

A REPRESENTATION THEOREM FOR MEASURABLE RELATION ALGEBRAS

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ABSTRACT. A relation algebra is called measurable when its identity is the sum of measurable atoms, where an atom is called measurable if its square is the sum of functional elements.

In this paper we show that atomic measurable relation algebras have rather strong structural properties: they are constructed from systems of groups, coordinated systems of isomorphisms between quotients of the groups, and systems of cosets that are used to "shift" the operation of relative multiplication. An atomic and complete measurable relation algebra is completely representable if and only if there is a stronger coordination between these isomorphisms induced by a scaffold (the shifting cosets are not needed in this case). We also prove that a measurable relation algebra in which the associated groups are all finite is atomic.

1. INTRODUCTION

The well-known pair of papers [7] and [8], by Jónsson and Tarski, were motivated by Tarski's efforts to prove that every model of his axiomatization of the calculus of relation algebras is representable, that is to say, every (abstract) relation algebra is isomorphic to a set relation algebra consisting of a universe of (binary) relations on some base set, under the standard set-theoretic operations on such relations. In the second of these papers, the authors proved several representation theorems for limited classes of relation algebras. In particular, they proved that an atomic relation algebra in which the atoms satisfy a specific "singleton inequality" is isomorphic to a set relation algebra. The singleton inequality is an inequality that is satisfied by a non-empty relation R and its converse in a set relation algebra of all binary relations on a base set if and only if R is a singleton relation in the sense that it has the form $R = \{(p, q)\}$ for some elements p and q in the base set.

Maddux [9] considerably strengthened this representation theorem. He eliminated the assumptions that the given relation algebra be atomic and that every atom satisfy the singleton inequality. Instead, he assumed only that the identity element be the sum of a set of non-zero *elements* satisfying the singleton inequality. Actually, he proved an even stronger version of this theorem by showing that every relation algebra in which the identity element is the sum of a set of non-zero elements satisfying the singleton inequality or a corresponding "doubleton inequality" is isomorphic to a set relation algebra. He called such relation algebras *pair dense*.

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The purpose of this paper is to prove a substantial generalization of Maddux’s theorem. The task is complicated by the fact that for no natural number $n \geq 3$ is there an equation or inequality that characterizes relations consisting of at most n pairs. This obstacle may be overcome by allowing oneself to use formulas from first-order logic instead of just equations and inequalities. In [2], an atom $x \leq 1$ is defined to be *measurable* if the square $x; 1; x$ is the sum of a set of *functions*, that is to say, a set of abstract elements f satisfying the functional inequality $f^\smile; f \leq 1$. These functions turn out to be abstract versions of permutations, and the set of these permutations that are non-zero and below the square $x; 1; x$ form a group. The size of the group gives a measure of the size of x . A relation algebra is said to be *measurable* if the identity element is the sum of measurable atoms, and *finitely measurable* if all of the atoms in this sum have finite measure.

In [2], a large class of examples of measurable set relation algebras is constructed using systems of groups and corresponding systems of isomorphisms between quotients of the groups. The resulting algebras are called (*generalized*) *group relation algebras*, and every such algebra is an example of a complete and atomic measurable relation algebra. The class of these examples, however, does not comprehend all possible examples of complete and atomic measurable relation algebras. In [1], the class of examples is expanded by using systems of cosets to “shift”, or change the value, of the operation of relational composition in group relation algebras. A characterization is given in [1] of when such “shifted” group relation algebras are relation algebras, and therefore examples of complete and atomic measurable relation algebras. They are called *coset relation algebras*. An example is given in [1] of a coset relation algebra—and therefore of an atomic, measurable relation algebra—that is not isomorphic to any set relation algebra, so not all atomic measurable relation algebras are representable in the classical sense of the word.

The purpose of the present paper is to prove that the class of coset relation algebras is adequate for the task of “representing in a wider sense” all atomic, measurable relation algebras. In the main theorem of the paper, we show that every atomic, measurable relation algebra \mathfrak{B} is *essentially isomorphic* to a coset relation algebra \mathfrak{C} in the sense that the completion (the minimal complete extension) of \mathfrak{B} is isomorphic to \mathfrak{C} . (The passage to the completion does not change the structure of \mathfrak{B} , it only fills in any missing infinite sums that are needed in order to obtain isomorphism with the complete relation algebra \mathfrak{C}). In particular, every measurable relation algebra that is finite is isomorphic to a coset relation algebra. If the algebra \mathfrak{B} is not finite, but is finitely measurable, then the assumption that \mathfrak{B} be atomic may be dropped. We also prove that a measurable relation algebra \mathfrak{B} is essentially isomorphic to a group relation algebra if and only if \mathfrak{B} has a “scaffold” of atoms, and this occurs if and only if \mathfrak{B} is completely representable.

Except for basic facts about groups, this article is intended to be self-contained. The definition of a relation algebra, and the relatively few basic relation algebraic laws that are needed to follow the proof in the paper are presented in Section 2. Readers who wish to learn more about the subject are recommended to look at Hirsch-Hodkinson [6], Maddux [10], or Givant [3].

2. RELATION ALGEBRAS

In the next few sections, most of the calculations will involve the arithmetic of relation algebras. This section provides a review the essential results that will be needed.

A relation algebra is an algebra of the form

$$\mathfrak{A} = (A, +, -, ;, \smile, 1'),$$

where $+$ and $;$ are binary operations called *addition* and *relative multiplication*, while $-$ and \smile are unary operations called *complement* and *converse*, and $1'$ is a distinguished constant called the *identity element*, such that the following axioms are satisfied for all elements r, s , and t in \mathfrak{A} .

- (R1) $r + s = s + r$.
- (R2) $r + (s + t) = (r + s) + t$.
- (R3) $-(-r + s) + -(-r + -s) = s$.
- (R4) $r ; (s ; t) = (r ; s) ; t$.
- (R5) $r ; 1' = r$.
- (R6) $r^{\smile\smile} = r$.
- (R7) $(r ; s)^{\smile} = s^{\smile} ; r^{\smile}$.
- (R8) $(r + s) ; t = r ; t + s ; t$.
- (R9) $(r + s)^{\smile} = r^{\smile} + s^{\smile}$.
- (R10) $r^{\smile} ; -(r ; s) + -s = -s$.

The axioms are commonly referred to by the following names: (R1) is the *commutative law for addition*, (R2) is the *associative law for addition*, (R3) is *Huntington's law*, (R4) is the *associative law for relative multiplication*, (R5) is the (right-hand) *identity law for relative multiplication*, (R6) is the *first involution law*, (R7) is the *second involution law*, (R8) is the (right-hand) *distributive law for relative multiplication*, (R9) is the *distributive law for converse*, and (R10) is *Tarski's law*. Under the assumption of the remaining axioms, (R10) is equivalent to the implication

$$(R11) \quad \text{if } (r ; s) \cdot t = 0, \quad \text{then } (r^{\smile} ; t) \cdot s = 0,$$

which we shall call the *cycle law*. It is this form of (R10) that we shall always use. Axioms (R1)–(R3) secure that $(A, +, -)$ is a Boolean algebra. It is called the *Boolean part* of \mathfrak{A} . We shall justify a consequence of these three axioms with the phrase *by Boolean algebra*. The Boolean operation of multiplication \cdot is defined in the usual way in terms of addition and complement. An element x in \mathfrak{A} is called a *subidentity element* if it is below the identity element, in symbols $x \leq 1'$.

Whenever parentheses indicating the order of performing operations are lacking, it is understood that unary operations have priority over binary operations, and multiplications have priority over addition.

Lemma 2.1. *The operation of converse is an automorphism of the Boolean part of a relation algebra. In particular, the following laws hold.*

- (i) $1^{\smile} = 1, \quad 0^{\smile} = 0, \quad 1'^{\smile} = 1'$.
- (ii) $(a \cdot b)^{\smile} = a^{\smile} \cdot b^{\smile}$.
- (iii) $(-a)^{\smile} = -(a^{\smile})$.
- (iv) $a \leq b$ if and only if $a^{\smile} \leq b^{\smile}$.
- (v) $a = 0$ if and only if $a^{\smile} = 0$.
- (vi) a is an atom if and only if a^{\smile} is an atom.

(vii) $x^\smile = x$ whenever x is a subidentity element.

Lemma 2.2. (i) $a ; 0 = 0 ; a = 0$.

(ii) $1 ; 1 = 1$.

(iii) If $a \leq b$ and $c \leq d$, then $a ; c \leq b ; d$.

(iv) $(a ; b) \cdot c = 0$ if and only if $(a^\smile ; c) \cdot b = 0$, if and only if $(c ; b^\smile) \cdot a = 0$.

(v) If a, b , and c are atoms, then

$$c \leq a ; b \text{ if and only if } b \leq a^\smile ; c, \text{ if and only if } a \leq c ; b^\smile.$$

(vi) $a \leq a ; 1$.

(vii) $(a ; 1) \cdot (1 ; b) = a ; 1 ; b$.

(viii) If x and y are subidentity atoms, then

$$1 ; x ; 1 = 1 ; y ; 1 \text{ if and only if } x ; 1 ; y \neq 0.$$

The laws in Lemma 2.1(iv) and Lemma 2.2(iii) are known as the *monotony laws* for converse and relative multiplication respectively. In referring to one of these laws to justify a step in some proof, we shall usually just say *by monotony*. The equivalences in Lemma 2.2(iv),(v) are usually called the *cycle laws* and the *cycle laws for atoms*, respectively—as opposed to the cycle law, which is (R11) and which is just one of the implications in (iv). Again, in using these equivalences to justify some step in a proof, we shall usually just say *by the cycle laws*.

The operations of relative multiplication and converse are *completely distributive* over addition in the sense that for any two sets of elements X and Y , if the sums $\sum X$ and $\sum Y$ exist, then the sums of the sets

$$X ; Y = \{a ; b : a \in X \text{ and } b \in Y\}, \quad Y^\smile = \{b^\smile : b \in Y\}$$

exist, and

$$(\sum X) ; (\sum Y) = \sum X ; Y, \quad \sum Y^\smile = (\sum Y)^\smile.$$

In referring to one of these laws to justify a step in some proof, we shall usually just say *by complete distributivity*.

The *domain* and *range* of an element a are defined to be the elements $(a ; 1) \cdot 1'$ and $(1 ; a) \cdot 1'$ respectively. Notice that they are subidentity elements. Every law about domains has a corresponding dual law about ranges. Therefore, only the law concerning domains will usually be given.

Lemma 2.3. Let x, y , and z be subidentity atoms. Every non-zero element $a \leq x ; 1 ; y$ has x as its domain and y as its range, and consequently the following laws hold.

(i) $x = (a ; a^\smile) \cdot 1' = (a ; 1) \cdot 1'$ and $y = (a^\smile ; a) \cdot 1' = (1 ; a) \cdot 1'$.

(ii) $x ; 1 = a ; 1$ and $1 ; y = 1 ; a$.

(iii) $x ; a = a$ and $a ; y = a$.

(iv) If $a \leq x ; 1 ; y$ and $b \leq y ; 1 ; z$, then $a ; b = 0$ if and only if $a = 0$ or $b = 0$. In particular, if $a \neq 0$, then $a ; a^\smile \neq 0$ and $a^\smile ; a \neq 0$.

An element f is called a *function*, or a *functional element*, if $f^\smile ; f \leq 1'$. If the converse of a function f is also a function, then f is said to be *bijective*. The element f is a *permutation*, or a *permutational element*, with domain x if it is bijective and if its domain and range are x .

Lemma 2.4. Let f, g be functions, and a, b arbitrary elements.

- (i) $f ; (a \cdot b) = (f ; a) \cdot (f ; b)$ and $(a \cdot b) ; f^\smile = (a ; f^\smile) \cdot (b ; f^\smile)$.
- (ii) $f ; g$ is a function.
- (iii) If $a \leq f$, then a is also a function.
- (iv) If f and g are bijective, then so are f^\smile and $f ; g$.
- (v) If f and g are permutations with domain x , then so are f^\smile and $f ; g$. Consequently, the permutations with domain x form a group under the operations of relative multiplication and converse, with x as the identity element of the group.
- (vi) A function is an atom if and only if its domain is an atom.

Part (i) of the preceding lemma says that if the left-hand argument of a relative product is a function, or the right-hand argument is the converse of a function, then the operation of relative multiplication distributes over multiplication. We shall refer to this law as the *distributive law for functions*. It plays an extremely important role in this work.

A *square* is an element of the form $x ; 1 ; x$ for some subidentity element x , and a *rectangle* is an element of the form $x ; 1 ; y$ for some subidentity elements x and y . The elements x and y are sometimes referred to as the *sides* of the rectangle

Lemma 2.5. *Let x, y, z, w be subidentity elements.*

- (i) $(x ; 1 ; y) \cdot 1' = x \cdot y$.
- (ii) $(x ; 1 ; y) \cdot (w ; 1 ; z) = (x \cdot w) ; 1 ; (y \cdot z)$.
- (iii) $(x ; 1 ; y)^\smile = y ; 1 ; x$.
- (iv) $(x ; 1 ; y) ; (y ; 1 ; z) \leq x ; 1 ; z$, and equality holds whenever x, y , and z are atoms such that $x ; 1 ; y$ and $y ; 1 ; z$ are both non-zero.

3. MEASURABLE ATOMS

Throughout this and the next few sections, we assume that all elements belong to an arbitrary, but fixed, relation algebra \mathfrak{A} with universe A . In order not to have to worry about the existence of certain infinite sums, we assume that \mathfrak{A} is complete. This assumption in no way restricts the applicability of the main results of the paper.

Definition 3.1. A subidentity atom x is *measurable* if the square $x ; 1 ; x$ with side x is a sum of functions. If this square is actually the sum of finitely many functions, then x is said to be *finitely measurable*.

It turns out that the set of non-zero functions below the square $x ; 1 ; x$ of a measurable atom x coincides with the set of atoms below the square, and the cardinality of this set is a measure of the “size” of x .

Lemma 3.2. *If x is a measurable atom, then an element below the square $x ; 1 ; x$ is an atom if and only if it is non-zero function.*

Proof. Let x be a measurable atom, and F the set of functions below $x ; 1 ; x$. The definition of measurability implies that

$$(1) \quad x ; 1 ; x = \sum F.$$

If f is an atom below $x ; 1 ; x$, then

$$0 < f = f \cdot (x ; 1 ; x) = f \cdot (\sum F) = \sum \{f \cdot g : g \in F\},$$

by Boolean algebra and (1), so there must be a function g in F such that $f \cdot g$ is non-zero. But then $f \leq g$, because f is an atom. Any element below a function is itself a function, by Lemma 2.4(iii), so f is a non-zero function.

On the other hand, if f is a non-zero function below $x ; 1 ; x$, then the domain of f is x , by Lemma 2.3, and x is an atom, by assumption, so f is an atom, by Lemma 2.4(vi). \square

The non-zero functions below the square on a measurable atom actually form a group under the operations of relative multiplication and converse.

Lemma 3.3. *If x is a measurable atom, then the set of non-zero functions below the square $x ; 1 ; x$ coincides with the set of permutations with domain x . This set forms a group under the operations $;$ and \smile , with identity element x .*

Proof. If f is a non-zero function below $x ; 1 ; x$, then f is an atom, by Lemma 3.2. The converse of an atom is an atom, by Lemma 2.1(vi), so $f\smile$ is also an atom. Apply Lemma 3.2 again to conclude that $f\smile$ is a function, and therefore f is a bijection. The element x is assumed to be an atom, so every non-zero element below the square $x ; 1 ; x$ has domain and range x , by Lemma 2.3. In particular, the domain and range of f are both x , so f is a permutation with domain x .

If g is an arbitrary permutation with domain x , then g is an atom, and hence non-zero, by Lemma 2.4(vi). Furthermore,

$$g \leq (g ; 1) \cdot (1 ; g) = (x ; 1) \cdot (1 ; x) = x ; 1 ; x,$$

by Lemmas 2.2(vi), 2.3(ii), and 2.2(vii), so g is a non-zero function below $x ; 1 ; x$. Conclusion: the set of non-zero functions below $x ; 1 ; x$ coincides with the set of permutations with domain x . This last set is a group under the operations of relative multiplication and converse, with x as the identity element, by Lemma 2.4(v), so the same must be true of the set of non-zero functions below $x ; 1 ; x$. \square

The preceding lemma justifies the following definition.

Definition 3.4. The group of non-zero functions below the square on a measurable atom x is denoted by G_x . The cardinality of this group is called the *measure* of x .

