\theta t\mathrm{d}_X + t\int_{(0,y_r)} \big(e^{\mathrm{i}\theta x} - 1\big)\Pi(\mathrm{d}x)\right) \times K_{1,r}(\theta, t, y_1, y_2, \dots, y_r). \tag{3.17}$$

The next corollary follows immediately from (3.7). Recall the definition of ρ_X in (2.5). For (3.18), replace r by $r+n$ and set $m = r$ in (3.7).

Corollary 3.2. *For $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, $y_r \geq \cdots \geq y_{r+n} > 0$, $\theta \in \mathbb{R}$, $t > 0$,*

$$\begin{aligned}&\mathrm{E}\left(\exp\left(\mathrm{i}\theta\frac{{}^{(r+n)}X_t - t\rho_X(\Delta X_t^{(r+n)})}{\Delta X_t^{(r+n)}}\right)\middle|\Delta X_t^{(k)} = y_k, r \leq k \leq r+n\right) = e^{-t\sigma^2\theta^2/2y_{r+n}^2}\times\\ &\exp\left(t\int_{(-\infty,1)} \big(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x\mathbf{1}_{\{|x|\leq 1\}}\big)\Pi\big(y_{r+n}\mathrm{d}x\big)\right) \times K_{r,r+n}(\theta/y_{r+n}, t, y_r, \dots, y_{r+n}).\end{aligned} \tag{3.18}$$

Suppose X is a subordinator with drift d_X. Then

$$\begin{aligned}&\mathrm{E}\left(\exp\left(\mathrm{i}\theta\frac{{}^{(r)}X_t - t\mathrm{d}_X}{\Delta X_t^{(r)}}\right)\middle|\, \Delta X_t^{(i)} = y_i,\, 1 \leq i \leq r\right)\\ &= \exp\left(t\int_{(0,1)} \big(e^{\mathrm{i}\theta x} - 1\big)\Pi\big(y_r\mathrm{d}x\big)\right) \times K_{1,r}(\theta/y_r, t, y_1, y_2, \dots, y_r).\end{aligned} \tag{3.19}$$

Proof of Corollaries 3.1 and 3.2: (3.16) follows from Theorem 3.1, using (3.3). Then (3.17) follows from (3.16) by rearranging the centering terms. (3.18) follows from (3.16) and (2.5), and (3.19) follows from (3.18). □

Another formula follows similarly from (3.17):

Corollary 3.3. *Suppose X is a subordinator with drift d_X. Then for $\theta \in \mathbb{R}$, $t > 0$, $r \in \mathbb{N}$, $n \in \mathbb{N}$,*

$$\begin{aligned} &\mathrm{E}\Big(\exp\Big(\mathrm{i}\theta\frac{^{(r+n)}X_t - t\mathrm{d}_X}{\Delta X_t^{(r)}}\Big)\Big|\Delta X_t^{(i)} = y_i,\, r \le i \le r+n\Big) \\ &= \exp\Big(t\int_{(0,y_{r+n})}(e^{\mathrm{i}\theta x/y_r} - 1)\Pi(\mathrm{d}x)\Big) \times K_{r,r+n}(\theta/y_r, t, y_r, \ldots, y_{r+n}). \end{aligned} \tag{3.20}$$

For the proofs in Section 4 we also need the following result.

Proposition 3.1. *Suppose $\overline{\Pi}(\cdot) \in RV_0(-\alpha)$ with $\alpha \in (0,2)$, and keep $r \in \mathbb{N}$ and $n = 2, 3, \ldots$. Take $x_k \ge 1$ for $1 \le k \le n-1$. Then for $x > 0$*

$$\begin{aligned} &\lim_{t\downarrow 0}\mathrm{P}\Big(\frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} > x_k,\ 1 \le k \le n-1\,\Big|\,\Delta X_t^{(r+n)} = x\overline{\Pi}^{+,\leftarrow}(1/t)\Big) \\ &= \ \mathrm{P}\Big(J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha}) > x_k, 1 \le k \le n-1\Big), \end{aligned} \tag{3.21}$$

where $J_{n-1}^{(1)}(u) \ge J_{n-1}^{(2)}(u) \ge \cdots \ge J_{n-1}^{(n-1)}(u)$ are the order statistics associated with the distribution in (2.6), $B_{r,n}$ is a Beta(r,n) random variable independent of $(J_i(u))_{1\le i\le n-1}$, and the limit is taken as $t \downarrow 0$ through points x such that $x\overline{\Pi}^{+,\leftarrow}(1/t)$ is a point of decrease of $\overline{\Pi}^{+}$.

When $r = 0$, (3.21) remains true if the RHS is replaced by

$$\mathrm{P}(L_{n-1}^{(k)} > x_k, 1 \le k \le n-1) \tag{3.22}$$

where $L_{n-1}^{(k)}$ are the order statistics associated with the distribution in (2.7).

Remark 3.2. (3.21) and (3.22) can be stated in a unified fashion if we make the convention that $B_{0,n} \equiv 0$ a.s., put $u = 0$ in (2.6), and identify $(J_i(0))$ with a sequence (L_i) of independent and identically distributed random variables each having the distribution in (2.7). Similarly for the corresponding statements in Theorem 2.1.