The following corollary is an immediate consequence of Definitions 3.1 and 3.4

Corollary 3.5. *If x is a measurable atom, then $x ; 1 ; x = \sum G_x$.*

Lemma 3.6. *If x and y are distinct measurable atoms, then the groups G_x and G_y are disjoint.*

Proof. The squares $x ; 1 ; x$ and $y ; 1 ; y$ are disjoint, because

$$(x ; 1 ; x) \cdot (y ; 1 ; y) = (x \cdot y) ; 1 ; (x \cdot y) = 0 ; 1 ; 0 = 0,$$

by Lemmas 2.5(ii) and 2.2(i). The groups G_x and G_y consist of non-zero elements below these respective squares, so they can have no elements in common. \square

Fix two measurable atoms x and y , and let f be an element in G_x , that is to say, let f be a non-zero function below $x ; 1 ; x$. Define a mapping ϑ_f on the set

$$A(x ; 1 ; y) = \{a \in A : a \leq x ; 1 ; y\}$$

by stipulating that

$$\vartheta_f(a) = f ; a$$

for every a in $A(x; 1; y)$. The relative product $f; a$ is called the *left translation of a by f* , so ϑ_f maps every element in $A(x; 1; y)$ to its left translation by f .

Lemma 3.7. *Each mapping ϑ_f is a permutation of the set $A(x; 1; y)$, and the correspondence $(f, a) \mapsto \vartheta_f(a)$ defines a left action of the group G_x on the set $A(x; 1; y)$ in the sense that*

$$\vartheta_x(a) = a \quad \text{and} \quad \vartheta_g(\vartheta_f(a)) = \vartheta_{g;f}(a)$$

for all a in $A(x; 1; y)$.

Proof. Consider elements f and g in G_x , and a in $A(x; 1; y)$. We have

$$\vartheta_f(a) = f; a \leq (x; 1; x); (x; 1; y) \leq x; 1; y,$$

by the definition of ϑ_f , the assumptions on f and a , monotony, and Lemma 2.5(iv). Consequently, $\vartheta_f(a)$ belongs to the set $A(x; 1; y)$. Also,

$$(1) \quad \vartheta_g(\vartheta_f(a)) = g; (f; a) = (g; f); a = \vartheta_{g;f}(a),$$

by the definitions of ϑ_f , ϑ_g , and $\vartheta_{g;f}$, and the associative law; and

$$(2) \quad \vartheta_x(a) = x; a = a,$$

by the definition of ϑ_x and Lemma 2.3(iii). Thus, the correspondence

$$(f, a) \mapsto \vartheta_f(a)$$

does define a left action of the group G_x on the set $A(x; 1; y)$. It follows that

$$a = \vartheta_x(a) = \vartheta_{f^-;f}(a) = \vartheta_{f^-}(\vartheta_f(a)),$$

by (2), Lemma 3.3, and (1), and dually,

$$a = \vartheta_f(\vartheta_{f^-}(a)),$$

so that the mappings ϑ_{f^-} and ϑ_f are inverses of one another. In particular, they must be one-to-one and onto, and hence permutations of the set $A(x; 1; y)$. \square

The set $A(x; 1; y)$ is closed under the binary operations of addition $+$ and multiplication \cdot in \mathfrak{A} , and also under the unary relativized complement operation $-_{x;1;y}$ that is defined by

$$-_{x;1;y}a = (x; 1; y) \cdot (-a)$$

for all a in $A(x; 1; y)$, where $-a$ is the complement of a in \mathfrak{A} . Under these operations, the set $A(x; 1; y)$ becomes a Boolean algebra, and actually a relativization of the Boolean part of \mathfrak{A} . Notice that an element belonging to this relativization is an atom in \mathfrak{A} just in case it is an atom in the relativization.

Lemma 3.8. *The mapping ϑ_f is an automorphism of the relativized Boolean algebra*

$$(A(x; 1; y), +, \cdot, -_{x;1;y}).$$

In particular, an element $a \leq x; 1; y$ is an atom if and only if $f; a$ is an atom.

Proof. The mapping ϑ_f is a permutation of the set $A(x; 1; y)$, by Lemma 3.7. The distributive law (R8) implies that ϑ_f preserves the operation of addition,

$$\vartheta_f(a + b) = f; (a + b) = f; a + f; b = \vartheta_f(a) + \vartheta_f(b).$$

The distributive law for functions, Lemma 2.4(i), and the assumption that f is a function, imply that ϑ_f preserves multiplication,

$$\vartheta_f(a \cdot b) = f ; (a \cdot b) = (f ; a) \cdot (f ; b) = \vartheta_f(a) \cdot \vartheta_f(b).$$

The element 0 is mapped to itself,

$$\vartheta_f(0) = f ; 0 = 0,$$

by Lemma 2.2(i). Finally, ϑ_f maps the unit $x ; 1 ; y$ of the relativization to itself,

$$\vartheta_f(x ; 1 ; y) = f ; x ; 1 ; y = f ; 1 ; y = x ; 1 ; y,$$

by the definition of ϑ_f , and Lemmas 3.3 and 2.3(ii). The operation of complement in the relativization can be defined in terms of addition and multiplication, with the help of the elements 0 and $x ; 1 ; y$, so ϑ_f must also preserve the operation of complement in the relativization. Conclusion: ϑ_f is an automorphism of the relativization. Automorphisms obviously map atoms to atoms, so the second assertion of the lemma follows at once from the first one, together with the remarks preceding the lemma. \square

One of the main points of Lemma 3.8 is that left translation by an element f in the group G_x maps the set of atoms of \mathfrak{A} that are below $x ; 1 ; y$ bijectively to itself.

Definition 3.9. For each element $a \leq x ; 1 ; y$, the *left stabilizer* of a in G_x under the group action of left translation is defined to be the set

$$\{f \in G_x : f ; a = a\}.$$

It will be denoted by H_a .

The next corollary is an immediate consequence of Lemma 3.7 and well-known basic facts about group actions.

Corollary 3.10. *For each $a \leq x ; 1 ; y$, the left stabilizer H_a is a subgroup of G_x . For any two elements f and g in G_x , we have $f ; a = g ; a$ if and only if f and g are in the same left coset of H_a .*

The preceding corollary implies that all elements in a coset H_ξ of H_a give rise to the same left translation of a . Write $H_\xi ; a$ to denote this left translation. This notation helps to avoid the cumbersome task of specifying in advance a representative f of the coset H_ξ , and writing $f ; a$. In a similar vein, for any subset X of G_x , write

$$X ; a = \{f ; a : f \in X\}, \quad \text{so that} \quad \sum X ; a = \sum \{f ; a : f \in X\}.$$

In complete analogy with the definition of ϑ_f , for each element g in G_y one can define a mapping ψ_g that sends every element a below $x ; 1 ; y$ to its *right translation* by g ,

$$\psi_g(a) = a ; g.$$

The mapping ψ_g is a permutation of the set $A(x ; 1 ; y)$, and the correspondence $(g, a) \mapsto \psi_g(a)$ defines a right action of the group G_y on the set $A(x ; 1 ; y)$. The mapping ψ_g is an automorphism of the relativized Boolean algebra corresponding to $A(x ; 1 ; y)$. In particular, an element $a \leq x ; 1 ; y$ is an atom if and only if $a ; g$ is an atom. The *right stabilizer* of an element $a \leq x ; 1 ; y$ under the group action of right translation is defined to be the set

$$\{g \in G_y : a ; g = a\},$$

and is denoted by K_a . The right stabilizer proves to be a subgroup of G_y , and for any two elements f and g in G_y , the right translations $a ; f$ and $a ; g$ are equal just in case f and g are in the same right coset of K_a . Consequently, if K_η is a right coset of K_a , then it makes sense to write $a ; K_\eta$ to denote the uniquely determined element that is the right translation of a by elements in K_η .

Every result about left translations and left stabilizers has a corresponding dual result about right translations and right stabilizers. In general, we will usually formulate only the left-hand version, while allowing ourselves to refer to the right-hand versions in later proofs that require it. The following easy lemma gives an example.

Lemma 3.11. *Let $x, y,$ and z be measurable atoms. If $a \leq x ; 1 ; y$ and $b \leq y ; 1 ; z$, then $H_a \subseteq H_{a;b}$.*

Proof. If f is in H_a , then $f ; a = a$, by the definition of H_a , and therefore

$$f ; a ; b = a ; b.$$

It follows that f is in $H_{a;b}$. □

4. LEFT-REGULAR AND RIGHT-REGULAR ELEMENTS

A special class of elements called regular elements plays an important role in the subsequent discussion. As will be seen, the prototypical regular element is an atom. More generally, if a is an atom below $x ; 1 ; y$, and if M is a subgroup of G_x that extends the left stabilizer H_a of a , then the element

$$b = \sum M ; a = \sum \{f ; a : f \in M\}$$

is a regular element.

Many of the properties of regular elements hold for broader classes of elements called left-regular elements and right-regular elements respectively. We begin with a study of these elements.

Definition 4.1. Let x and y be measurable atoms. An element $a \leq x ; 1 ; y$ is called *left-regular* or *right-regular* respectively, according to whether

$$a ; a^\smile = \sum H_a \quad \text{or} \quad a^\smile ; a = \sum K_a,$$

and a is called *regular* if it is both left and right-regular. □

The next lemma and the remarks following it are intended to clarify this definition.

Lemma 4.2. *Let x and y be measurable atoms. For any elements a and b below $x ; 1 ; y$, there are uniquely determined sets $E \subseteq G_x$ and $F \subseteq G_y$ such that*

$$a ; b^\smile = \sum E \quad \text{and} \quad a^\smile ; b = \sum F.$$

If $a = b \neq 0$, then E and F contain x and y respectively and are closed under the operation of converse.

Proof. Assume

$$(1) \quad 0 \leq a, b \leq x ; 1 ; y.$$

Use (1), Lemma 2.1(i), monotony, and Lemma 2.5(iii) to obtain

$$(2) \quad 0 \leq b^\smile \leq (x ; 1 ; y)^\smile = y ; 1 ; x.$$

Use (1), (2), monotony, and Lemma 2.5(iv) to arrive at

$$(3) \quad 0 \leq a; b^\smile \leq (x; 1; y); (y; 1; x) \leq x; 1; x.$$

The set $A(x; 1; x)$ of elements below $x; 1; x$ is a Boolean algebra with unit $x; 1; x$, under the operations of addition and complement relativized to $x; 1; x$, by the remarks preceding Lemma 3.8. The unit $x; 1; x$ is the sum of the set G_x , by Corollary 3.5 and the assumption that x is measurable. Moreover, G_x coincides with the set of atoms that are below $x; 1; x$, by Lemma 3.2. It follows that the relativized Boolean algebra $A(x; 1; x)$ is atomic, and its set of atoms is G_x . In an atomic Boolean algebra, each element is the sum of a uniquely determined set of atoms. Combine these remarks to conclude that each element below $x; 1; x$ is the sum of a uniquely determined subset of G_x . This applies in particular to the element $a; b^\smile$, by (3), so there must be a unique subset E of G_x such that

$$(4) \quad a; b^\smile = \sum E.$$

Assume next that $a = b \neq 0$, and observe that $a; a^\smile \neq 0$, by (1), (2) (with a in place of b), and Lemma 2.3(iv). The element $a; a^\smile$ is left fixed by converse, because

$$(5) \quad (a; a^\smile)^\smile = a^{\smile\smile}; a^\smile = a; a^\smile,$$

by the involution laws (R7) and (R6). Consequently,

$$(6) \quad \sum E = a; a^\smile = (a; a^\smile)^\smile = (\sum E)^\smile = \sum \{f^\smile : f \in E\},$$

by (4) (with a in place of b), (5), (4), and complete distributivity. Two sums of sets of atoms are equal just in case the sets themselves are equal, so (6) implies that

$$E = \{f^\smile : f \in E\}.$$

Thus, the set E is closed under converse. The element a is non-zero, by assumption, and below $x; 1; y$, by (1), so it has as its domain the atom x , by Lemma 2.3. It follows that $x \leq a; a^\smile$, by the assumption on x and Lemma 2.3(i), and therefore x is in E , by the definition of E .

The proof for $a^\smile; b$ is just the dual of the preceding argument. \square

Fix an element a below $x; 1; y$. Because of the preceding lemma there is always a unique set of atoms $E \subseteq G_x$ such that $a; a^\smile = \sum E$. In what follows, this set will be denoted by X_a . Similarly, there is a unique set of atoms $F \subseteq G_y$ such that $a^\smile; a = \sum F$, and this set will be denoted by Y_a . This notation and the definitions of left- and right-regular elements immediately imply the following corollary.

Corollary 4.3. *An element $a \leq x; 1; y$ is left-regular or right-regular if and only if $X_a = H_a$ or $Y_a = K_a$ respectively.*

Lemma 4.4. *A left-regular or right-regular element is always non-zero.*

Proof. The stabilizer H_0 of the zero element 0 is G_x , because $f; 0 = 0$ for all $f \in G_x$, by Lemma 2.2(i). Notice that this stabilizer is not empty since it contains, for example, the element x . On the other hand,

$$0; 0^\smile = 0 = \sum \emptyset,$$

by Lemma 2.2(i), so the set X_0 of atoms below $0; 0^\smile$ is empty. It follows that the sets H_0 and X_0 cannot be equal, and therefore 0 cannot be left-regular, by Corollary 4.3. A dual argument proves the corresponding result for right-regular elements. \square

In the remainder of this section, we shall usually only formulate lemmas and theorems for left-regular element, leaving the formulations and proofs of the dual results for right-regular elements to the reader. When there is a need to refer to such a result, we shall simply refer to “the right-regular version of ...”.

Lemma 4.5. *Suppose x and y are measurable atoms, and $0 < a \leq x; 1; y$.*

- (i) *An element f in G_x belongs to X_a if and only if $(f; a) \cdot a \neq 0$.*
- (ii) *$H_a \subseteq H_{a; a^\sim} \subseteq X_a \subseteq G_x$.*

Proof. For any element f in G_x ,

$$\begin{aligned} f \in X_a & \quad \text{if and only if} & \quad f \leq a; a^\sim, \\ & \quad \text{if and only if} & \quad f \cdot (a; a^\sim) \neq 0, \\ & \quad \text{if and only if} & \quad (f; (a^\sim)^\sim) \cdot a \neq 0, \\ & \quad \text{if and only if} & \quad (f; a) \cdot a \neq 0, \end{aligned}$$

by the definition of X_a , the fact that f is an atom (by Lemmas 3.2, 3.3, and the definition of G_x), the cycle laws, and the first involution law. This proves (i).

To establish the first inclusion in (ii), recall that

$$a^\sim \leq (x; 1; y)^\sim = y; 1; x,$$

by the assumption on a , monotony, and Lemma 2.5(iii). Apply Lemma 3.11 (with a^\sim and x in place of b and z respectively) to arrive at the desired inclusion. To establish the second inclusion, assume that f is in $H_{a; a^\sim}$. The second assertion of Lemma 4.2 implies that x is below $a; a^\sim$. Consequently,

$$(1) \quad f = f; x \leq f; a; a^\sim = a; a^\sim = \sum X_a,$$

by Lemma 3.3, monotony, the assumption that f is in the stabilizer of $a; a^\sim$, and the definition of X_a . Since f is an atom, and X_a a set of atoms, it follows from (1) that f must belong to X_a . The final inclusion in (ii) is a consequence of the definition of X_a . \square

It is of some interest to conclude from Lemma 4.5 that the set X_a is a union of left cosets of H_a in G_x . Indeed, if f and g are in the same left coset of H_a , then $f; a = g; a$ by Corollary 3.10, and therefore

$$(f; a) \cdot a \neq 0 \quad \text{if and only if} \quad (g; a) \cdot a \neq 0.$$

It follows from this equivalence and part (i) of the preceding lemma that f is in X_a if and only if g is in X_a . In other words, if one element of a left coset of H_a belongs to X_a , then the entire coset is included in X_a .

Lemma 4.6. *Let x and y be measurable atoms. For each non-zero $a \leq x; 1; y$, the following are equivalent.*

- (i) *a is left-regular.*
- (ii) *For any f in G_x , either $f; a = a$ or $(f; a) \cdot a = 0$.*
- (iii) *For any f and g in G_x , either $f; a = g; a$ or $(f; a) \cdot (g; a) = 0$.*

Proof. Let f and g be elements of G_x , and a a non-zero element below $x; 1; y$. To establish the implication from (i) to (ii), assume that a is left-regular, and observe that

$$(1) \quad H_a = X_a,$$

by Corollary 4.3. If f is in H_a , then $f ; a = a$, by definition of the left stabilizer, and if f is not in H_a , then f is not in X_a , by (1), and consequently $(f ; a) \cdot a = 0$, by Lemma 4.5(i). Thus, (ii) holds.

To derive (iii) from (ii), observe first that

$$(2) \quad f ; a = g ; a \quad \text{if and only if} \quad g^\sim ; f ; a = a.$$

Indeed, if $f ; a = g ; a$, then

$$g^\sim ; f ; a = g^\sim ; g ; a = x ; a = a,$$

by Lemmas 3.3 and 2.3(iii). On the other hand, if $g^\sim ; f ; a = a$, then $g^\sim ; f$ is in the left stabilizer H_a of a , so that f and g must belong to the same left coset of H_a . Use Corollary 3.10 to conclude that $f ; a = g ; a$. From (2), (ii) applied to the element $g^\sim ; f$, and the cycle laws, it follows that

$$\begin{aligned} f ; a \neq g ; a & \quad \text{if and only if} & \quad g^\sim ; f ; a \neq a, \\ & \quad \text{if and only if} & \quad (g^\sim ; f ; a) \cdot a = 0, \\ & \quad \text{if and only if} & \quad (g ; a) \cdot (f ; a) = 0. \end{aligned}$$

The implication from (iii) to (ii) is trivial: just take g to be the element x , and use Lemma 2.3(iii).

Finally, to derive (i) from (ii), assume that (ii) holds. Certainly, H_a is included in the set X_a , by Lemma 4.5(ii). For the reverse inclusion, consider an element f in X_a . Use Lemma 4.5(i) to see that $(f ; a) \cdot a \neq 0$, and then invoke (ii) to obtain $f ; a = a$. This implies that f is in the left stabilizer H_a , so X_a is included in H_a . Thus, (1) holds, so a is left-regular, by Corollary 4.3. \square

The next corollary implies that in a measurable relation algebra, atoms are always regular elements. This will play a very important role in the proof of the representation theorem for measurable relation algebras.

Corollary 4.7. *Let x and y be measurable atoms. Every atom below $x ; 1 ; y$ is regular.*

Proof. Let a be an atom below $x ; 1 ; y$. For each f in G_x the left translation $f ; a$ is also an atom by Lemma 3.8, so $f ; a = a$ or $(f ; a) \cdot a = 0$. Apply Lemma 4.6 to conclude that a is left-regular. A dual argument, involving the version of Lemma 4.6 that applies to right-regular elements, shows that a is right-regular. Consequently, a is regular. \square

Lemma 4.8. *Let x and y be measurable atoms, and a and b left-regular elements below $x ; 1 ; y$. If $a \leq b$, then $H_a \subseteq H_b$.*

Proof. If $a \leq b$, then

$$(1) \quad \sum H_a = \sum X_a = a ; a^\sim \leq b ; b^\sim = \sum X_b = \sum H_b,$$

by Corollary 4.3 and the assumption that a is left-regular, the definition of X_a , the assumption that $a \leq b$ and monotony, the definition of X_b , and Corollary 4.3 and the assumption that b is left-regular. The desired inclusion follows from (1) and the fact that H_a and H_b are sets of atoms. \square

Suppose x and y are measurable atoms and $0 < a \leq x ; 1 ; y$. Fix a left coset system in G_x of the left stabilizer H_a , and denote it by $\langle H_{a,\xi} : \xi < \kappa_a \rangle$ (where κ_a is, say, an ordinal number that is defined to coincide with the set of its predecessors).

When no confusion will arise, we shall drop the reference to a and write simply $\langle H_\xi : \xi < \kappa \rangle$. Similarly, fix a right coset system in G_y of the right stabilizer K_a , and denote it by $\langle K_{a,\eta} : \eta < \lambda_a \rangle$ or simply by $\langle K_\eta : \eta < \lambda \rangle$. The next lemma uses the notation $H_\xi ; a$ that was introduced after Corollary 3.10 to denote the element $f ; a$ for f in H_ξ .

Lemma 4.9 (First Partition Lemma). *Let x and y be measurable atoms. If $a \leq x ; 1 ; y$ is left-regular, then $\langle H_\xi ; a : \xi < \kappa \rangle$ forms a partition of $x ; 1 ; y$.*

Proof. It must be shown that the elements $H_\xi ; a$ are non-zero, pairwise disjoint, and sum to $x ; 1 ; y$. They are pairwise distinct by Corollary 3.10, and therefore pairwise disjoint by Lemma 4.6(iii). The left-regular element a is non-zero, by Lemma 4.4, so its translation $H_\xi ; a$ is non-zero by Lemma 3.8. Finally,

$$\begin{aligned} x ; 1 ; y &= x ; 1 ; a = x ; 1 ; x ; a = \left(\sum G_x \right) ; a \\ &= \sum \{ h ; a : h \in G_x \} = \sum \{ H_\xi ; a : \xi < \kappa \}, \end{aligned}$$

by Lemmas 2.3(ii),(iii), Corollary 3.5, and complete distributivity. \square

Corollary 4.10. *Let x and y be measurable atoms, and $a \leq x ; 1 ; y$ a left-regular element. If X and Y are unions of left cosets of H_a , then*

$$\sum X ; a \leq \sum Y ; a \quad \text{if and only if} \quad X \subseteq Y,$$

and consequently,

$$\sum X ; a = \sum Y ; a \quad \text{if and only if} \quad X = Y.$$

Proof. The proof of the implication from right to left in the first assertion is trivial. To prove the reverse implication, assume that

$$(1) \quad \sum X ; a \leq \sum Y ; a,$$

and consider a left coset H_ξ that is included in X . Obviously,

$$H_\xi ; a \leq \sum X ; a \leq \sum Y ; a,$$

by (1) and the fact that the element $H_\xi ; a$ belongs to the set $X ; a$. Two left translations of a by cosets of H_a are either equal or disjoint, by Lemma 4.9, so there must be a left coset H_η included in Y such that $H_\xi ; a = H_\eta ; a$. Distinct left cosets of H_a give rise to disjoint left translations of a , again by Lemma 4.9, so $\xi = \eta$. Thus, every left coset of H_a that is included in X is also included in Y , and therefore X must be included in Y . This proves the first assertion of the corollary. The second is an immediate consequence of the first. \square

A sense of the importance of Partition Lemma 4.9 can be gained from the following consequence.

Lemma 4.11 (Atomic Partition Lemma). *Let x and y be measurable atoms. If $a \leq x ; 1 ; y$ is an atom, then $\langle H_\xi ; a : \xi < \kappa \rangle$ is a listing of the distinct atoms below $x ; 1 ; y$ and these atoms sum to $x ; 1 ; y$. Consequently, every element below $x ; 1 ; y$ is the sum of a unique subset of these atoms.*

Proof. An atom a below $x ; 1 ; y$ is a regular element, by Corollary 4.7, so the elements in the system

$$(1) \quad \langle H_\xi ; a : \xi \leq \kappa \rangle$$

form a partition of $x;1;y$, by Partition Lemma 4.9. In particular, they are mutually disjoint and sum to the unit $x;1;y$ of the relativized Boolean algebra $A(x;1;y)$. Moreover, they are all atoms, by Lemma 3.8. It follows by Boolean algebra that the relativized Boolean algebra $A(x;1;y)$ is atomic, and therefore each of its elements is the sum of a unique set of atoms from (1). \square

The preceding lemma says that if there is an atom a below $x;1;y$, then every left translation of a is again an atom and these atoms partition $x;1;y$. The same is of course true for the right translations of a .

Suppose x and y are measurable atoms, and a and b left-regular elements below $x;1;y$ with $a \leq b$. By Lemma 4.8, the left stabilizer H_a is a subgroup of the left stabilizer H_b . Let

$$\langle H_{a,\xi} : \xi < \kappa \rangle \quad \text{and} \quad \langle H_{b,\eta} : \eta < \lambda \rangle$$

be left coset systems for H_a and H_b respectively in G_x . As is well known from group theory, there must be a partition $\langle \Gamma_\eta : \eta < \lambda \rangle$ of the index set $\{\xi : \xi < \kappa\}$ such that

$$H_{b,\eta} = \bigcup \{H_{a,\xi} : \xi \in \Gamma_\eta\}$$

for each $\eta < \lambda$. The next lemma, a generalization of the First Partition Lemma, refers to these assumptions.

Lemma 4.12 (Second Partition Lemma). *Let x and y be measurable atoms, and a and b left-regular elements below $x;1;y$. If $a \leq b$, then $\langle H_{a,\xi} ; a : \xi \in \Gamma_\eta \rangle$ is a partition of $H_{b,\eta} ; b$, and in particular,*

$$H_{b,\eta} ; b = \sum H_{b,\eta} ; a = \sum \{H_{a,\xi} ; a : \xi \in \Gamma_\eta\}$$

for each $\eta < \lambda$.

Proof. By assumption, $a \leq b$. Also, for each ξ in Γ_η , the left coset $H_{a,\xi}$ is a subset of the left coset $H_{b,\eta}$, by the remarks preceding the lemma. Use monotony and Corollary 3.10 to obtain

$$(1) \quad H_{a,\xi} ; a \leq H_{a,\xi} ; b = H_{b,\eta} ; b.$$

The system $\langle H_{b,\eta} ; b : \eta < \lambda \rangle$ is a partition of $x;1;y$, by Partition Lemma 4.9, so the elements $H_{b,\zeta} ; b$ and $H_{b,\eta} ; b$ are disjoint for indices $\zeta, \eta < \lambda$ with $\zeta \neq \eta$. Consequently, for ξ in Γ_ζ ,

$$(2) \quad (H_{a,\xi} ; a) \cdot (H_{b,\eta} ; b) \leq (H_{a,\xi} ; b) \cdot (H_{b,\eta} ; b) = (H_{b,\zeta} ; b) \cdot (H_{b,\eta} ; b) = 0,$$

by (1) (with ζ in place of η) and Boolean algebra. The system

$$(3) \quad \langle H_{a,\xi} ; a : \xi < \kappa \rangle$$

is also a partition of $x;1;y$, by Lemma 4.9, so the elements of this system are non-zero, mutually disjoint, and

$$\sum \{H_{a,\xi} ; a : \xi < \kappa\} = x;1;y.$$

Multiply both sides of this last equation by $H_{b,\eta} ; b$, and use (1), (2), and Boolean algebra to obtain

$$(4) \quad \sum \{H_{a,\xi} ; a : \xi \in \Gamma_\eta\} = H_{b,\eta} ; b.$$

Conclusion: $\langle H_{a,\xi} : \xi \in \Gamma_\eta \rangle$ is a partition of $H_{b,\eta} ; b$.

The second assertion of the lemma is an almost immediate consequence of the first:

$$H_{b,\eta}; b = \sum\{H_{a,\xi}; a : \xi \in \Gamma_\eta\} = \sum(\bigcup\{H_{a,\xi} : \xi \in \Gamma_\eta\}); a = \sum H_{b,\eta}; a,$$

by (4), complete distributivity, and the remarks preceding the lemma. \square

Non-zero products of regular elements play an important role in the analysis of the behavior of regular elements.

Lemma 4.13 (First Product Lemma). *Let x and y be measurable atoms, and a and b left-regular elements below $x; 1; y$. If $a \cdot b \neq 0$, then $a \cdot b$ is left-regular and $H_{a \cdot b} = H_a \cap H_b$.*

Proof. Observe that

$$(a \cdot b); (a \cdot b)^\sim = (a \cdot b); (a^\sim \cdot b^\sim) \leq (a; a^\sim) \cdot (b; b^\sim),$$

by Lemma 2.1(ii) and monotony, so

$$(1) \quad \sum X_{a \cdot b} \subseteq (\sum X_a) \cdot (\sum X_b),$$

by the definitions of the sets $X_{a \cdot b}$, X_a , and X_b . Since these are all sets of atoms, it follows from (1) by Boolean algebra that

$$(2) \quad X_{a \cdot b} \subseteq X_a \cap X_b.$$

Use Lemma 4.5(ii) (with $a \cdot b$ in place of a), the assumption that $a \cdot b \neq 0$, (2), the assumed left-regularity of a and b , and Corollary 4.3 to obtain

$$(3) \quad H_{a \cdot b} \subseteq X_{a \cdot b} \subseteq X_a \cap X_b = H_a \cap H_b.$$

On the other hand, if f is in $H_a \cap H_b$, then f is also in $H_{a \cdot b}$, because

$$f; (a \cdot b) = (f; a) \cdot (f; b) = a \cdot b,$$

by the distributive law for functions. Consequently,

$$(4) \quad H_a \cap H_b \subseteq H_{a \cdot b}.$$

Combine (3) with (4) to arrive at

$$H_{a \cdot b} = X_{a \cdot b} = H_a \cap H_b.$$

The left-regularity of the product $a \cdot b$ is an immediate consequence of the first of these equalities and Corollary 4.3 (with $a \cdot b$ in place of a). \square

Lemma 4.14. *Let x and y be measurable atoms, and a and b left-regular elements below $x; 1; y$. If $a \cdot b \neq 0$, then*

$$H_a \subseteq H_b \quad \text{if and only if} \quad a \leq b,$$

and consequently

$$H_a = H_b \quad \text{if and only if} \quad a = b.$$

Proof. Assume that $a \cdot b \neq 0$. If $a \leq b$, then H_a is included in H_b , by Lemma 4.8. To establish the reverse implication, suppose that $H_a \subseteq H_b$. Use this assumption, the assumption that $a \cdot b \neq 0$, and Product Lemma 4.13 to see that $a \cdot b$ is a left-regular element, and that

$$(1) \quad H_{a \cdot b} = H_a \cap H_b = H_a.$$

Let $\langle f_\xi : \xi < \kappa \rangle$ be a system of representatives for the left cosets of H_a in G_x . Partition Lemma 4.9 (with f_ξ in placed of H_ξ) says that

$$(2) \quad \langle f_\xi ; a : \xi < \kappa \rangle$$

is a partition of $x ; 1 ; y$. Now $\langle f_\xi : \xi < \kappa \rangle$ is also a system of representatives for the left cosets of $H_{a \cdot b}$, by (1), so

$$(3) \quad \langle f_\xi ; (a \cdot b) : \xi < \kappa \rangle$$

is a partition of $x ; 1 ; y$, by Lemma 4.9 and the left-regularity of $a \cdot b$. Since

$$(4) \quad f_\xi ; (a \cdot b) \leq f_\xi ; a$$

for each ξ , by monotony, the partitions in (2) and (3) must be equal. Therefore, equality must actually hold in (4). Use this observation and the distributive law for functions to obtain

$$f_\xi ; a = f_\xi ; (a \cdot b) = (f_\xi ; a) \cdot (f_\xi ; b).$$

It follows that

$$f_\xi ; a \leq f_\xi ; b.$$

Take $\xi = 0$ in this inequality, and use Lemma 2.3(iii), together with the convention that $f_0 = x$, to arrive at $a \leq b$. This completes the proof of the first equivalence in the lemma.

The second equivalence is an immediate consequence of the first. \square

The preceding lemma leads naturally to the question, for two left-regular elements a and b below $x ; 1 ; y$, when is the product $a \cdot b$ non-zero? A necessary and sufficient condition for this to happen is given below in the Second Product Lemma.

The next lemma says that any translation, left or right, of a left-regular element a is again left-regular, and the left stabilizer of such a translation can be computed from the left stabilizer of a .

Lemma 4.15 (First Translation Lemma). *Let x and y be measurable atoms, and $a \leq x ; 1 ; y$ a left-regular element.*

- (i) *For every f in G_x , the left translation $f ; a$ is left-regular, and its left stabilizer is*

$$H_{f;a} = f ; H_a ; f^\sim.$$

If H_a is a normal subgroup of G_x , then $H_{f;a} = H_a$.

- (ii) *For every element g in G_y , the right translation $a ; g$ is left-regular and its left stabilizer is $H_{a;g} = H_a$.*

Proof. Consider an element f in G_x . Use the definition of the set $X_{f;a}$, the second involution law and the associative law for relative multiplication, the definition of the set X_a , the assumed left-regularity of the element a , together with Corollary 4.3, and complete distributivity to obtain

$$(1) \quad \begin{aligned} \sum X_{f;a} &= (f ; a) ; (f ; a)^\sim = f ; a ; a^\sim ; f^\sim = f ; (\sum X_a) ; f^\sim \\ &= f ; (\sum H_a) ; f = \sum (f ; H_a ; f^\sim). \end{aligned}$$

It is easy to check, and it follows from Lemma 3.7 and group theory, that the left stabilizer of the left translation $f ; a$ is the subgroup

$$(2) \quad H_{f;a} = f ; H_a ; f^\sim.$$

Combine (1) with (2) to arrive at

$$\sum X_{f;a} = \sum H_{f;a}.$$

Since $X_{f;a}$ and $H_{f;a}$ are both sets of atoms, the preceding equation implies that the two sets must be equal. Use Corollary 4.3 (with $f ; a$ in place of a) to conclude that $f ; a$ is left-regular. If, in addition, H_a is a normal subgroup of G_x , then this subgroup must coincide with $f ; H_a ; f^\sim$, and therefore also with $H_{f;a}$, by (2). This proves (i).