Proof of Proposition 3.1: This is a variant of the proof of Theorem 3.1. Assume $\overline{\Pi}(\cdot) \in RV_0(-\alpha)$ with $\alpha \in (0,2)$ and choose $r \in \mathbb{N}_0$, $n = 2, 3, \ldots$, $x_k \ge 1$. For brevity write $q_t := \overline{\Pi}^{+,\leftarrow}(1/t)$, $t > 0$. First suppose $\overline{\Pi}^{+}$ is discontinuous at xq_t, $x > 0$, so

$$\mathrm{P}(\Delta X_t^{(r+n)} = xq_t) = \mathrm{P}(\Gamma_{r+n} \in [a_t(x), b_t(x)),$$

where $a_t(x) := t\overline{\Pi}^+(xq_t) < b_t(x) := t\overline{\Pi}^+(xq_t-)$, and consider the ratio

$$\frac{\mathrm{P}\big(\Delta X_t^{(r+k)} > x_k \Delta X_t^{(r+n)},\, 1 \le k \le n-1,\, \Delta X_t^{(r+n)} = xq_t\big)}{\mathrm{P}\big(\Delta X_t^{(r+n)} = xq_t\big)}$$
$$= \frac{\mathrm{P}\big(\overline{\Pi}^{+,\leftarrow}(\Gamma_{r+k}/t) > x_k x q_t,\, 1 \le k \le n-1,\, a_t(x) \le \Gamma_{r+n} < b_t(x)\big)}{\mathrm{P}\big(a_t(x) \le \Gamma_{r+n} < b_t(x)\big)}. \quad (3.23)$$

With $f_{r+n}(y)$ as the bounded, continuous, density of Γ_{r+n}, the denominator in (3.23) is, by the mean value theorem,

$$\int_{a_t(x)}^{b_t(x)} f_{r+n}(y)\mathrm{d}y = (b_t(x) - a_t(x)) f_{r+n}(\xi_t(x)), \quad (3.24)$$

for some $\xi_t(x) \in [a_t(x), b_t(x))$. Let $c_t(x_k, x) := t\overline{\Pi}^+(x_k x q_t)$. Recalling (3.2), the numerator in (3.23) can be written as

$$\mathrm{P}\big(\Gamma_{r+k} < t\overline{\Pi}^+(x_k x q_t),\, 1 \le k \le n-1,\, a_t(x) \le \Gamma_{r+n} < b_t(x)\big)$$
$$= \int_{a_t(x)}^{b_t(x)} \mathrm{P}\big(\Gamma_{r+k} < c_t(x_k, x),\, 1 \le k \le n-1 \,\big|\, \Gamma_{r+n} = y\big) f_{r+n}(y)\mathrm{d}y$$
$$= \int_{a_t(x)}^{b_t(x)} \mathrm{P}\left(\frac{\Gamma_{r+k}}{\Gamma_{r+n}} < \frac{c_t(x_k, x)}{y},\, 1 \le k \le n-1\right) f_{r+n}(y)\mathrm{d}y. \quad (3.25)$$

In the last equation we used that $(\Gamma_{r+k}/\Gamma_{r+n})_{1 \le k \le n-1}$ is independent of Γ_{r+n} (using "beta-gamma algebra"; see, e.g., Pitman (2006, p.11)).

Again using the mean value theorem the last expression in (3.25) equals

$$(b_t(x) - a_t(x)) f_{r+n}(\eta_t(x)) \mathrm{P}\left(\frac{\Gamma_{r+k}}{\Gamma_{r+n}} < \frac{c_t(x_k, x)}{\eta_t(x)},\, 1 \le k \le n-1\right) \quad (3.26)$$

for some $\eta_t(x) \in [a_t(x), b_t(x))$. Recall (2.2) and that $q_t := \overline{\Pi}^{+,\leftarrow}(1/t)$ to see that each of $a_t(x)$, $b_t(x)$, $\xi_t(x)$ and $\eta_t(x)$ tends to $x^{-\alpha}$, that $f_{r+n}(\xi_t(x))$ and $f_{r+n}(\eta_t(x))$ both tend to $f_{r+n}(x^{-\alpha})$, and that $c_t(x_k, x)$ tends to $(x_k x)^{-\alpha}$, all as $t \downarrow 0$. Take the ratio of the numerator of (3.23) in the form (3.26) to the denominator in the form (3.24), and let $t \downarrow 0$ to get the limit of the ratio in (3.23) as

$$\mathrm{P}\left(\frac{\Gamma_{r+k}}{\Gamma_{r+n}} < x_k^{-\alpha},\, 1 \le k \le n-1\right). \quad (3.27)$$

This gives an expression for the limits on the lefthand side of (3.21) and (3.22). To write them in the forms of the righthand sides of (3.21) and (3.22), first take $r \in \mathbb{N}$, and use the fact that, conditionally on $\Gamma_r/\Gamma_{r+n} = s > 0$,

$$\left(\frac{\Gamma_{r+1}}{\Gamma_{r+n}}, \ldots, \frac{\Gamma_{r+n-1}}{\Gamma_{r+n}}\right) \stackrel{\mathrm{D}}{=} \left(U_{n-1}^{(1)}, \ldots, U_{n-1}^{(n-1)}\right), \quad (3.28)$$

where $(U^{(i)}_{n-1})_{1\le i\le n-1}$ are the order statistics of a sample $(s+(1-s)U_i)_{1\le i\le n-1}$, with $(U_i)_{1\le i\le n-1}$ i.i.d. uniform $[0,1]$. Thus for $0<s<1\le x$

$$\mathrm{P}(s+(1-s)U_1\le x^{-\alpha})=\mathrm{P}\Big(U_1\le \frac{x^{-\alpha}-s}{1-s}\Big)=\frac{x^{-\alpha}-s}{1-s}.$$

This equals $\mathrm{P}(J_1(s^{1/\alpha})\le x^{-\alpha})$ as calculated from (2.6) so we get the required representation in (3.21). When $r=0$, (3.28) remains true with $(U^{(i)}_{n-1})_{1\le i\le n-1}$ the order statistics of $(U_i)_{1\le i\le n-1}$ i.i.d. uniform $[0,1]$, and since $\mathrm{P}(U_1\le x^{-\alpha})=x^{-\alpha}=\mathrm{P}(L_1>x)$, with L_1 as in (2.7), we get (3.22).

Next suppose $\overline{\Pi}^+$ is continuous at xq_t, $x>0$, and xq_t is a point of decrease of $\overline{\Pi}^+$. Hold $t>0$ fixed. The continuous case analogue of the ratio in (3.23) is

$$\lim_{\varepsilon\downarrow 0}\frac{\mathrm{P}\big(\Delta X_t^{(r+k)}>x_k\Delta X_t^{(r+n)},\, 1\le k\le n-1,\, xq_t-\varepsilon<\Delta X_t^{(r+n)}\le xq_t+\varepsilon\big)}{\mathrm{P}\big(xq_t-\varepsilon<\Delta X_t^{(r+n)}\le xq_t+\varepsilon\big)}. \tag{3.29}$$

Letting $a_t(x,\varepsilon):=t\overline{\Pi}^+(xq_t+\varepsilon)<b_t(x,\varepsilon):=t\overline{\Pi}^+(xq_t-\varepsilon)$ for $\varepsilon\in(0,xq_t)$, the denominator in (3.29) is

$$\int_{a_t(x,\varepsilon)}^{b_t(x,\varepsilon)} f_{r+n}(y)\mathrm{d}y=(b_t(x,\varepsilon)-a_t(x,\varepsilon))f_{r+n}(\xi_t(x,\varepsilon)), \tag{3.30}$$

for some $\xi_t(x,\varepsilon)\in[a_t(x,\varepsilon),b_t(x,\varepsilon))$. Note that the righthand side of (3.30) is positive since xq_t is a point of decrease of $\overline{\Pi}^+$. Let $c_t(x_k,x,\varepsilon):=t\overline{\Pi}^+(x_k(xq_t-\varepsilon))$. By a similar calculation as in (3.25) (but noting the inequalities in (3.29) as opposed to the equality in (3.23)), the numerator in (3.29) is not greater than

$$\begin{aligned}&\int_{a_t(x,\varepsilon)}^{b_t(x,\varepsilon)}\mathrm{P}\Big(\frac{\Gamma_{r+k}}{\Gamma_{r+n}}<\frac{c_t(x_k,x,\varepsilon)}{y},\,1\le k\le n-1\Big)f_{r+n}(y)\mathrm{d}y\\ &=(b_t(x,\varepsilon)-a_t(x,\varepsilon))f_{r+n}(\eta_t(x,\varepsilon))\mathrm{P}\Big(\frac{\Gamma_{r+k}}{\Gamma_{r+n}}<\frac{c_t(x_k,x,\varepsilon)}{\eta_t(x,\varepsilon)},\,1\le k\le n-1\Big)\end{aligned}$$

where $\eta_t(x,\varepsilon)\in[a_t(x,\varepsilon),b_t(x,\varepsilon))$. Letting $\varepsilon\downarrow 0$ we find an upper bound of the form

$$\begin{aligned}&\mathrm{P}\Big(\Delta X_t^{(r+k)}>x_k\Delta X_t^{(r+n)},\,1\le k\le n-1\big|\Delta X_t^{(r+n)}=xq_t\Big)\\ &\le \mathrm{P}\Big(\frac{\Gamma_{r+k}}{\Gamma_{r+n}}<\frac{t\overline{\Pi}^+(x_kxq_t-)}{t\overline{\Pi}^+(xq_t)},\,1\le k\le n-1\Big)\end{aligned}$$

at points $t>0$, $x>0$, such that xq_t is a point of decrease of $\overline{\Pi}^+$. Similarly we get a lower bound with $t\overline{\Pi}^+(x_kxq_t)$ replacing $t\overline{\Pi}^+(x_kxq_t-)$. Then as $t\downarrow 0$, on account of the regular variation of $\overline{\Pi}^+$, both bounds approach the expression in (3.27), which can be re-expressed in terms of the J_i and L_i, as shown. Having reached the same limit in both cases, we have proved Proposition 3.1. □

4 Proofs for Section 2

Throughout this section X will be a Lévy process in the domain of attraction at 0 of a non-normal stable random variable. Thus the Lévy tail $\overline{\Pi}$ is regularly varying of index $-\alpha$, $\alpha \in (0,2)$, at 0, and the balance condition (2.1) holds at 0. Since $a_+ > 0$ in (2.1), also $\overline{\Pi}^+ \in RV_0(-\alpha)$ at 0.