To prove (ii), assume that g is in G_y . Use the definition of the set $X_{a;g}$, the second involution law and the associative law for relative multiplication, Lemma 3.3 (with y in place of x), Lemma 2.3(iii), and the left-regularity of a to arrive at

$$\sum X_{a;g} = (a ; g) ; (a ; g)^\sim = a ; g ; g^\sim ; a^\sim = a ; y ; a^\sim = a ; a^\sim = \sum H_a.$$

Since $X_{a;g}$ and H_a are sets of atoms, we may conclude from this computation that

$$(3) \quad X_{a;g} = H_a.$$

Use Lemmas 3.3 and 2.3(iii) to obtain

$$\begin{aligned} f ; a ; g = a ; g & \quad \text{if and only if} & \quad f ; a ; g ; g^\sim = a ; g ; g^\sim, \\ & \quad \text{if and only if} & \quad f ; a ; y = a ; y, \\ & \quad \text{if and only if} & \quad f ; a = a. \end{aligned}$$

These equivalences show that f belongs to the left stabilizer $H_{a;g}$ if and only if it belongs to the left stabilizer H_a , so that

$$(4) \quad H_{a;g} = H_a.$$

Combine (3) and (4) to arrive at

$$X_{a;g} = H_{a;g} = H_a,$$

and use these equalities together with Corollary 4.3 (with $a ; g$ in place of a) to conclude that $a ; g$ is left-regular and its left stabilizer is H_a . \square

The following corollary is a very important consequence of Atomic Partition Lemma 4.11 and Translation Lemma 4.15.

Corollary 4.16. *Let x and y be measurable atoms. If $a \leq x ; 1 ; y$ is an atom, then its left and right stabilizers H_a and K_a are normal subgroups of G_x and G_y respectively. If $b \leq x ; 1 ; y$ is also an atom, then $H_b = H_a$ and $K_b = K_a$.*

Proof. Let a and b be arbitrary atoms below $x ; 1 ; y$. The version of Lemma 4.11 for right-regular elements says that the right translations of a constitute all of the atoms below $x ; 1 ; y$. In particular, there must be an element g in G_y such that $b = a ; g$. Use this equality and part (ii) of Lemma 4.15 to obtain

$$(1) \quad H_b = H_{a;g} = H_a.$$

For any element f in G_x , the left translation $f ; a$ is an atom below $x ; 1 ; y$, by Lemma 4.11. Take this element for b in (1), and use part (i) of Lemma 4.15 to arrive at

$$(2) \quad H_a = H_{f;a} = f ; H_a ; f^\sim.$$

The equality of the first and last terms in (2) for every f in G_x implies that the subgroup H_a is normal in G_x , and (1) implies that every atom $b \leq x ; 1 ; y$ has the

same left stabilizer as the atom a . A dual argument yields the corresponding result for the right stabilizer K_a . \square

If a left-regular element $a \leq x ; 1 ; y$ has a normal left stabilizer, then part (i) of Translation Lemma 4.15 implies that any left translation of a is a left-regular element with the same left stabilizer as a . The next lemma implies that any left-regular element below $x ; 1 ; y$ with the same left stabilizer as a must in fact be a left translation of a . Thus, the left translations of a are precisely the left-regular elements below $x ; 1 ; y$ with the same left stabilizer as a . In fact, this property characterizes left-regular elements with normal left-stabilizers.

Lemma 4.17 (Second Translation Lemma). *Let x and y be measurable atoms. For every left-regular element $a \leq x ; 1 ; y$, the following assertions are equivalent.*

- (i) H_a is a normal subgroup of G_x .
- (ii) For every left-regular element $b \leq x ; 1 ; y$, we have $H_b \subseteq H_a$ if and only if b is below some left translation of a .
- (iii) For every left-regular element $b \leq x ; 1 ; y$, we have $H_a \subseteq H_b$ if and only if b is above some left translation of a .
- (iv) For every left-regular element $b \leq x ; 1 ; y$, we have $H_b = H_a$ if and only if b is equal to some left translation of a .

Proof. Assume $a \leq x ; 1 ; y$ is left-regular. For each element f in G_x , the left translation $f ; a$ is left-regular, and

$$(1) \quad H_{f;a} = f ; H_a ; f^\sim,$$

by part (i) of Translation Lemma 4.15.

To establish the implication from (i) to each of (ii), (iii), and (iv), assume that the left stabilizer H_a is a normal subgroup, and use (1) to obtain

$$(2) \quad H_{f;a} = H_a.$$

Consider any left-regular element $b \leq x ; 1 ; y$. Partition Lemma 4.9 implies that

$$x ; 1 ; y = \sum \{f ; a : f \in G_x\},$$

and b is non-zero, by Lemma 4.4, so

$$(3) \quad b \cdot (f ; a) \neq 0$$

for some f in G_x . Use (2), and then use Lemma 4.14 (with b and $f ; a$ in place of a and b respectively) and (3), to arrive at

$$\begin{array}{lll} H_b \subseteq H_a & \text{if and only if} & H_b \subseteq H_{f;a}, \\ & \text{if and only if} & b \leq f ; a. \end{array}$$

A similar argument yields

$$\begin{array}{lll} H_a \subseteq H_b & \text{if and only if} & H_{f;a} \subseteq H_b, \\ & \text{if and only if} & f ; a \leq b. \end{array}$$

Combine these equivalences to conclude that

$$H_b = H_a \quad \text{if and only if} \quad f ; a = b.$$

To establish the implication from (ii) to (i), assume that (ii) holds, and consider an arbitrary element f in G_x . The element $b = f ; a$ is left-regular, by the initial observation of this proof, and obviously $b \leq f ; a$, so (ii) implies that

$$(4) \quad H_b \subseteq H_a.$$

Use (1), the choice of b , and (4) to see that

$$(5) \quad f ; H_a ; f^\sim = H_{f;a} = H_b \subseteq H_a.$$

The inclusion of the left side of (5) in the right side holds for all f in G_x , so H_a must be a normal subgroup of G_x .

The proof of the implication from (iii) to (i) is similar to the preceding argument, but uses the fact that the subgroup H_a is normal just in case

$$H_a \subseteq f ; H_a ; f^\sim$$

for every element f in G_x . The implication from (iv) to (i) is a consequence of the implication from (ii) to (i). \square

Product Lemma 4.13 has as a hypothesis that the product of the two left-regular elements a and b be non-zero. The next lemma gives necessary and sufficient conditions for this hypothesis to be satisfied, under the additional assumption that the left stabilizers are normal. It also characterizes the product subgroup $H_a ; H_b$ as the left stabilizer of a specific element. Recall from Lemma 4.13 that

$$H_{a \cdot b} = H_a \cap H_b,$$

so that the coset system for $H_{a \cdot b}$ coincides with the coset system for $H_a \cap H_b$, which is

$$\langle H_{a,\xi} \cap H_{b,\eta} : \xi < \kappa \text{ and } \eta < \lambda \rangle,$$

where

$$\langle H_{a,\xi} : \xi < \kappa \rangle \quad \text{and} \quad \langle H_{b,\eta} : \eta < \lambda \rangle$$

are respectively coset systems for H_a and H_b in $H_a ; H_b$.

Lemma 4.18 (Second Product Lemma). *Let x and y be measurable atoms, and a and b left-regular elements below $x ; 1 ; y$ with normal stabilizers H_a and H_b .*

- (i) $a \cdot b \neq 0$ if and only if $a ; a^\sim ; b = b ; b^\sim ; a$.
- (ii) If $a \cdot b \neq 0$, then the product subgroup $H_a ; H_b$ is the left stabilizer of the element $b ; b^\sim ; a$, and the system of left translations

$$\langle (H_{a,\xi} \cap H_{b,\eta}) ; (a \cdot b) : \xi < \kappa \text{ and } \eta < \lambda \rangle$$

is a partition of $b ; b^\sim ; a$, where $\langle H_{a,\xi} : \xi < \kappa \rangle$ and $\langle H_{b,\eta} : \eta < \lambda \rangle$ are cosets systems for H_a and H_b in $H_a ; H_b$. Different left translations of $a \cdot b$ coincide with the different products of the left translations of a and b in the sense that

$$(H_{a,\xi} \cap H_{b,\eta}) ; (a \cdot b) = (H_{a,\xi} ; a) \cdot (H_{b,\eta} ; b)$$

for every $\xi < \kappa$ and $\eta < \lambda$.

Proof. To prove (i), assume first that $a \cdot b \neq 0$. The product $a \cdot b$ is then a left regular element, by Product Lemma 4.13. Consequently,

$$(1) \quad a ; a^\sim ; (a \cdot b) = (\sum H_a) ; (a \cdot b) = \sum H_a ; (a \cdot b) = H_a ; a = a$$

by the assumed left-regularity of a , complete distributivity, the final assertion of Partition Lemma 4.12 (with $a \cdot b$ and a in place of a and b respectively, and H_a in

place of $H_{b,\eta}$), and the fact that H_a is the stabilizer of a . Use (1) and monotony to get

$$a \leq a; a^\sim; b.$$

Form the relative product of both sides of this inequality on the left with $b; b^\sim$, and then use monotony, the definition of a regular element and the assumed left-regularity of a and b , complete distributivity, the assumption that H_a is normal, the definition of H_b as the stabilizer of b , complete distributivity, and the left-regularity of a , to arrive at

$$\begin{aligned} b; b^\sim; a \leq b; b^\sim; a; a^\sim; b &= (\sum H_b); (\sum H_a); b = \sum H_b; H_a; b \\ &= \sum H_a; H_b; b = \sum H_a; b = (\sum H_a); b = a; a^\sim; b. \end{aligned}$$

A symmetric argument yields the reverse inequality. This establishes the implication from left to right in part (i).

To establish the reverse implication, assume that

$$(2) \quad a; a^\sim; b = b; b^\sim; a.$$

Use the definition of a left-regular element and the assumed left-regularity of b , complete distributivity, the definition of H_a as the stabilizer of a , and the assumption that this stabilizer is a normal subgroup to get

$$(3) \quad b; b^\sim; a = (\sum H_b); a = \sum H_b; a = \sum H_b; H_a; a = \sum H_a; H_b; a.$$

The product subgroup $H_a; H_b$ is the union of the cosets $H_{a,\xi}$ of H_a in $H_a; H_b$, by assumption, so

$$(4) \quad \sum H_a; H_b; a = \sum (\bigcup_\xi H_{a,\xi}); a = \sum_\xi H_{a,\xi}; a,$$

by complete distributivity. Also, the elements $H_{a,\xi}; a$ are non-zero and pairwise disjoint, by Partition Lemma 4.9. Combine this observation with (3) and (4) to see that

$$(5) \quad \langle H_{a,\xi}; a : \xi < \kappa \rangle$$

is a partition of $b; b^\sim; a$. A similar argument shows that

$$(6) \quad \langle H_{b,\eta}; b : \eta < \lambda \rangle$$

is a partition of $a; a^\sim; b$.

Use (2), Lemma 2.3(i), monotony, Lemma 2.3(iii), the left-regularity of b , and Lemma 4.4 to obtain

$$(7) \quad b; b^\sim; a = a; a^\sim; b \geq x; b = b > 0.$$

Since (5) is a partition of $b; b^\sim; a$, it follows from (7) that there must be an index $\gamma < \kappa$ such that

$$(H_{a,\gamma}; a) \cdot b \neq 0.$$

Put $\bar{a} = H_{a,\gamma}; a$ and write the preceding inequality as

$$(8) \quad \bar{a} \cdot b \neq 0.$$

The element \bar{a} is, by definition, a left translation of the left-regular element a with a normal left stabilizer, so \bar{a} is itself left-regular with the same normal left stabilizer H_a , by part (i) of Translation Lemma 4.15. In view of (8), Product Lemma 4.13 may be applied (with \bar{a} in place of a) to conclude that $\bar{a} \cdot b$ is left-regular with left stabilizer $H_a \cap H_b$.

As is well known from group theory, the normal subgroup $H_a \cap H_b$ has the coset system

$$(9) \quad \langle H_{a,\xi} \cap H_{b,\eta} : \xi < \kappa \text{ and } \eta < \lambda \rangle$$

in $H_a ; H_b$. Every left translation of a left-regular element by a left coset of its left stabilizer is again left-regular, by part (i) of Translation Lemma 4.15. In particular, each left translation

$$(H_{a,\xi} \cap H_{b,\eta}) ; (\bar{a} \cdot b)$$

of $\bar{a} \cdot b$ is left-regular, and therefore non-zero, by Lemma 4.4. Choose ξ so that $H_{a,\xi}$ is the coset inverse of $H_{a,\gamma}$ in the quotient group G_x/H_a , and use monotony, the definition of \bar{a} , the inverse property from group theory, and the definition of H_a as the left stabilizer of a to obtain

$$(10) \quad (H_{a,\xi} \cap H_{b,\eta}) ; (\bar{a} \cdot b) \leq H_{a,\xi} ; \bar{a} = H_{a,\xi} ; H_{a,\gamma} ; a = H_a ; a = a.$$

Similarly, take $\eta = 0$, so that $H_{b,\eta}$ coincides with the identity coset H_b , and use monotony and the definition of H_b to obtain

$$(11) \quad (H_{a,\xi} \cap H_{b,\eta}) ; (\bar{a} \cdot b) \leq H_{b,\eta} ; b = H_b ; b = b.$$

Form the products of the left and right sides of (10) and (11), and use Boolean algebra to arrive at

$$0 < (H_{a,\xi} \cap H_{b,\eta}) ; (\bar{a} \cdot b) \leq a \cdot b.$$

Conclusion: $a \cdot b \neq 0$, as was to be shown.

To prove (ii), assume that $a \cdot b \neq 0$. It follows from part (i) of the lemma that (2) also holds, so it makes sense to write

$$(12) \quad c = b ; b^\sim ; a = a ; a^\sim ; b.$$

The first task is to check that the product subgroup $H_a ; H_b$ coincides with the left stabilizer H_c . Consider an element h in G_x . If h is in the product subgroup, then there must be elements f in H_a and g in H_b such that $h = f ; g$, by the definition of the product subgroup. Consequently,

$$(13) \quad h ; c = h ; b ; b^\sim ; a = f ; g ; b ; b^\sim ; a = f ; b ; b^\sim ; a \\ = f ; a ; a^\sim ; b = a ; a^\sim ; b = b ; b^\sim ; a = c,$$

by (12), the assumptions on h , the assumption that g is in the left stabilizer of b , part (i) of the lemma, which implies that (2) holds, the assumption that f is in the left stabilizer of a , and (2) again. It follows from (13) that h belongs to the left stabilizer H_c .

On the other hand, if h belongs to H_c , then

$$(14) \quad h ; b ; b^\sim ; a = h ; c = c = b ; b^\sim ; a,$$

by (12) and the definition of H_c . Form the relative product of the left and right sides of (14), on the right, with a^\sim to obtain

$$(15) \quad h ; b ; b^\sim ; a ; a^\sim = b ; b^\sim ; a ; a^\sim.$$

Use complete distributivity, the left-regularity of a and b , (15), the left-regularity of a and b again, and complete distributivity to get

$$(16) \quad \sum H_b ; H_a = (\sum H_b) ; (\sum H_a) = b ; b^\sim ; a ; a^\sim = h ; b ; b^\sim ; a ; a^\sim \\ = h ; (\sum H_b) ; (\sum H_a) = h ; (\sum H_b ; H_a) = \sum h ; H_b ; H_a.$$

The sets involved in the first and last sums of (16) are sets of atoms, so the two sets must be equal. Combine this with the assumption that H_a is normal to arrive at

$$H_a ; H_b = H_b ; H_a = h ; H_b ; H_a = h ; H_a ; H_b.$$

Thus, the left coset of the product subgroup $H_a ; H_b$ determined by each element h belonging to the left stabilizer H_c coincides with $H_a ; H_b$, so each such h belongs to the product subgroup. Conclusion:

$$(17) \quad H_c = H_a ; H_b.$$

Product Lemma 4.13, and the assumption that a and b are left-regular elements with $a \cdot b \neq 0$ imply that $a \cdot b$ is a left-regular element with left stabilizer $H_a \cap H_b$. Recall that (9) is a coset system for this left stabilizer in $H_a ; H_b$. Consequently, H_c is the union of the cosets $H_{a,\xi} \cap H_{b,\eta}$ for $\xi < \kappa$ and $\eta < \lambda$, by (17). The assumption that $a \cdot b \neq 0$, together with Boolean algebra, (7), and (12), implies that

$$0 < a \cdot b \leq b \leq c,$$

In view of these observations, Partition Lemma 4.12 may be applied (with $a \cdot b$ and c in place of a and b respectively, and with $H_{a,\xi} \cap H_{b,\eta}$ and H_c in place of $H_{a,\xi}$ and $H_{b,\eta}$ respectively) to conclude that

$$(18) \quad \langle (H_{a,\xi} \cap H_{b,\eta}) ; (a \cdot b) : \xi < \kappa \text{ and } \eta < \lambda \rangle$$

is a partition of $H_c ; c$, and therefore a partition of c , by the definition of H_c as the left stabilizer of c . In particular, the elements in this system are non-zero and pairwise disjoint.