Proof of Theorem 2.1: (i) Take $r \in \mathbb{N}_0$, $n \in \mathbb{N}$, and choose $x_1 \geq \cdots \geq x_{n-1} \geq 1$, $x_n = 1$, $\theta_k \in \mathbb{R}$, $1 \leq k \leq n$, and $v > 0$. For shorthand, write $M_t^{(r+n)}$ for $\rho_X(\Delta X_t^{(r+n)})$. We proceed by finding the limit as $t \downarrow 0$ of the conditional characteristic function

$$\mathrm{E}\left(\exp\left(\mathrm{i} \sum_{k=1}^{n} \frac{\theta_k ({}^{(r)}X_t - tM_t^{(r+n)})}{\Delta X_t^{(r+k)}} \right) \middle| \frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} = x_k, 1 \leq k < n, \frac{\Delta X_t^{(r+n)}}{\overline{\Pi}^{+,\leftarrow}(1/t)} = v^{-1/\alpha} \right)$$
$$= \mathrm{E}\left(\exp\left(\mathrm{i} \sum_{k=1}^{n} \frac{\theta_k}{x_k} \frac{({}^{(r)}X_t - tM_t^{(r+n)})}{\Delta X_t^{(r+n)}} \right) \middle| \frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} = x_k, 1 \leq k \leq n-1, \frac{\Delta X_t^{(r+n)}}{\overline{\Pi}^{+,\leftarrow}(1/t)} = v^{-1/\alpha} \right). \tag{4.1}$$

Decompose ${}^{(r)}X_t$ as follows:

$$\frac{{}^{(r)}X_t}{\Delta X_t^{(r+n)}} = \sum_{k=1}^{n} \frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} + \frac{{}^{(r+n)}X_t}{\Delta X_t^{(r+n)}}, \tag{4.2}$$

and recall the definitions of x_{n+} and $\widetilde{\theta}_{n+}$ in (2.10). Given the conditioning in (4.1), the first component on the RHS of (4.2) equals $\sum_{k=1}^{n} x_k = x_{n+}$, so we can write the RHS of (4.1) as

$$e^{\mathrm{i}\widetilde{\theta}_{n+}x_{n+}} \times \mathrm{E}\left(\exp\left(\mathrm{i}\widetilde{\theta}_{n+} \frac{{}^{(r+n)}X_t - tM_t^{(r+n)}}{\Delta X_t^{(r+n)}} \right) \middle| \frac{\Delta X_t^{(r+k)}}{\overline{\Pi}^{+,\leftarrow}(1/t)} = x_k v^{-1/\alpha},\ 1 \leq k \leq n \right) \tag{4.3}$$

(recall $x_n = 1$). Then by (3.18) with θ replaced by $\widetilde{\theta}_{n+}$, y_k replaced by $y_k(t) := x_k v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)$, and $\sigma^2 = 0$, the expression in (4.3) equals

$$e^{\mathrm{i}\widetilde{\theta}_{n+}x_{n+}} \times \exp\left(\int_{(-\infty,1)} \left(e^{\mathrm{i}\widetilde{\theta}_{n+}x} - 1 - \mathrm{i}\widetilde{\theta}_{n+}x\mathbf{1}_{\{|x|\leq 1\}}\right) t\Pi\left(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)\mathrm{d}x\right) \right)$$
$$\times K_{r+1,r+n}(\widetilde{\theta}_{n+}/y_{r+n}(t),\, t,\, y_{r+1}(t), \ldots, y_{r+n}(t)) \tag{4.4}$$

(again, recall $x_n = 1$). By (2.2), we have $t\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)) \to v$, and hence $t\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)\mathrm{d}x) \to v\Lambda(\mathrm{d}x)$, $x > 0$, vaguely, as $t \downarrow 0$. The limit of the second factor in (4.4) can then be found straightforwardly using integration by parts and applying (2.1) and (2.2).

The term containing K in (4.4) is negligible here, as follows. Note that $\overline{\Pi}^+$ in $RV_0(-\alpha)$ implies $\Delta\overline{\Pi}^+(x) := \overline{\Pi}^+(x-)-\overline{\Pi}^+(x) = o(\overline{\Pi}^+(x))$ as $x \downarrow 0$. Recall $K_{r,r+n}$ is defined in (3.5), and $\kappa(y,v) = \overline{\Pi}^+(y-) - v$. Substituting $y_{r+n}(t) = v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)$ for y gives

$$
\begin{aligned}
t\kappa(y_{r+n}(t), v/t) &= t\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)-) - v \\
&= t\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)) - v + t\Delta\big(\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t))\big) \\
&= t\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)) - v + o\big(t\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t))\big) \\
&\to v - v = 0, \text{ as } t \downarrow 0. \qquad (4.5)
\end{aligned}
$$

Furthermore, $G_t^{(y_{r+n}(t),\Gamma_{r+n}/t)}/y_{r+n}(t)$ has the distribution of $Y_{t\kappa(y_{r+n}(t),\Gamma_{r+n}/t)}$ and hence tends to 0 in probability when $t \downarrow 0$. So we can ignore the K term in (4.4).

We conclude that the expression in (4.4) tends as $t \downarrow 0$ to

$$
e^{\mathrm{i}\widetilde{\theta}_{n+}x_{n+}} \times \exp\left(v\int_{(-\infty,1)} \big(e^{\mathrm{i}\widetilde{\theta}_{n+}x} - 1 - \mathrm{i}\widetilde{\theta}_{n+}x\mathbf{1}_{\{|x|\le 1\}}\big)\Lambda(\mathrm{d}x)\right). \qquad (4.6)
$$

Thus, by (4.2), to find the limit as $t \downarrow 0$ of

$$
\mathrm{E}\exp\left(\mathrm{i}\sum_{k=1}^{n}\frac{\theta_k({}^{(r)}X_t - tM_t^{(r+n)})}{\Delta X_t^{(r+k)}}\right),
$$

we multiply (4.6) by the limit, as $t \downarrow 0$ through points v such that $v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)$ is a point of decrease of $\overline{\Pi}^+$, of

$$
\mathrm{P}\left(\frac{\Delta X_t^{(r+k)}}{\Delta X_t^{(r+n)}} \in \mathrm{d}x_k,\, 1 \le k \le n-1 \,\middle|\, \frac{\Delta X_t^{(r+n)}}{\overline{\Pi}^{+,\leftarrow}(1/t)} = v^{-1/\alpha}\right) \times \mathrm{P}\left(\frac{\Delta X_t^{(r+n)}}{\overline{\Pi}^{+,\leftarrow}(1/t)} \in \mathrm{d}(v^{-1/\alpha})\right),
$$

and then integrate over v and the x_k.[3]

From (3.21) when $r \in \mathbb{N}$ and from (3.22) when $r = 0$ we see that the limit of the conditional probability depends only on the $J_{n-1}^{(k)}$ or $L_{n-1}^{(k)}$ and $B_{r,n}$, and not on v, while by (2.2)

$$
\begin{aligned}
\mathrm{P}\big(\Delta X_t^{(r+n)} > v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)\big) &= \mathrm{P}\big(\overline{\Pi}^{+,\leftarrow}(\Gamma_{r+n}/t) > v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t)\big) \\
&= \mathrm{P}\big(\Gamma_{r+n} < t\overline{\Pi}^+(v^{-1/\alpha}\overline{\Pi}^{+,\leftarrow}(1/t))\big) \\
&\to \mathrm{P}\big(\Gamma_{r+n} \le v\big), \text{ as } t \downarrow 0.
\end{aligned}
$$

[3]We use the result: $\int f_t(\omega)P_t(\mathrm{d}\omega) \to \int f(\omega)\mathrm{P}(\mathrm{d}\omega)$ when $P_t \xrightarrow{w} P$ are probability measures and $f_t \to f$, f continuous, $|f| \le 1$. In (4.1), the f_t are characteristic functions and the limit distribution P in (4.7) is continuous in all its variables.

Putting the RHS of (3.21) or (3.22) together with the expression in (4.6) we can write the limiting characteristic function of the n-vector on the LHS of (2.11) as

$$\int_{\mathbf{x}^{\uparrow}\geq 1} e^{\mathrm{i}\widetilde{\theta}_{n+}x_{n+}} \int_0^\infty \exp\Big(v\int_{-\infty}^1 \big(e^{\mathrm{i}\widetilde{\theta}_{n+}x} - 1 - \mathrm{i}\widetilde{\theta}_{n+}x\mathbf{1}_{\{|x|\leq 1\}}\big)\Lambda(\mathrm{d}x)\Big)\mathrm{P}\left(\Gamma_{r+n}\in \mathrm{d}v\right)$$
$$\times\, \mathrm{P}\big(J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha}) \in \mathrm{d}x_k, 1\leq k\leq n-1\big) \quad (4.7)$$

when $r\in\mathbb{N}$, and with each $J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha})$ replaced by $L_{n-1}^{(k)}$ when $r=0$. Recall that $\int_{\mathbf{x}^{\uparrow}\geq 1}$ denotes integration over the region $\{x_1\geq x_2\geq\cdots\geq x_{n-1}\geq 1\}$.

Note that, with Λ defined as in (2.3), $\overline{\Lambda}(x)\in RV_0(-\alpha)$ and $\overline{\Lambda}^+$ and $\overline{\Lambda}^-$ satisfy (2.1). So exactly the same calculation[4] with $\Delta S_1^{(k)}$ replacing $\Delta X_t^{(k)}$, for $r+1\leq k\leq r+n$, and Λ replacing Π (and no limit on t is necessary), shows that the characteristic function of the vector of stable ratios on the RHS of (2.11) equals (4.7) when $r\in\mathbb{N}$ or the corresponding version when $r=0$.

(ii) To derive (2.12) and the corresponding version when $r=0$, observe that the exponent inside the integral in (4.7) is the characteristic function of a Lévy process $(W_v)_{v\geq 0}$ having Lévy triplet $(0,0,\Lambda(\mathrm{d}x)\mathbf{1}_{(-\infty,1)})$, that is, of a Stable($\alpha$) process with jumps truncated below 1. So the integral with respect to v in (4.7) is

$$\int_{v>0}\mathrm{E}\big(e^{\mathrm{i}\widetilde{\theta}_{n+}W_v}\big)\mathrm{P}\left(\Gamma_{r+n}\in\mathrm{d}v\right) = \mathrm{E}\big(e^{\mathrm{i}\widetilde{\theta}_{n+}W_{\Gamma_{r+n}}}\big),$$

and thus we obtain (2.12) when $r\in\mathbb{N}$ and the corresponding version when $r=0$ with the J_i replaced by L_i.