Monotony implies that

$$(H_{a,\xi} \cap H_{b,\eta}) ; (a \cdot b) \leq H_{a,\xi} ; a \quad \text{and} \quad (H_{a,\xi} \cap H_{b,\eta}) ; (a \cdot b) \leq H_{b,\eta} ; b,$$

and therefore

$$(19) \quad (H_{a,\xi} \cap H_{b,\eta}) ; (a \cdot b) \leq (H_{a,\xi} ; a) \cdot (H_{b,\eta} ; b),$$

by Boolean algebra. The elements on the left side of this inequality are left-regular and therefore non-zero, so the products on the right must also be non-zero. It has already been shown that, on the basis of (2), the systems of left translations in (5) and in (6) are both partitions of c . Combine this with the preceding observation, and use Boolean algebra, to conclude that the system

$$(20) \quad \langle (H_{a,\xi} ; a) \cdot (H_{b,\eta} ; b) : \xi < \kappa \text{ and } \eta < \lambda \rangle$$

is also a partition of c . Summarizing, (18) and (20) are both partitions of c . The inequality in (19) therefore implies that the two partitions must coincide, so that equality holds in (19). This completes the proof of (ii). \square

5. REGULAR ELEMENTS WITH NORMAL STABILIZERS

Suppose x and y are measurable atoms, and $a \leq x ; 1 ; y$ a regular element with normal stabilizers. The assumption on the stabilizers implies, in particular, that it is possible to form the quotient groups G_x/H_a and G_y/K_a . It turns out that the element a induces in a canonical fashion an isomorphism between these two quotients. To prove this, we begin with a lemma.

Lemma 5.1. *Let x and y be measurable atoms. If an element $a \leq x;1;y$ is regular with normal stabilizers, then for every f in G_x and every g in G_y ,*

$$(f; a) \cdot (a; g) \neq 0 \quad \text{if and only if} \quad f; a = a; g.$$

Proof. The implication from right to left is obvious, since translations of regular elements are regular, by Translation Lemma 4.15, and regular elements are never zero, by Lemma 4.4. To derive the reverse implication, assume that the hypotheses of the lemma are satisfied, and suppose that

$$(f; a) \cdot (a; g) \neq 0.$$

Both $f; a$ and $a; g$ are regular elements below $x;1;y$, by Translation Lemma 4.15 and its right-regular version, so the product

$$(2) \quad c = (f; a) \cdot (a; g)$$

must be a regular element below $x;1;y$, by the Product Lemma 4.13 and its right-regular version. Moreover, this product has the same left and right stabilizers as a , because

$$(3) \quad H_c = H_{f;a} \cap H_{a;g} = (f; H_a; f^\smile) \cap H_a = H_a,$$

by (2), Product Lemma 4.13, Translation Lemma 4.15, and the assumption that H_a is normal, and similarly,

$$K_c = K_{f;a} \cap K_{a;g} = K_a \cap (g^\smile; K_a; g) = K_a,$$

by the right-regular versions of Lemmas 4.13 and 4.15.

Use the assumption that H_a is normal, the regularity of c , (3), and the implication from (i) to (iv) in Translation Lemma 4.17 (with c in place of b) to get that c is a left translation of a , in symbols,

$$(4) \quad c = (f; a) \cdot (a; g) = h; a$$

for some h in G_x . In particular, $h; a \leq f; a$. Any two left translations of a are equal or disjoint, by Lemma 4.6(iii), so $h; a = f; a$. It follows from this equation, (4), and Boolean algebra that $f; a \leq a; g$. A dual argument using right-regularity establishes the reverse inequality. \square

Corollary 5.2. *Let x and y be measurable atoms. If $a \leq x;1;y$ is a regular element with normal stabilizers, then every right translation of a is also a left translation, and conversely.*

Proof. If an element a satisfies the hypotheses of the corollary, then a and all of its translations are regular elements below $x;1;y$, by Translation Lemma 4.15 and its right-regular version, and in particular they are not 0, by Lemma 4.4. Let g be any element in G_y . Since the left translations $f; a$, for f in G_x , sum to $x;1;y$, by Partition Lemma 4.9, there must be an element f in G_x such that

$$(f; a) \cdot (a; g) \neq 0.$$

Consequently,

$$(f; a) = (a; g),$$

by Lemma 5.1. In other words, the right translation $a; g$ can be written as a left translation $f; a$. The converse is proved in a similar way. \square

We turn now to the task of constructing a group triple in an atomic, measurable relation algebra, and verifying the semi-frame conditions for this triple. The definitions of these notions, which are from [1], are not needed in this section, but they are needed in the next section. We give them here in order to motivate the discussion. The reader may choose to ignore them for now and refer back to them at the appropriate moment in the next section. A *group triple*

$$\mathcal{F} = (G, \varphi, C)$$

consists of a system

$$G = \langle G_x : x \in I \rangle$$

of disjoint groups, a system

$$\varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle$$

of associated quotient isomorphisms, with φ_{xy} mapping a quotient group G_x/H_{xy} to a quotient group G_y/K_{xy} for each pair (x, y) in a fixed equivalence relation \mathcal{E} on the group index set I , and finally a system

$$C = \langle C_{xyz} : (x, y, z) \in \mathcal{E}_3 \rangle$$

of associated cosets, with C_{xyz} a coset of the normal subgroup $H_{xy} \circ H_{xz}$ in G_x for each triple (x, y, z) in the set \mathcal{E}_3 of triples (x, y, z) such that (x, y) and (y, z) are both in \mathcal{E} .

Definition 5.3. A group triple

$$\mathcal{F} = (G, \varphi, C)$$

is a (*coset*) *semi-frame* if the following *semi-frame conditions* are satisfied.

- (i) φ_{xx} is the identity automorphism of $G_x/\{e_x\}$ for all x in I .
- (ii) $\varphi_{yx} = \varphi_{xy}^{-1}$ whenever (x, y) is in \mathcal{E} .
- (iii) $\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}$ whenever (x, y, z) is in \mathcal{E}_3 .
- (iv) $\hat{\varphi}_{xy} | \hat{\varphi}_{yz} = \tau | \hat{\varphi}_{xz}$ whenever (x, y, z) is in \mathcal{E}_3 ,

where $\hat{\varphi}_{xy}$, $\hat{\varphi}_{yz}$, and $\hat{\varphi}_{xz}$ are the quotient isomorphisms induced on

$$G_x/(H_{xy} \circ H_{xz}), \quad G_y/(K_{xy} \circ H_{yz}), \quad G_x/(H_{xy} \circ H_{xz})$$

by φ_{xy} , φ_{yz} , φ_{xz} respectively, and τ is the inner automorphism of $G_x/(H_{xy} \circ H_{xz})$ determined by the coset C_{xyz} . \square

The global assumption of measurability is not needed in order to accomplish most of the task of constructing a group triple and verifying the semi-frame conditions, so for now we continue with the local assumptions that x and y are measurable atoms, and $a \leq x ; 1 ; y$ is a regular element with normal stabilizers.

Theorem 5.4 (First Isomorphism Theorem). *Let x and y be measurable atoms. If $a \leq x ; 1 ; y$ is a regular element with normal stabilizers, and in particular, if a is an atom, then the quotient groups G_x/H_a and G_y/K_a are isomorphic. In fact, if*

$$\langle H_\xi : \xi < \kappa \rangle \quad \text{and} \quad \langle K_\eta : \eta < \lambda \rangle$$

are coset systems of H_a in G_x and K_a in G_y respectively, then there is a unique bijection φ from κ to λ such that

$$H_\xi ; a = a ; K_{\varphi(\xi)},$$

and the mapping $H_\xi \mapsto K_{\varphi(\xi)}$ is the desired isomorphism.

Proof. Consider a regular element $a \leq x; 1; y$ with normal stabilizers. The left and right translations of a are just the elements in the systems

$$\langle H_\xi; a : \xi < \kappa \rangle \quad \text{and} \quad \langle a; K_\eta : \eta < \lambda \rangle$$

respectively. For any fixed $\xi < \kappa$, there is a $\xi' < \lambda$ such that

$$(1) \quad H_\xi; a = a; K_{\xi'},$$

by Corollary 5.2. If $\xi'' < \lambda$ is any other index such that

$$H_\xi; a = a; K_{\xi''},$$

then

$$a; K_{\xi'} = a; K_{\xi''},$$

and therefore $\xi' = \xi''$, because distinct cosets of K_a lead to disjoint right translations of a , by the right-regular version of Lemma 4.6(iii). Thus, there is a unique $\xi' < \lambda$ such that (1) holds. Define $\varphi(\xi)$ to be ξ' . The preceding remarks imply that φ is a well-defined mapping from κ into λ , and

$$(2) \quad H_\xi; a = a; K_{\varphi(\xi)}$$

for each $\xi < \kappa$, by (1) and the definition of φ .

In a completely analogous fashion, define a mapping ψ from λ into κ such that

$$(3) \quad a; K_\eta = H_{\psi(\eta)}; a$$

for each $\eta < \lambda$. In particular,

$$(4) \quad H_\xi; a = a; K_{\varphi(\xi)} = H_{\psi(\varphi(\xi))}; a,$$

by (2) and (3) (with $\varphi(\xi)$ in place of η). Distinct cosets of H_a lead to disjoint left translations of a , by Partiton Lemma 4.9, so $\psi(\varphi(\xi)) = \xi$ for each $\xi < \kappa$, by (4). A symmetric argument shows that $\varphi(\psi(\eta)) = \eta$ for each $\eta < \lambda$. It follows that φ must be a bijection from κ to λ , with ψ as its inverse. Consequently, the correspondence

$$(5) \quad H_\xi \longmapsto K_{\varphi(\xi)}$$

is a bijection from G_x/H_a to G_y/K_a .

Suppose that

$$(6) \quad H_\xi; H_\eta = H_\mu.$$

Use (2), (6), and two more applications of (2) to arrive at

$$(7) \quad a; K_{\varphi(\mu)} = H_\mu; a = H_\xi; H_\eta; a = H_\xi; a; K_{\varphi(\eta)} = a; K_{\varphi(\xi)}; K_{\varphi(\eta)}.$$

Distinct cosets of K_a lead to distinct right translations of a , by the right-regular version of Lemma 4.9, so (7) implies that

$$K_{\varphi(\xi)}; K_{\varphi(\eta)} = K_{\varphi(\mu)},$$

This argument shows that the mapping (5) preserves the quotient group composition operation of forming the relative product of two cosets.

In groups, the identity element and the operation of forming inverses are both definable in terms of the group composition operation, so any bijection that preserves composition is automatically a group isomorphism. In particular, it follows from the preceding observations that the mapping (5) must be an isomorphism between the quotient groups. \square

Convention 5.5. It is clear from the First Isomorphism Theorem that we may assume $\kappa = \lambda$, and we may reindex the coset system $\langle K_\xi : \xi < \kappa \rangle$ so that $\varphi(\xi) = \xi$, that is to say, so that

$$(1) \quad H_\xi \longmapsto K_\xi$$

is the canonical isomorphism from G_x/H_a to G_y/K_a determined by the equation

$$(2) \quad H_\xi ; a = a ; K_\xi.$$

Furthermore, it may be supposed that $H_0 = H_a$, and consequently that $K_0 = K_a$ (where 0 denotes an index zero, and not the zero element in the relation algebra \mathfrak{A} that has been fixed throughout the discussion.) In the subsequent development, these notational conventions shall be adopted. When an explicit reference to a is required, we write $H_{a,\xi}$ and $K_{a,\xi}$ for H_ξ and K_ξ respectively, and we denote the canonical isomorphism (1) by φ_a , and call it the *isomorphism determined by a* . \square

The particular isomorphism described in Isomorphism Theorem 5.4 is determined by the element a . Different translations of a may determine different isomorphisms. How are these various isomorphisms related to one other? Assume that b is, for example, a left translation of a , say $b = H_\eta ; a$. In this case, b is also a regular element with the same normal stabilizer as a , by Translation Lemma 4.15 and its right-regular version. Moreover, if φ_a maps H_ξ to K_ξ for each $\xi < \kappa$, as in Convention 5.5, or in different words, if equation (2) of the convention holds for each $\xi < \kappa$, then

$$(3) \quad b = H_\eta ; a = a ; K_\eta,$$

and therefore

$$(4) \quad \begin{aligned} H_\xi ; b &= H_\xi ; H_\eta ; a = H_\xi ; a ; K_\eta = a ; K_\xi ; K_\eta \\ &= a ; K_\eta ; K_\eta^\sim ; K_\xi ; K_\eta = b ; K_\eta^\sim ; K_\xi ; K_\eta, \end{aligned}$$

by (3) and (2). The fourth equality uses quotient group identity and inverse properties:

$$K_\xi = K_a ; K_\xi = K_\eta ; K_\eta^\sim ; K_\xi.$$

It follows from (4) and the definition of the isomorphism determined by b that

$$(5) \quad \varphi_b(H_\xi) = K_\eta^\sim ; K_\xi ; K_\eta$$

for each ξ .

If τ is the inner automorphism of G_x/H_a determined by

$$(6) \quad \tau(H_\xi) = H_\eta^\sim ; H_\xi ; H_\eta$$

for each ξ , then

$$(7) \quad \varphi_b = \tau | \varphi_a.$$

The verification of (7) is an easy computation:

$$\begin{aligned} (\tau | \varphi_a)(H_\xi) &= \varphi_a(\tau(H_\xi)) = \varphi_a(H_\eta^\sim ; H_\xi ; H_\eta) = \varphi_a(H_\eta)^\sim ; \varphi_a(H_\xi) ; \varphi_a(H_\eta) \\ &= K_\eta^\sim ; K_\xi ; K_\eta = \varphi_b(H_\xi), \end{aligned}$$

by the definition of relational composition, (6), the isomorphism properties of φ_a , the definition of φ_a , and (5). The isomorphism φ_b agrees with the isomorphism φ_a

on a coset H_ξ just in case the images of H_ξ under these two mappings are equal, that is to say, just in case

$$K_\xi = K_\eta^\sim ; K_\xi ; K_\eta,$$

or, equivalently, just in case

$$H_\xi = H_\eta^\sim ; H_\xi ; H_\eta.$$

This is the same as saying that H_η commutes with H_ξ .

The preceding argument proves the following theorem.

Theorem 5.6 (Second Isomorphism Theorem). *Let x and y be measurable atoms, and $a \leq x ; 1 ; y$ a regular element with normal stabilizers. If b is the left translation of a by a coset H_η of H_a , then φ_b is just the relational composition*

$$\varphi_b = \tau | \varphi_a,$$

where τ is the inner automorphism of G_x/H_a determined by

$$\tau(H_\xi) = H_\eta^\sim ; H_\xi ; H_\eta.$$

The isomorphism φ_b agrees with φ_a on precisely those cosets of H_a that commute with H_η .

The theorem allows us to characterize when the isomorphisms determined by a left translation b of a regular element a coincides with the isomorphism φ_a .

Corollary 5.7. *Under the hypotheses of the First Isomorphism Theorem, if b is the left translation of a by a coset H_η of H_a , then*

$$\varphi_b = \varphi_a$$

just in case H_η is in the center of G_x/H_a .