When $r\in\mathbb{N}$ and $n=2,3,\ldots$, the alternative representation in (2.13) is obtained by evaluating the dv integral in (4.7), resulting in (recall $\psi(\cdot)$ defined in (2.8)):

$$\begin{aligned}
&\mathrm{E}\exp\left(\mathrm{i}\sum_{k=1}^n \frac{\theta_k\big({}^{(r)}X_t - t\rho_X(\Delta X_t^{(r+n)})\big)}{\Delta X_t^{(r+n)}}\right)\\
&\to \mathrm{E}\exp\left(\mathrm{i}\sum_{k=1}^n \frac{\theta_k\big({}^{(r)}S_1 - \rho_S(\Delta S_1^{(r+n)})\big)}{\Delta S_1^{(r+n)}}\right)\\
&= \int_{\mathbf{x}^{\uparrow}\geq 1} e^{\mathrm{i}\widetilde{\theta}_{n+}x_{n+}}\int_{v>0}\frac{v^{r+n-1}e^{-v\left(1-\psi(\widetilde{\theta}_{n+})\right)}}{\Gamma(r+n)}\mathrm{d}v\\
&\qquad\times\,\mathrm{P}\big(J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha})\in\mathrm{d}x_k, 1\leq k\leq n-1\big),
\end{aligned}\quad (4.8)$$

equal to the expression in (2.13). When $r\in\mathbb{N}_0$, $n=1$, similar working shows that (4.7) can be replaced by

$$\lim_{t\downarrow 0}\mathrm{E}\big(e^{\mathrm{i}\theta\left({}^{(r)}X_t - t\rho_X(\Delta X_t^{(r+1)})\right)/\Delta X_t^{(r+1)}}\big)$$

[4]This easy correspondence is the reason for adopting the nonstandard centering in (2.4).

$$= e^{\mathrm{i}\theta} \times \int_{v>0} \exp\left(v \int_{(-\infty,1)} \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\le 1\}}\right)\Lambda(\mathrm{d}x)\right) \mathrm{P}\left(\Gamma_{r+1} \in \mathrm{d}v\right)$$

$$= e^{\mathrm{i}\theta} \times \mathrm{E}\big(e^{\mathrm{i}\theta W_{\Gamma_{r+1}}}\big),\ \theta \in \mathbb{R}. \tag{4.9}$$

(iii) Finally, to prove (2.15) when $r \in \mathbb{N}$, set $\theta_1 = \cdots = \theta_{n-1} = 0$, $\theta_n = \theta$ (so $\widetilde{\theta}_{n+} = \theta$ and, recall, $x_{n+} = x_1 + \cdots + x_{n-1} + 1$) in (4.8) to get the characteristic function of the RHS of (2.14) equal to

$$\int_{\mathbf{x}^{\uparrow} \ge 1} \frac{e^{\mathrm{i}\theta x_{n+}}}{(1-\psi(\theta))^{r+n}} \mathrm{P}\big(J_{n-1}^{(k)}(\mathrm{B}_{r,n}^{1/\alpha}) \in \mathrm{d}x_k, 1 \le k \le n-1\big)$$

$$= \frac{e^{\mathrm{i}\theta}}{(1-\psi(\theta))^{r+n}} \int_{0<u<1} \mathrm{E}\exp\left(\mathrm{i}\theta \sum_{k=1}^{n-1} J_{n-1}^{(k)}(u)\right) \mathrm{P}\big(\mathrm{B}_{r,n}^{1/\alpha} \in \mathrm{d}u\big)$$

$$= \frac{e^{\mathrm{i}\theta}}{(1-\psi(\theta))^{r+n}} \int_{0<u<1} \big(\mathrm{E}e^{\mathrm{i}\theta J_1(u)}\big)^{n-1} \mathrm{P}\big(\mathrm{B}_{r,n}^{1/\alpha} \in \mathrm{d}u\big)$$

$$= \frac{e^{\mathrm{i}\theta}}{(1-\psi(\theta))^{r+n}} \mathrm{E}\big(\phi^{n-1}(\theta, \mathrm{B}_{r,n}^{1/\alpha})\big),$$

where $\phi(\theta, u) = \mathrm{E}e^{\mathrm{i}\theta J_1(u)}$ as in (2.9), with $|\psi(\theta)| < 1$ when $|\theta| \le \theta_0$. Similarly, (4.9) can alternatively be written as $e^{\mathrm{i}\theta}$ times the expression in (2.17). The $r = 0$ case follows as before. □

Proof of Theorem 2.2: In this proof X is a driftless subordinator whose Lévy tail measure is in $RV_0(-\alpha)$, $0 < \alpha < 1$. From (3.2) we obtain the Laplace transform

$$\mathrm{E}\exp\left(-\lambda \frac{{}^{(r+n)}X_t}{\Delta X_t^{(r)}}\right) = \int_{y>0}\int_{w>y} e^{-t\int_{(0,a)}\left(1-e^{-\lambda x/b}\right)\Pi(\mathrm{d}x) - t\kappa(w/t)(1-e^{-\lambda a/b})}$$
$$\times \mathrm{P}\left(\Gamma_r \in \mathrm{d}y, \Gamma_{r+n} \in \mathrm{d}w\right), \tag{4.10}$$

where $\lambda > 0$ and for brevity

$$a = a(w,t) := \overline{\Pi}^{\leftarrow}(w/t) \le b = b(y,t) := \overline{\Pi}^{\leftarrow}(y/t),\ t > 0,\ w > y > 0$$

(we can write $\overline{\Pi}$ and $\overline{\Pi}^{\leftarrow}$ for $\overline{\Pi}^{+}$ and $\overline{\Pi}^{+,\leftarrow}$ in (3.20)). We derive an upper bound for the exponent in (4.10) as follows. Keep $0 < t \le t_0$ for a fixed $t_0 > 0$, throughout.

First, the integral in the exponent of (4.10) is

$$t\int_{(0,a)} \left(1 - e^{-\lambda x/b}\right)\Pi(\mathrm{d}x) \le t(\lambda/b)\int_0^a e^{-\lambda x/b}\overline{\Pi}(x)\mathrm{d}x \quad \text{(integrate by parts)}$$
$$= t\lambda \int_0^{a/b} e^{-\lambda x}\overline{\Pi}(bx)\mathrm{d}x. \tag{4.11}$$

Now $a(w,t) \to \overline{\Pi}^{\leftarrow}(+\infty) = 0$ as $w \to \infty$ or $t \downarrow 0$, and $b(y,t) \to \overline{\Pi}^{\leftarrow}(+\infty) = 0$ as $y \to \infty$ or $t \downarrow 0$. To compare the magnitudes of a and b we use the Potter bounds

(Bingham, Goldie and Teugels (1987, p.25)). Since $\overline{\Pi} \in RV_0(-\alpha)$ with $0 < \alpha < 1$, given $\eta > 0$ there are constants $c > 0$ and $z_0 = z_0(\eta) > 0$ such that

$$\frac{\overline{\Pi}(\mu z)}{\overline{\Pi}(z)} \le c\mu^{-\alpha-\eta} \text{ for all } \mu \in (0,1),\ z \in (0, z_0); \tag{4.12}$$

and since $\overline{\Pi}^{\leftarrow} \in RV_\infty(-1/\alpha)$ we also have

$$\frac{\overline{\Pi}^{\leftarrow}(\mu z)}{\overline{\Pi}^{\leftarrow}(z)} \le c\mu^{-1/\alpha+\eta} \text{ for all } \mu > 1,\ z > 1/z_0 \tag{4.13}$$

(where c and z_0 may be chosen the same in both cases, and $\eta < 1/\alpha$). Thus for $0 < x \le a/b \le 1$ and $0 < b \le z_0$, using (4.12),

$$t\overline{\Pi}(bx) \le ctx^{-\alpha-\eta}\overline{\Pi}(b) = ctx^{-\alpha-\eta}\overline{\Pi}(\overline{\Pi}^{\leftarrow}(y/t)) \le cyx^{-\alpha-\eta},$$

and we have $b \le z_0$ if $\overline{\Pi}^{\leftarrow}(y/t) \le z_0$, i.e., if $y/t \ge \overline{\Pi}(z_0)$. For $w > y$ and $y/t \ge 1/z_0$, using (4.13),

$$\frac{a}{b} = \frac{\overline{\Pi}^{\leftarrow}(w/t)}{\overline{\Pi}^{\leftarrow}(y/t)} = \frac{\overline{\Pi}^{\leftarrow}((w/y)(y/t))}{\overline{\Pi}^{\leftarrow}(y/t)} \le c\left(\frac{w}{y}\right)^{-1/\alpha+\eta} = c\left(\frac{y}{w}\right)^{1/\alpha-\eta}. \tag{4.14}$$

Now keep $y/t \ge z_1 := \overline{\Pi}(z_0) \vee (1/z_0)$ and $0 < \eta < 1 - \alpha$ (so also $\eta < 1/\alpha$). Then by (4.11)

$$\begin{aligned} t\int_{(0,a)} \left(1 - e^{-\lambda x/b}\right)\Pi(\mathrm{d}x) &\le t\lambda \int_0^{a/b} e^{-\lambda x}\overline{\Pi}(bx)\mathrm{d}x \\ &\le c\lambda y \int_0^{a/b} x^{-\alpha-\eta}\mathrm{d}x = \frac{c\lambda y}{1-\alpha-\eta}\left(\frac{a}{b}\right)^{1-\alpha-\eta} \\ &\le c'\lambda y \left(\frac{y}{w}\right)^{\beta} =: \lambda g_1(w,y), \end{aligned} \tag{4.15}$$

where $c' := c^{2-\alpha-\eta}/(1-\alpha-\eta) > 0$ and $\beta := (1-\alpha-\eta)(1/\alpha-\eta) > 0$.

Alternatively, when $y/t < z_1$, we have $b = \overline{\Pi}^{\leftarrow}(y/t) \ge \overline{\Pi}^{\leftarrow}(z_1)$, while $t \le t_0$ implies $a = \overline{\Pi}^{\leftarrow}(w/t) \le \overline{\Pi}^{\leftarrow}(w/t_0)$. Then

$$\begin{aligned} t\int_{(0,a)} \left(1 - e^{-\lambda x/b}\right)\Pi(\mathrm{d}x) &\le t(\lambda/b)\int_{(0,a)} x\Pi(\mathrm{d}x) \\ &\le t_0(\lambda/\overline{\Pi}^{\leftarrow}(z_1))\int_{(0,\overline{\Pi}^{\leftarrow}(w/t_0))} x\Pi(\mathrm{d}x) \\ &=: \lambda g_2(w). \end{aligned} \tag{4.16}$$

For the term containing κ in (4.10), we have, for all $x \in (0, z_0)$,

$$\Delta\overline{\Pi}(x) = \overline{\Pi}(x-) - \overline{\Pi}(x) \le \overline{\Pi}(x/2) - \overline{\Pi}(x) \le 2^{\alpha+\eta}c\overline{\Pi}(x)$$

by (4.12). Thus for all $t > 0$ and $w > y > 0$, using (4.5),

$$t\kappa(w/t) \le t\Delta\overline{\Pi}(\overline{\Pi}^{\leftarrow}(w/t)) \le 2^{\alpha+\eta}ct\overline{\Pi}(\overline{\Pi}^{\leftarrow}(w/t)) \le 2^{\alpha+\eta}cw,$$