Corollary 5.8. *Under the hypotheses of the First Isomorphism Theorem, if G_x/H_a is abelian, then $\varphi_b = \varphi_a$ for every translation b of a .*

We now explore some of the important consequences of the Isomorphism Theorems. Let x and y be measurable atoms. A regular element $a \leq x ; 1 ; y$ with normal stabilizers determines an isomorphism φ_a from G_x/H_a to G_y/K_a , by Isomorphism Theorem 5.4. Another regular element $b \leq x ; 1 ; y$ with normal stabilizers (with b not necessarily a translation of a) determines an isomorphism φ_b from G_x/H_b to G_y/K_b . What is the relationship between these two isomorphisms when $a \leq b$? To answer this question, we use the notation and conventions discussed in Convention 5.5. In particular,

$$\langle H_{a,\xi} : \xi < \kappa \rangle \quad \text{and} \quad \langle H_{b,\eta} : \eta < \lambda \rangle$$

are assumed to be coset systems for H_a and H_b respectively in G_x , and

$$\langle K_{a,\xi} : \xi < \kappa \rangle \quad \text{and} \quad \langle K_{b,\eta} : \eta < \lambda \rangle$$

are assumed to be the corresponding coset systems for K_a and K_b respectively in G_y .

If $a \leq b$, then

$$(1) \quad H_a \subseteq H_b \quad \text{and} \quad K_a \subseteq K_b,$$

by Lemma 4.8 and its right-regular version, so there are partitions $\langle \Gamma_\eta : \eta < \lambda \rangle$ and $\langle \Delta_\eta : \eta < \lambda \rangle$ of κ such that

$$(2) \quad H_{b,\eta} = \bigcup \{H_{a,\xi} : \xi \in \Gamma_\eta\} \quad \text{and} \quad K_{b,\eta} = \bigcup \{K_{a,\xi} : \xi \in \Delta_\eta\}$$

for each $\eta < \lambda$. Apply Partition Lemma 4.12 and its right-regular version to see that

$$(3) \quad \langle H_{a,\xi} ; a : \xi \in \Gamma_\eta \rangle \quad \text{and} \quad \langle a ; K_{a,\xi} : \xi \in \Delta_\eta \rangle$$

partition $H_{b,\eta} ; b$ and $b ; K_{b,\eta}$ respectively. For each $\eta < \lambda$ and $\xi < \kappa$,

$$(4) \quad H_{b,\eta} ; b = b ; K_{b,\eta} \quad \text{and} \quad H_{a,\xi} ; a = a ; K_{a,\xi},$$

by Convention 5.5. Use the partition properties of (3), together with (4), to obtain

$$b ; K_{b,\eta} = H_{b,\eta} ; b = \sum \{H_{a,\xi} ; a : \xi \in \Gamma_\eta\} = \sum \{a ; K_{a,\xi} : \xi \in \Gamma_\eta\}.$$

It follows from this computation that the translations $a ; K_{a,\xi}$ with indices ξ in Γ_η partition $b ; K_{b,\eta}$. Compare this with the second part of (3), and use the fact that the right translations of a by distinct cosets of K_a are pairwise disjoint, by the right-regular version of Partition Lemma 4.9, to conclude that $\Gamma_\eta = \Delta_\eta$. Consequently,

$$(5) \quad K_{b,\eta} = \bigcup \{K_{a,\xi} : \xi \in \Gamma_\eta\},$$

by (2).

In view of (1) and (2), the isomorphism φ_a from G_x/H_a to G_y/K_a induces in a natural way an isomorphism $\hat{\varphi}_a$ from G_x/H_b to G_y/K_b that is defined by

$$\hat{\varphi}_a(H_{b,\eta}) = \bigcup \{\varphi_a(H_{a,\xi}) : \xi \in \Gamma_\eta\}$$

for each η . It is easy to check that this induced isomorphism coincides with φ_b , because

$$\hat{\varphi}_a(H_{b,\eta}) = \bigcup \{\varphi_a(H_{a,\xi}) : \xi \in \Gamma_\eta\} = \bigcup \{K_{a,\xi} : \xi \in \Gamma_\eta\} = K_{b,\eta} = \varphi_b(H_{b,\eta}),$$

by the definition of $\hat{\varphi}_a$, the definition of φ_a and Convention 5.5, (5), and the definition of φ_b and Convention 5.5 (with b in place of a). The following theorem has been proved.

Theorem 5.9 (Refinement Theorem). *Let x and y be measurable atoms, and a and b regular elements below $x ; 1 ; y$ with normal stabilizers. If $a \leq b$, then H_a and K_a are subgroups of H_b and K_b respectively, and the isomorphism φ_a from G_x/H_a to G_y/K_a induces an isomorphism $\hat{\varphi}_a$ from G_x/H_b to G_y/K_b , that coincides with φ_b .*

A measurable atom x is certainly a regular element with normal stabilizers, by Corollaries 4.7 and 4.16 (with $y = x$). By definition, the left stabilizer H_x consists of those elements f in G_x such that $f ; x = x$. Since $f ; x = f$ for all f in G_x , by Lemma 3.3, it follows that

$$f \in H_x \quad \text{if and only if} \quad f = x,$$

and therefore $H_x = \{x\}$. Similarly, $K_x = \{x\}$. The cosets of these stabilizers are the singletons $\{f\}$ for f in G_x . The quotient isomorphism φ_x maps G_x/H_x to G_x/K_x , and takes $\{f\}$ to $\{g\}$ if and only if $f ; x = x ; g$, that is to say, if and only if $f = g$, by Lemma 3.3. Thus, φ_x is the identity automorphism of $G_x/\{x\}$. This argument proves the following theorem.

Theorem 5.10 (Identity Theorem). *A measurable atom x is a regular element with left and right stabilizers both equal to the trivial normal subgroup $\{x\}$. The quotient isomorphism φ_x is the identity automorphism of $G_x/\{x\}$.*

The next theorem says that if $a \leq x ; 1 ; y$ is a regular element with normal stabilizers, then so is a^\smile , and φ_{a^\smile} is just the inverse isomorphism of φ_a .

Theorem 5.11 (Converse Theorem). *Let x and y be measurable atoms. If a is a regular element below $x ; 1 ; y$ with normal stabilizers, and in particular, if a is an atom, then a^\smile is a regular element below $y ; 1 ; x$ with normal stabilizers*

$$H_{a^\smile} = K_a \quad \text{and} \quad K_{a^\smile} = H_a,$$

and the quotient isomorphism $\varphi_{a^\smile} = \varphi_a^{-1}$.

Proof. Assume the hypotheses of the theorem, write $b = a^\smile$, and observe that

$$(1) \quad b = a^\smile \leq (x ; 1 ; y)^\smile = y ; 1 ; x,$$

by the definition of b , the assumption that $a \leq x ; 1 ; y$, monotony, and Lemma 2.5(iii).

For each element f in G_y ,

$$\begin{aligned} f \in H_b & \quad \text{if and only if} & \quad f ; b = b, \\ & \quad \text{if and only if} & \quad f ; a^\smile = a^\smile, \\ & \quad \text{if and only if} & \quad a ; f^\smile = a, \\ & \quad \text{if and only if} & \quad f^\smile \in K_a, \\ & \quad \text{if and only if} & \quad f \in K_a, \end{aligned}$$

by the definition of H_b , the definition of b , the two involution laws (R6) and (R7), the definition of K_a , and the fact that K_a is a subgroup of G_y and therefore closed under converse. An analogous argument applies to f in G_x , with K_b and H_a in place of H_b and K_a respectively, so that

$$(2) \quad H_b = K_a \quad \text{and} \quad K_b = H_a.$$

Similarly, for each element f in G_y ,

$$\begin{aligned} f \in X_b & \quad \text{if and only if} & \quad f \leq b ; b^\smile, \\ & \quad \text{if and only if} & \quad f \leq a^\smile ; a, \\ & \quad \text{if and only if} & \quad f \in Y_a, \end{aligned}$$

by the definitions of the set X_b , the element b , and the set Y_a . An analogous argument applies to f in G_x , with Y_b and X_a in place of X_b and Y_a respectively, so that

$$(3) \quad X_b = Y_a \quad \text{and} \quad Y_b = X_a.$$

Combine (2) with (3), and use the assumed regularity of a , together with Corollary 4.3, to arrive at

$$(4) \quad H_b = K_a = Y_a = X_b \quad \text{and} \quad K_b = H_a = X_a = Y_b.$$

With the help of Corollary 4.3 (with b in place of a), conclude from (1), (4), and (2) that b is a regular element below $y ; 1 ; x$, with left and right stabilizers as in (2).

To prove the final assertion of the theorem, consider cosets H_ξ and K_η of H_a and K_a respectively. The equations in (2) imply that H_ξ and K_η are also cosets of K_b and H_b respectively. Use the definition of φ_a in Convention 5.5, the involution

laws, the definition of b , the definition of φ_b , and the isomorphism properties of φ_b to obtain

$$\begin{aligned}
\varphi_a(H_\xi) = K_\eta & \quad \text{if and only if} & \quad H_\xi ; a = a ; K_\eta, \\
& \quad \text{if and only if} & \quad a^\sim ; H_\xi^\sim = K_\eta^\sim ; a^\sim, \\
& \quad \text{if and only if} & \quad b ; H_\xi^\sim = K_\eta^\sim ; b, \\
& \quad \text{if and only if} & \quad \varphi_b(K_\eta^\sim) = H_\xi^\sim, \\
& \quad \text{if and only if} & \quad \varphi_b(K_\eta) = H_\xi.
\end{aligned}$$

The equivalence of the first and last equations implies that the isomorphisms φ_b and φ_a are inverses of one another. \square

Convention 5.12. Continue with the assumption that $a \leq x ; 1 ; y$ is a regular element with normal stabilizers, and write $b = a^\sim$. Converse Theorem 5.11 implies that

$$H_b = K_a \quad \text{and} \quad K_b = H_a.$$

It also implies that if

$$\langle H_{a,\xi} : \xi < \kappa \rangle \quad \text{and} \quad \langle K_{a,\xi} : \xi < \kappa \rangle$$

are coset systems for H_a and K_a in G_x and G_y respectively such that

$$\varphi_a(H_{a,\xi}) = K_{a,\xi}$$

for each ξ , then by putting

$$(1) \quad H_{b,\xi} = K_{a,\xi} \quad \text{and} \quad K_{b,\xi} = H_{a,\xi}$$

for each ξ , we arrive at cosets systems

$$\langle H_{b,\xi} : \xi < \kappa \rangle \quad \text{and} \quad \langle K_{b,\xi} : \xi < \kappa \rangle$$

for H_b and K_b in G_y and G_x respectively such that

$$\varphi_b(H_{b,\xi}) = K_{b,\xi}$$

for each ξ . In what follows, we shall always assume that the coset systems for H_b and K_b have been chosen so that (1) holds. \square

The next goal is to prove, for the relative product of two regular elements with normal stabilizers, the analogue of Theorems 5.10 and 5.11. In more detail, suppose that

$$a \leq x ; 1 ; y \quad \text{and} \quad b \leq y ; 1 ; z$$

are regular elements with normal stabilizers. The right stabilizer K_a of a , and the left stabilizer H_b of b , are normal subgroups G_y , and therefore so is the product subgroup $K_a ; H_b$, which includes both K_a and H_b as subgroups. The quotient isomorphisms φ_a^{-1} from G_y/K_a to G_x/H_a , and φ_b from G_y/H_b to G_z/K_b induce isomorphisms $\hat{\varphi}_a^{-1}$ and $\hat{\varphi}_b$ on the quotient group $G_y/(K_a ; H_b)$. The goal is to prove that $a ; b$ is a regular element below $x ; 1 ; z$ with normal stabilizers

$$H_{a;b} = \hat{\varphi}_a^{-1}[K_a ; H_b] \quad \text{and} \quad K_{a;b} = \hat{\varphi}_b[K_a ; H_b],$$

and the isomorphism $\varphi_{a;b}$ from $G_x/H_{a;b}$ to $G_z/K_{a;b}$ coincides with the relational composition $\hat{\varphi}_a | \hat{\varphi}_b$ of the two induced isomorphisms $\hat{\varphi}_a$ and $\hat{\varphi}_b$.

We begin with a lemma.

Lemma 5.13. *Let $x, y,$ and z be measurable atoms. If*

$$a \leq x ; 1 ; y \quad \text{and} \quad b \leq y ; 1 ; z$$

are regular elements with normal stabilizers, then for all elements f in G_x and h in G_z the following conditions are equivalent.

- (i) $f ; a ; b = a ; b ; h.$
- (ii) *There are elements g_1 and g_2 belonging to the same coset of $K_a ; H_b$ in G_y such that*

$$f ; a = a ; g_1 \quad \text{and} \quad g_2 ; b = b ; h.$$

- (iii) *There is an element g in G_y such that*

$$f ; a = a ; g \quad \text{and} \quad g ; b = b ; h.$$

Proof. Let f and h be elements of G_x and G_z respectively. It is easy to establish the implication from (iii) to (i): if

$$f ; a = a ; g \quad \text{and} \quad g ; b = b ; h,$$

then

$$f ; a ; b = a ; g ; b = a ; b ; h.$$

Assume now that (i) holds, with the goal of establishing (ii). The element f is in G_x , and therefore belongs to some coset H_ξ of H_a . Apply Isomorphism Theorem 5.4 and Convention 5.5 to write

$$(1) \quad H_\xi ; a = a ; K_\xi.$$

Take g_1 to be any element in K_ξ , and observe that the first equation in (ii) holds, by (1) and the remark following Corollary 3.10. An analogous argument produces an element g_2 in G_y such that the second equation in (ii) holds. It remains to verify that g_1 and g_2 belong to the same coset of $K_a ; H_b$. First of all,

$$(2) \quad a^- ; f ; a ; b ; b^- = a^- ; a ; g_1 ; b ; b^- \\ = (\sum K_a) ; g_1 ; (\sum H_b) = \sum K_a ; g_1 ; H_b,$$

by the choice of g_1 , the assumed regularity of a and b , and complete distributivity. An analogous argument shows that

$$(3) \quad a^- ; a ; b ; h ; b^- = a^- ; a ; g_2 ; b ; b^- \\ = (\sum K_a) ; g_2 ; (\sum H_b) = \sum K_a ; g_2 ; H_b.$$

The assumption in (i) implies that

$$(4) \quad a^- ; f ; a ; b ; b^- = a^- ; a ; b ; h ; b^-.$$

Combine (2)–(4) to arrive at

$$\sum K_a ; g_1 ; H_b = \sum K_a ; g_2 ; H_b.$$

Sums of sets of atoms can only be equal when the sets themselves are equal, so the preceding equation implies that

$$(5) \quad K_a ; g_1 ; H_b = K_a ; g_2 ; H_b.$$

Use (5) and the assumption that the right stabilizer K_a is a normal subgroup to conclude that

$$(6) \quad g_1 ; K_a ; H_b = K_a ; g_1 ; H_b = K_a ; g_2 ; H_b = g_2 ; K_a ; H_b.$$

The equality of the first and last terms implies that g_1 and g_2 belong to the same coset of $K_a ; H_b$.

Turn, finally, to the implication from (ii) to (iii). If g_1 and g_2 are in the same coset of $K_a ; H_b$, then by interchanging the first two terms, and also the last two terms, in (6), one checks that (5) holds. The element g_1 belongs to the coset $K_a ; g_1 ; H_b$, because the identity element y of G_y belongs to both K_a and H_b , and

$$g_1 = y ; g_1 ; y.$$

It follows from (5) that g_1 belongs to $K_a ; g_2 ; H_b$, so there must be elements g_a in K_a and g_b in H_b such that

$$g_1 = g_a ; g_2 ; g_b.$$

Put

$$(7) \quad g = g_a^\sim ; g_1 = g_2 ; g_b.$$

Use the assumption in (ii), the assumption that g_a , and hence also g_a^\sim , belongs to K_a , the definition of K_a as the right stabilizer of a , and (7) to obtain

$$f ; a = a ; g_1 = a ; g_a^\sim ; g_1 = a ; g.$$

Similarly, use (ii), the assumption that g_b is in H_b , the definition of H_b as the left stabilizer of b , and (7) to obtain

$$b ; h = g_2 ; b = g_2 ; g_b ; b = g ; b$$

Thus, (iii) holds. \square

In the statement of the next theorem, φ_a is assumed to be an isomorphism from G_x/H_a to G_y/K_a that induces an isomorphism $\hat{\varphi}_a$ from $G_x/H_{a;b}$ to $G_y/(K_a ; H_b)$, while φ_b is an isomorphism from G_y/H_b to G_z/K_b that induces an isomorphism $\hat{\varphi}_b$ from $G_y/(K_a ; H_b)$ to $G_z/K_{a;b}$.

Theorem 5.14 (First Relative Product Theorem). *Let x , y , and z be measurable atoms. If*

$$a \leq x ; 1 ; y \quad \text{and} \quad b \leq y ; 1 ; z$$

are regular elements with normal stabilizers, and in particular if they are atoms, then the relative product $a;b$ is a regular element below $x;1;z$ with normal stabilizers

$$H_{a;b} = \varphi_a^{-1}[K_a ; H_b] \quad \text{and} \quad K_{a;b} = \varphi_b[K_a ; H_b],$$

and with quotient isomorphism

$$\varphi_{a;b} = \hat{\varphi}_a \mid \hat{\varphi}_b.$$

Proof. The stabilizers H_a and K_b are assumed to be normal subgroups of G_y , so the product group $K_a ; H_b$ is also a normal subgroup of G_y . It can of course be written as the union of distinct cosets K_ξ of K_a , say with $\xi < \kappa$, so that

$$(1) \quad K_a ; H_b = \bigcup_{\xi < \kappa} K_\xi.$$

Write

$$(2) \quad H_\xi = \varphi_a^{-1}(K_\xi)$$

for $\xi < \kappa$, and use the definition of φ_a and Convention 5.5 to obtain

$$(3) \quad H_\xi ; a = a ; K_\xi$$

for each ξ . The first goal is to prove that $a ; b$ is left-regular, by showing that

$$(4) \quad X_{a;b} = \bigcup_{\xi < \kappa} H_\xi = H_{a;b}.$$

Since K_ξ is a coset of K_a in $K_a ; H_b$, there must be an element h_ξ in H_b such that

$$(5) \quad K_\xi = K_a ; h_\xi.$$

Use (3), (5), the definition of K_a as the right stabilizer of a , and the assumption that h_ξ belongs to the left stabilizer of b , to get

$$H_\xi ; a ; b = a ; K_\xi ; b = a ; K_a ; h_\xi ; b = a ; h_\xi ; b = a ; b$$

This argument shows that H_ξ is included in the left stabilizer $H_{a;b}$ for every $\xi < \kappa$, and therefore

$$(6) \quad \bigcup_{\xi < \kappa} H_\xi \subseteq H_{a;b}.$$

Compute:

$$(7) \quad \begin{aligned} \sum X_{a;b} &= (a ; b) ; (a ; b)^\smile = a ; b ; b^\smile ; a^\smile \\ &= a ; (\sum H_b) ; a^\smile = a ; K_a ; (\sum H_b) ; a^\smile \\ &= \sum a ; K_a ; H_b ; a^\smile = \sum a ; (\bigcup_{\xi < \kappa} K_\xi) ; a^\smile \\ &= \sum_{\xi < \kappa} a ; K_\xi ; a^\smile = \sum_{\xi < \kappa} H_\xi ; a ; a^\smile \\ &= \sum_{\xi < \kappa} H_\xi ; (\sum H_a) = \sum (\bigcup_{\xi < \kappa} (H_\xi ; H_a)) = \sum (\bigcup_{\xi < \kappa} H_\xi), \end{aligned}$$

by the definition of $X_{a;b}$, the second involution law and the associative law, the left-regularity of b , the definition of K_a as the right stabilizer of a , complete distributivity, (1), complete distributivity, (3), the left-regularity of a , complete distributivity, and the fact that H_a is the identity coset of the quotient G_x/H_a . Since $X_{a;b}$ and $\bigcup_{\xi < \kappa} H_\xi$ are both sets of atoms, it follows from (7) that these sets must be equal. Consequently,

$$\bigcup_{\xi < \kappa} H_\xi \subseteq H_{a;b} \subseteq X_{a;b} = \bigcup_{\xi < \kappa} H_\xi$$

by (6), Lemma 4.5(ii), and the remark following (7). The first and last terms are the same, so equality holds everywhere, which proves (4).

Use (4) and Corollary 4.3 to conclude that the relative product $a ; b$ is left-regular. Also,

$$(8) \quad H_{a;b} = \bigcup_{\xi < \kappa} H_\xi = \bigcup_{\xi < \kappa} \varphi_a^{-1}(K_\xi) = \varphi_a^{-1}[\bigcup_{\xi < \kappa} K_\xi] = \varphi_a^{-1}[K_a ; H_b],$$

by (4), (2), the preservation of unions under inverse images, and (1).

The group $K_a ; H_b$ is normal in G_y , so the quotient $(K_a ; H_b)/K_a$ is a normal subgroup of G_y/K_a . Because φ_a maps G_x/H_a isomorphically to G_y/K_a , the inverse image of $(K_a ; H_b)/K_a$ under φ_a , which is $H_{a;b}/H_a$, by (8) and the definition of φ_a , must be a normal subgroup of G_x/H_a , and therefore $H_{a;b}$ must be a normal subgroup of G_x , by group theory.

It has been shown that the relative product $a ; b$ is left-regular and that its left stabilizer $H_{a;b}$ is normal in G_x and coincides with $\varphi_a^{-1}[K_a ; H_b]$. A symmetric argument shows that $a ; b$ is right-regular, and that its right stabilizer $K_{a;b}$ is normal in G_z and coincides with $\varphi_b[K_a ; H_b]$. It remains to prove that

$$(9) \quad \varphi_{a;b} = \hat{\varphi}_a \mid \hat{\varphi}_b.$$

Write $c = a ; b ; b^\smile$, and observe that $c \leq x ; 1 ; y$, since

$$\begin{aligned} c = a ; b ; b^\smile &\leq (x ; 1 ; y) ; (y ; 1 ; z) ; (y ; 1 ; z)^\smile \\ &= (x ; 1 ; y) ; (y ; 1 ; z) ; (z ; 1 ; y) \leq x ; 1 ; y, \end{aligned}$$

by the definition of c , the assumptions on a and b , monotony, and Lemma 2.5(iii),(iv). Also,

$$(10) \quad c = a ; b ; b^\smile = a ; (\sum H_b) = \sum a ; H_b,$$

by the left-regularity of b and complete distributivity. Compute:

$$\begin{aligned} (11) \quad \sum X_c = c ; c^\smile &= (a ; b ; b^\smile) ; (a ; b ; b^\smile)^\smile \\ &= a ; b ; b^\smile ; b ; b^\smile ; a^\smile = a ; b ; (\sum K_b) ; b^\smile ; a^\smile \\ &= \sum a ; b ; K_b ; b^\smile ; a^\smile = a ; b ; b^\smile ; a^\smile \\ &= (a ; b) ; (a ; b)^\smile = \sum H_{a;b}, \end{aligned}$$

by the definition of X_c , (10), the two involution laws and the associative law, the right-regularity of b , complete distributivity, the definition of K_b as the right stabilizer of b , the second involution law and the associative law, and the left-regularity of $a ; b$. Because X_c and $H_{a;b}$ are sets of atoms, it follows from (11) that $X_c = H_{a;b}$. Use this observation, the definition of c and Lemma 3.11 (with $a ; b$ and b^\smile in place of a and b respectively), and Lemma 4.5(ii) (with c in place of a) to arrive at

$$X_c = H_{a;b} \subseteq H_c \subseteq X_c.$$

The first and last sets are equal, so equality holds everywhere. Apply Corollary 4.3 to conclude that c is a left-regular element with normal left stabilizer $H_c = H_{a;b}$.

The proof that c is right-regular with normal right stabilizer $K_c = K_a ; H_b$ involves a similar computation. In more detail,

$$\begin{aligned} (12) \quad \sum Y_c = c^\smile ; c &= (a ; b ; b^\smile)^\smile ; (a ; b ; b^\smile) \\ &= b ; b^\smile ; a^\smile ; a ; b ; b^\smile = (\sum H_b) ; (\sum K_a) ; (\sum H_b) \\ &= \sum H_b ; K_a ; H_b = \sum K_a ; H_b ; H_b = \sum K_a ; H_b, \end{aligned}$$

by the definition of Y_c , the definition of c , the two involution laws and the associative law, the right-regularity of a and the left-regularity of b , complete distributivity, the assumption that K_a is a normal subgroup of G_y , and Corollary 3.10 (with b in place of a), which implies that H_b is a subgroup of G_y and hence closed under the group operation of relative multiplication. The sets Y_c and $K_a ; H_b$ consist of atoms in G_y , so (12) implies that these sets are equal,

$$(13) \quad Y_c = K_a ; H_b.$$

The next step is to check that

$$(14) \quad K_a ; H_b \subseteq K_c.$$

Consider an element g in $K_a ; H_b$, and let k in K_a and h in H_b be elements such that

$$(15) \quad g = k ; h.$$

Compute:

$$\begin{aligned} c ; g &= \sum a ; H_b ; g = \sum a ; H_b ; k ; h \\ &= \sum a ; k ; H_b ; h = \sum a ; H_b ; h = \sum a ; H_b = c \end{aligned}$$

by (10) and complete distributivity, (15), the assumption that H_b is normal in G_y , the assumption that k is in the right stabilizer K_a of a , the assumption that h is in the subgroup H_b of b , which is closed under the group operation of relative multiplication, and (10). It follows from this computation that g belongs to the right stabilizer K_c , which completes the proof of (14).

Use (13), (14), and the right-regular version of Lemma 4.5(ii) to arrive at

$$Y_c = K_a ; H_b \subseteq K_c \subseteq Y_c.$$

The first and last sets are equal, so equality holds everywhere. Apply Corollary 4.3 (with c in place of a) to conclude that c is a right-regular element with normal right stabilizer $K_c = K_a ; H_b$.

The element $a \leq x ; 1 ; y$ is regular with normal stabilizers, by assumption, and the same properties have been established for the element c . Moreover,

$$a = a ; y \leq a ; b ; b^\smile = c$$

by Lemma 2.3(iii),(i) (with b, y , and z in place of a, x , and y respectively), and the definition of c . Apply Refinement Theorem 5.9 (with c in place of b) to conclude that the isomorphism $\hat{\varphi}_a$ from G_x/H_c to G_y/K_c induced by φ_a coincides with φ_c , in symbols,

$$(16) \quad \varphi_c = \hat{\varphi}_a.$$

Now write $d = a^\smile ; a ; b$. In an entirely analogous fashion, one shows that d is a regular element below $y ; 1 ; z$ with normal left and right stabilizers

$$H_d = K_a ; H_b \quad \text{and} \quad K_d = K_{a;b},$$

and therefore, since $b \leq d$,

$$(17) \quad \varphi_d = \hat{\varphi}_b,$$

by Refinement Theorem 5.9.

The last step in the link is to prove that

$$(18) \quad \varphi_{a;b} = \varphi_c \upharpoonright \varphi_d.$$

Recall that $\varphi_{a;b}$ maps $G_x/H_{a;b}$ isomorphically to $G_z/K_{a;b}$. Fix an element f in G_x , and select an element h in G_z so that the coset $f ; H_{a;b}$ of the normal subgroup $H_{a;b}$ is mapped by $\varphi_{a;b}$ to the coset $K_{a;b} ; h$ of the normal subgroup $K_{a;b}$. It follows from the definition of $\varphi_{a;b}$ and Convention 5.5 (with $a ; b$ in place of a) that

$$(19) \quad f ; H_{a;b} ; a ; b = a ; b ; K_{a;b} ; h.$$

Use the left- and right-regular versions of Corollary 3.10 and the remark following it, together with (19), to arrive at

$$(20) \quad f ; a ; b = (f ; H_{a;b}) ; a ; b = a ; b ; (K_{a;b} ; h) = a ; b ; h.$$

Use (20) and the implication from (i) to (iii) in Lemma 5.13 to obtain an element g in G_y such that

$$(21) \quad f ; a = a ; g \quad \text{and} \quad g ; b = b ; h.$$

Compute:

$$(22) \quad f ; c = f ; a ; (\sum H_b) = a ; g ; (\sum H_b) \\ = \sum a ; g ; H_b = \sum a ; H_b ; g = a ; (\sum H_b) ; g = c ; g$$

by (10), (21), complete distributivity, the assumption that H_b is normal, complete distributivity, and (10). Use the left- and right-regular versions of Corollary 3.10 and the remark following it, together with (22), to arrive at

$$(23) \quad (f ; H_c) ; c = f ; c = c ; g = c ; (K_c ; g).$$

Apply the definition of φ_c and Convention 5.5 (with c in place of a) to conclude from (23) that φ_c maps the coset $f ; H_c$ of H_c to the coset $K_c ; g$ of K_c . Use the equations

$$H_c = H_{a;b} \quad \text{and} \quad K_c = K_a ; H_b$$

established earlier in the proof to conclude that

$$\varphi_c(f ; H_{a;b}) = \varphi_c(f ; H_c) = K_c ; g = K_a ; H_b ; g.$$

In an entirely analogous fashion, one shows that

$$\varphi_d(g ; K_a ; H_b) = K_{a;b} ; h.$$

The subgroup $K_b ; H_b$ is normal in G_y , so the cosets $g ; K_a ; H_b$ and $K_a ; H_b ; g$ are equal. Combine this observation with the preceding equations to see that the composite isomorphism $\varphi_c | \varphi_d$ maps the coset $f ; H_{a;b}$ to the coset $K_{a;b} ; h$, just as does the isomorphism $\varphi_{a;b}$, by (19) and Convention 5.5 (with $a ; b$ in place of a). Consequently, the two isomorphisms coincide. This proves (18).

Together, (16)–(18) immediately yield (9):

$$\varphi_{a;b} = \varphi_c | \varphi_d = \hat{\varphi}_a | \hat{\varphi}_b.$$

This completes the proof of the theorem. \square

Relative Product Theorem 5.14 can be combined with Refinement Theorem 5.9 and Isomorphism Theorem 5.6 to obtain an important characterization of the relational composition of two quotient isomorphisms. Recall the notation introduced before Theorem 5.6: the inner automorphism of a quotient group G_x/H_c determined by a coset H_η of H_c is the mapping τ defined by

$$\tau(H_\xi) = H_\eta^{-1} ; H_\xi ; H_\eta$$

for all cosets H_ξ of H_c . This inner automorphism induces an inner automorphism $\hat{\tau}$ of the quotient group $G_x/H_{a;b}$ under the assumption that H_c is a subgroup of $H_{a;b}$. In the statement of the next theorem, the mappings $\hat{\varphi}_a$ and $\hat{\varphi}_b$ are the isomorphism described before Relative Product Theorem 5.14, while $\hat{\varphi}_c$ is the isomorphism from $G_x/H_{a;b}$ to $G_z/K_{a;b}$ induced by the isomorphism φ_c from G_x/H_c to G_z/K_c , and $\hat{\tau}$ is the inner automorphism of $G_x/H_{a;b}$ induced by an inner automorphism τ of G_x/H_c .

Theorem 5.15 (Second Relative Product Theorem). *Let x , y , and z be measurable atoms, and*

$$a \leq x ; 1 ; y, \quad b \leq y ; 1 ; z, \quad c \leq x ; 1 ; z$$

regular elements with normal stabilizers. In particular, they may be atoms. If

$$H_\eta ; c \leq a ; b,$$

for some coset H_η of H_c , then

$$H_a; H_c \subseteq \varphi_a^{-1}[K_a; H_b] \quad \text{and} \quad K_b; K_c \subseteq \varphi_b[K_a; H_b],$$

and

$$\hat{\varphi}_a | \hat{\varphi}_b = \hat{\tau} | \hat{\varphi}_c,$$

where τ is the inner automorphism of G_x/H_c determined by H_η .

Proof. The assumption that a and b are regular elements with normal stabilizers implies that $a; b$ is a regular element with normal stabilizers

$$(1) \quad H_{a;b} = \varphi_a^{-1}[K_a; H_b] \quad \text{and} \quad K_{a;b} = \varphi_b[K_a; H_b],$$

and that

$$(2) \quad \varphi_{a;b} = \hat{\varphi}_a | \hat{\varphi}_b,$$

by Relative Product Theorem 5.14.