because we kept $y/t > \overline{\Pi}(z_0)$, as a consequence of which $\overline{\Pi}^{\leftarrow}(w/t) \le \overline{\Pi}^{\leftarrow}(y/t) \le z_0$. Then $t\kappa(w/t)(1-e^{-\lambda a/b}) \le c2^{\alpha+\eta}w\lambda a/b$. When $w > y$ and $y/t \ge 1/z_0$, $t\kappa(w/t)(1-e^{-\lambda a/b}) \le 2^{\alpha+\eta}c^2\lambda w(y/w)^{1/\alpha-\eta}$ by (4.14). When $y/t < z_1$, so $b \ge \overline{\Pi}^{\leftarrow}(z_1)$, $t\kappa(w/t)(1-e^{-\lambda a/b}) \le 2^{\alpha+\eta}cw\lambda\overline{\Pi}^{\leftarrow}(w/t_0)/\overline{\Pi}^{\leftarrow}(z_1)$. So an overall upper bound for the term containing κ in (4.10) is

$$\begin{aligned} & t\kappa(w/t)(1-e^{-\lambda a/b}) \\ & \le \lambda g_3(w,y) := 2^{\alpha+\eta}\max\big(c^2\lambda w(y/w)^{1/\alpha-\eta},\ cw\lambda\overline{\Pi}^{\leftarrow}(w/t_0)/\overline{\Pi}^{\leftarrow}(z_1)\big). \end{aligned} \tag{4.17}$$

Combine (4.15)–(4.17) to get an upper bound for the negative of the exponent in (4.10) of the form

$$\lambda g(w,y) := \lambda\big(\max(g_1(w,y), g_2(w)) + g_3(w,y)\big).$$

So, for all $0 < t \le t_0$, $n \in \mathbb{N}$,

$$\begin{aligned} \mathrm{E}\exp\left(-\lambda\frac{{}^{(r+n)}X_t}{\Delta X_t^{(r)}}\right) &\ge \int_{y>0}\int_{w>y} e^{-t_0\lambda g(w,y)}\mathrm{P}\left(\Gamma_r \in \mathrm{d}y, \Gamma_{r+n} \in \mathrm{d}w\right) \\ &= \mathrm{E}\big(e^{-t_0\lambda g(\Gamma_{r+n},\Gamma_r)}\big). \end{aligned} \tag{4.18}$$

Now when $w \to \infty$, $g_1(w,y) \to 0$ for each $y > 0$ (see (4.15)) and $g_2(w) \to 0$ as $w \to \infty$ because $\overline{\Pi}^{\leftarrow}(w) \to 0$ as $w \to \infty$ (see (4.16)); while $g_3(w,y) \to 0$ for each $y > 0$ because $\overline{\Pi}^{\leftarrow} \in RV_\infty(-1/\alpha)$ and $0 < \alpha < 1$ (see (4.17)).

Finally, since $\Gamma_{r+n} \stackrel{P}{\to} \infty$ as $n \to \infty$ for each $r \in \mathbb{N}$, we can let $n \to \infty$ and use Fatou's lemma in (4.18) to see that

$$\mathrm{E}\exp\left(-\lambda\frac{{}^{(r+n)}X_t}{\Delta X_t^{(r)}}\right) \ \to\ 1, \text{ as } n \to \infty,$$

for each $r \in \mathbb{N}$, uniformly in $\lambda > 0$ and $t \in (0, t_0]$. We deduce convergence in probability in (2.21) uniformly in $t \in (0, t_0]$ from this.

References

Bingham, N. H., Goldie, C. M., & Teugels, J. L. (1987). *Regular Variation.* Cambridge University Press.

Buchmann, B., Fan, Y., & Maller, R. A. (2016). Distributional representations and dominance of a Lévy process over its maximal jump processes. *Bernoulli*, *22*(4), 2325–2371.

Buchmann, B., Maller, R. A., & Resnick, S. I. (2016). Processes of rth largest. *arXiv:1607.08674*.

Gregoire, G. (1984). Negative binomial distributions for point processes. *Stochastic Process. Appl.*, *16*(2), 179–188.

Ipsen, Y. F. & Maller, R. A. (2016). Generalised Poisson-Dirichlet distributions and the negative binomial point process. *arXiv:1611.09980.*

Ipsen, Y. F. & Maller, R. A. (2018). Negative binomial construction of random discrete distributions on the infinite simplex. *arXiv:1802.02655.*

Ipsen, Y. F., Maller, R. A., & Resnick, S. I. (2017). Ratios of ordered points of point processes with regularly varying intensity measures. *arXiv:1707.09653.*

Kevei, P. & Mason, D. M. (2013). Randomly weighted self-normalized Lévy processes. *Stochastic Processes and their Applications*, *123*(2), 490–522.

Kevei, P. & Mason, D. M. (2014). The limit distribution of ratios of jumps and sums of jumps of subordinators. *Lat. Am. J. Probab. Math. Stat.*, *11*(2), 631–642.

Kingman, J. F. C. (1975). Random discrete distributions. *J. R. Stat. Soc. Series B Stat. Methodol.*, *37*(1), 1–22.

Perman, M. (1993). Order statistics for jumps of normalised subordinators. *Stochastic Process. Appl.*, *46*(2), 267–281.

Pitman, J. (2006). *Combinatorial Stochastic Processes.* Springer.

Pitman, J. & Yor, M. (1997). The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator. *Annals of Probability*, *25*(2), 855–900.

Zaanen, A.C. (1958). *An Introduction to the Theory of Integration.* North-Holland Pub. Co.