The element c is assumed to be regular with normal stabilizers, so its left translation

$$(3) \quad d = H_\eta; c \leq a; b,$$

is also regular with normal stabilizers, and in fact,

$$(4) \quad H_d = H_c \quad \text{and} \quad K_d = K_c,$$

by Translation Lemma 4.15(i). These observations show that Refinement Theorem 5.9 may be applied to (3) to conclude, first, that

$$(5) \quad H_d \subseteq H_{a;b} \quad \text{and} \quad K_d \subseteq K_{a;b},$$

and, second, that the isomorphism φ_d from G_x/H_d to G_z/K_d induces an isomorphism $\hat{\varphi}_d$ from $G_x/H_{a;b}$ to $G_z/K_{a;b}$ with the property

$$(6) \quad \hat{\varphi}_d = \varphi_{a;b}.$$

Combine (2) and (6) to arrive at

$$(7) \quad \hat{\varphi}_d = \hat{\varphi}_a | \hat{\varphi}_b.$$

Isomorphism Theorem 5.6 (with z , c , and d in place of y , a , and b respectively) says that

$$(8) \quad \varphi_d = \tau | \varphi_c,$$

where τ is the inner automorphism of G_x/H_c determined by the coset H_η . The inclusions in (4) and (5) imply that

$$(9) \quad H_c \subseteq H_{a;b} \quad \text{and} \quad K_c \subseteq K_{a;b},$$

so the inner automorphism τ of G_x/H_c induces an inner automorphism $\hat{\tau}$ of $G_x/H_{a;b}$, and the isomorphism φ_c from G_x/H_c to G_z/K_c induces an isomorphism $\hat{\varphi}_c$ from $G_x/H_{a;b}$ to $G_z/K_{a;b}$, and

$$(10) \quad \hat{\varphi}_d = \hat{\tau} | \hat{\varphi}_c,$$

by (8). Combine (7) and (10) to arrive at

$$\hat{\varphi}_a | \hat{\varphi}_b = \hat{\tau} | \hat{\varphi}_c.$$

Together, (1), (4), and (5) show that H_c and K_c are subgroups of

$$(11) \quad \varphi_a^{-1}[K_a; H_b] \quad \text{and} \quad \varphi_b[K_a; H_b]$$

respectively. Since K_a and H_b are subgroups of $K_a; H_b$, the two groups in (11) also include $\varphi_a^{-1}[K_a]$ and $\varphi_b[H_b]$ respectively, that is to say, they include H_a and K_b , by the definitions of φ_a and φ_b . Use these observation to arrive at

$$H_a; H_c \subseteq \varphi_a^{-1}[K_a; H_b] \quad \text{and} \quad K_b; K_c \subseteq \varphi_b[K_a; H_b].$$

□

The special case of the theorem in which c is below $a; b$ is important enough to merit separate formulation. In this case, the translating coset H_η is the identity coset H_c , and therefore the inner automorphism τ determined by this coset is the identity automorphism of G_x/H_c .

Corollary 5.16. *Let x, y , and z be measurable atoms, and*

$$a \leq x; 1; y, \quad b \leq y; 1; z, \quad c \leq x; 1; z$$

regular elements with normal stabilizers. If $c \leq a; b$, then $\hat{\varphi}_a | \hat{\varphi}_b = \hat{\varphi}_c$.

Another special case of Relative Product Theorem 5.15 is when one of the quotient groups

$$G_x/H_{a;b}, \quad G_y/(K_a; H_b), \quad G_z/K_{a;b}$$

is abelian (and hence all of them are abelian, since they are isomorphic to one another). In this case, the inner automorphism $\hat{\tau}$ mentioned in the theorem is again the identity automorphism, and therefore may be omitted from the final equation of the theorem.

Corollary 5.17. *Let x, y , and z be measurable atoms, and*

$$a \leq x; 1; y, \quad b \leq y; 1; z, \quad \text{and} \quad c \leq x; 1; z$$

regular elements with normal stabilizers. If the quotient group $G_x/H_{a;b}$ is abelian, then

$$\hat{\varphi}_a | \hat{\varphi}_b = \hat{\varphi}_c$$

whenever some translation of c is below $a; b$.

Relative Product Theorem 5.14 says that, under suitable hypotheses, the inverse image of $K_a; H_b$ under φ_a^{-1} is $H_{a;b}$, and the image of $K_a; H_b$ under φ_b is $K_{a;b}$. The following corollary gives, for atoms, an alternative description of these images.

Theorem 5.18 (Image Theorem). *Let x, y , and z be measurable atoms. If*

$$a \leq x; 1; y, \quad b \leq y; 1; z, \quad c \leq x; 1; z$$

are all atoms, then

$$\varphi_a[H_a; H_c] = K_a; H_b, \quad \varphi_b[K_a; H_b] = K_b; K_c,$$

and

$$\varphi_c[H_a; H_c] = K_b; K_c.$$

In particular,

$$H_{a;b} = H_a; H_c \quad \text{and} \quad K_{a;b} = K_b; K_c.$$

Proof. Consider elements a , b , and c satisfying the hypotheses of the theorem. All three are regular elements with normal stabilizers, by Corollaries 4.7 and 4.16. The relative product $a ; b$ is a regular element below $x ; 1 ; z$ with normal stabilizers, by Relative Product Theorem 5.14, and in particular, it is non-zero, by Lemma 4.4. The left-translations of c are atoms that partition $x ; 1 ; z$, by Atomic Partition Lemma 4.11, so one of these left translations must be below the non-zero element $a ; b$. Apply Relative Product Theorem 5.15 to obtain

$$(1) \quad H_a ; H_c \subseteq \varphi_a^{-1}[K_a ; H_b] \quad \text{and} \quad K_b ; K_c \subseteq \varphi_b[K_a ; H_b].$$

The converses a^\smile , b^\smile , and c^\smile are atoms below

$$y ; 1 ; x, \quad z ; 1 ; y, \quad \text{and} \quad z ; 1 ; x$$

respectively, by Lemmas 2.1(vi) and 2.5(iii), the assumptions on a , b , and c , and monotony. Apply the argument from the preceding paragraph to the atoms a^\smile , c^\smile , and b^\smile in place of a , b , and c respectively to get

$$(2) \quad H_{a^\smile} ; H_b \subseteq \varphi_{a^\smile}^{-1}[K_{a^\smile} ; H_c] \quad \text{and} \quad K_c ; K_b \subseteq \varphi_c[K_{a^\smile} ; H_c].$$

Apply the same argument again to the atoms b^\smile , c^\smile , and a^\smile in place of a , b , and c respectively to get

$$(3) \quad H_b ; H_{a^\smile} \subseteq \varphi_b^{-1}[K_b ; H_{c^\smile}] \quad \text{and} \quad K_{c^\smile} ; K_{a^\smile} \subseteq \varphi_{c^\smile}[K_b ; H_{c^\smile}].$$

The equations

$$(4) \quad \varphi_{a^\smile} = \varphi_a^{-1}, \quad H_{a^\smile} = K_a, \quad K_{a^\smile} = H_a,$$

$$(5) \quad \varphi_{c^\smile} = \varphi_c^{-1}, \quad H_{c^\smile} = K_c, \quad K_{c^\smile} = H_c$$

are all valid, by Converse Theorem 5.11, so the inclusions in (2) and (3) may be rewritten in the forms

$$(6) \quad K_a ; H_b \subseteq \varphi_a[H_a ; H_c] \quad \text{and} \quad K_c ; K_b \subseteq \varphi_c[H_a ; H_c],$$

and

$$(7) \quad H_b ; K_a \subseteq \varphi_b^{-1}[K_b ; K_c] \quad \text{and} \quad H_c ; H_a \subseteq \varphi_c^{-1}[K_b ; K_c].$$

For example, to obtain the first inclusion in (6), use the second equation in (4), the first inclusion in (2), the first equation in (4), and the fact that the inverse of the inverse of φ_a is φ_a , to arrive at

$$K_a ; H_b = H_{a^\smile} ; H_b \subseteq \varphi_{a^\smile}^{-1}[K_{a^\smile} ; H_c] = (\varphi_a^{-1})^{-1}[H_a ; H_c] = \varphi_a[H_a ; H_c].$$

The first equations in (1) and (6) yield

$$(8) \quad \varphi_a[H_a ; H_c] = K_a ; H_b.$$

Similarly, the second equation in (1) and the first equation in (7), together with the assumption that the stabilizers are normal subgroups, yield

$$(9) \quad \varphi_b[K_a ; H_b] = K_b ; K_c.$$

Finally, the second equations in (6) and (7), and the assumption that the stabilizers are normal, yield

$$(10) \quad \varphi_c[H_a ; H_c] = K_b ; K_c.$$

To obtain the first equation in the final assertion of the theorem, use Relative Product Theorem 5.14 and (8),

$$H_{a;b} = \varphi_a^{-1}[K_a ; H_b] = H_a ; H_c.$$

The second equation is obtained in a similar fashion, using (9) instead of (8). \square

6. SEMI-SCAFFOLDS AND SCAFFOLDS

The previous sections take a “local” perspective, and study properties of certain types of elements that lie below rectangles with measurable atoms for sides. The most important elements of this kind are regular elements with normal stabilizers, and in particular, atoms. This and the next sections take a “global” perspective, and focus on the entire measurable relation algebra.

Definition 6.1. A relation algebra is called *measurable* if the identity element is the sum of measurable atoms. If the identity element is the sum of finitely measurable atoms, then the algebra is said to be *finitely measurable*. \square

Fix a complete and atomic, measurable relation algebra \mathfrak{A} for the remainder of the discussion. The assumption of completeness is only needed in order not to have to worry about the existence of certain infinite sums. All elements and operations are assumed to be those of \mathfrak{A} . Let I be the set of measurable atoms in \mathfrak{A} . The assumption of measurability is expressed symbolically by the equation $1' = \sum I$. Define a binary relation \mathcal{E} on the set I by putting

$$\mathcal{E} = \{(x, y) : x, y \in I \text{ and } x ; 1 ; y \neq 0\}.$$

For atoms x and y , the conditions

$$x ; 1 ; y \neq 0 \quad \text{and} \quad 1 ; x ; 1 = 1 ; y ; 1$$

are equivalent, by Lemma 2.2(viii), so an alternative definition of \mathcal{E} is given by

$$\mathcal{E} = \{(x, y) : x, y \in I \text{ and } 1 ; x ; 1 = 1 ; y ; 1\}.$$

This second way of defining \mathcal{E} makes clear that \mathcal{E} is an equivalence relation on the set I .

The principal notion that will be in this section is that of a semi-scaffold.

Definition 6.2. A system $a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ of atoms is called a *semi-scaffold* if, for all pairs (x, y) in \mathcal{E} , and for all atoms x in I , the following conditions are satisfied.

- (i) $a_{xy} \leq x ; 1 ; y$.
- (ii) $a_{xx} = x$.
- (iii) $a_{yx} = a_{xy}$.

A semi-scaffold is called a *scaffold* if, for all pairs (x, y) and (y, z) in \mathcal{E} ,

- (iv) $a_{xz} \leq a_{xy} ; a_{yz}$.

\square

The conditions defining a semi-scaffold and a scaffold can be weakened somewhat. To this end, it is helpful to assume that the set I of measurable atoms is linearly ordered, say by a relation $<$.

Lemma 6.3. *A system $a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ of atoms is a semi-scaffold if and only if the following assumptions are satisfied.*

- (i) $a_{xy} \leq x ; 1 ; y$ for all (x, y) in \mathcal{E} with $x < y$.
- (ii) $a_{xx} = x$ for all x in I .
- (iii) $a_{yx} = a_{xy}^\sim$ for all (x, y) in \mathcal{E} with $x < y$.

A semi-scaffold is a scaffold if and only if

- (iv) $a_{xz} \leq a_{xy} ; a_{yz}$ for all (x, y) and (y, z) in \mathcal{E} with $x < y < z$.

Proof. It is obvious that a system of atoms satisfying conditions (i)–(iii) in the definition of a semi-scaffold satisfies assumptions (i)–(iii) of the lemma. To establish the reverse implication, assume that a is a system of atoms satisfying assumptions (i)–(iii) of the lemma. Fix a pair (x, y) in \mathcal{E} . The first step is to verify that semi-scaffold condition (i) holds for this pair. If $x < y$, then condition (i) holds by assumption. If $x = y$, then

$$a_{xy} = a_{xx} = x \leq x ; 1 ; x = x ; 1 ; y,$$

by the hypothesis of this case, assumption (ii) of the lemma, and Lemma 2.5(i) (with $y = x$). If $x > y$, then $a_{yx} \leq y ; 1 ; x$, by assumption (i) of the lemma (with x and y interchanged), and consequently

$$a_{xy} = a_{yx}^\sim \leq (y ; 1 ; x)^\sim = x ; 1 ; y,$$

by assumption (iii) of the lemma (with x and y interchanged), the preceding observation, monotony, and Lemma 2.5(iii). This completes the verification of semi-scaffold condition (i).

Semi-scaffold condition (ii) coincides with assumption (ii) of the lemma, so there is nothing to verify. Turn now to the verification of semi-scaffold condition (iii). If $x < y$, then condition (iii) holds by assumption (iii) of the lemma. If $x = y$, then

$$a_{yx} = a_{xx} = x = x^\sim = a_{xx}^\sim = a_{xy}^\sim,$$

by the hypothesis $x = y$, assumption (ii) of the lemma, and Lemma 2.1(vii). If $x > y$, then $a_{xy} = a_{yx}^\sim$, by assumption (iii) of the lemma (with x and y interchanged), and therefore

$$a_{yx} = (a_{yx}^\sim)^\sim = a_{xy}^\sim,$$

by the first involution law and the preceding observation.

To prove the second assertion of the theorem, assume that the system of atoms a satisfies assumptions (i)–(iii) of the lemma. It then satisfies semi-scaffold conditions (i)–(iii), by the arguments of the previous paragraphs, so it may be assumed that these conditions hold for a . The goal is to derive condition (iv) from these conditions and assumption (iv) of the lemma. Consider pairs (x, y) and (y, z) in \mathcal{E} . The atom a_{xz} is below $x ; 1 ; z$, by semi-scaffold condition (i), so its domain is x , and $x ; a_{xz} = a_{xz}$, by Lemma 2.3. Consequently, if $x = y$, then

$$a_{xy} ; a_{yz} = a_{xx} ; a_{xz} = x ; a_{xz} = a_{xz},$$

by the hypothesis $x = y$, semi-scaffold condition (ii), and the preceding observation. A similar argument applies if $y = z$. If $x = z$, then

$$a_{xy} ; a_{yz} = a_{xy} ; a_{yx} = a_{xy} ; a_{xy}^\sim = \sum H_{a_{xy}} \geq x = a_{xx} = a_{xz},$$

by the hypothesis $x = z$, semi-scaffold condition (iii), the assumption that a_{xy} is an atom and hence regular, by Corollary 4.7, the fact that the stabilizer $H_{a_{xy}}$ is

a subgroup of G_x and therefore contains the group identity element x , and semi-scaffold condition (ii). Assume now that x , y , and z are distinct. If $x < y < z$, then condition (iv) holds by assumption (iv) of the lemma. If $x < z < y$, then

$$a_{xy} \leq a_{xz}; a_{zy} = a_{xz}; a_{yz}^{\sim},$$

by the hypothesis of this case, assumption (iv) of the lemma (with y and z interchanged), and semi-scaffold condition (iii). Apply the cycle laws for atoms in Lemma 2.2(v) to arrive at $a_{xz} \leq a_{xy}; a_{yz}$. Similarly, if say $z < x < y$, then

$$a_{zy} \leq a_{zx}; a_{xy} = a_{xz}^{\sim}; a_{xy},$$

by the hypothesis of this case, assumption (iv) of the lemma (with z , x , and y in place of x , y , and z respectively) and semi-scaffold condition (iii) (with z in place of y). Form the converse of both sides of this inequality, and use semi-scaffold condition (iii) (with z in place of x), monotony, and the two involution laws to obtain

$$a_{yz} = a_{zy}^{\sim} \leq (a_{xz}^{\sim}; a_{xy})^{\sim} = a_{xy}^{\sim}; (a_{xz}^{\sim})^{\sim} = a_{xz}^{\sim}; a_{xz}.$$

Apply the cycle laws for atoms in Lemma 2.2(v) to arrive at $a_{xz} \leq a_{xy}; a_{yz}$. The remaining three cases of condition (iv) are treated in a similar fashion. \square

With the help of the preceding lemma, it is easy to show that every atomic, measurable relation algebra has a semi-scaffold.

Lemma 6.4 (Semi-scaffold Existence Lemma). *Every atomic, measurable relation algebra has a semi-scaffold.*

Proof. Assume that the set I of measurable atoms is linearly ordered, say by a relation $<$. For each x in I , put

$$a_{xx} = x,$$

and observe that assumption (ii) of Lemma 6.3 holds. For each pair (x, y) in \mathcal{E} with $x < y$, choose an atom $b \leq x; 1; y$, and put

$$a_{xy} = b \quad \text{and} \quad a_{yx} = b^{\sim}.$$

Such an atom b exists because the relation algebra is assumed to be atomic, and the definition of \mathcal{E} , together with the assumption that (x, y) is in \mathcal{E} , imply that the rectangle $x; 1; y$ is non-zero. Observe that a_{yx} is also an atom, by Lemma 2.1(vi). The choice of a_{xy} and the definition of a_{yx} imply that assumptions (i) and (iii) of Lemma 6.3 hold. Consequently, the system

$$a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$$

of atoms is a semi-scaffold, by Lemma 6.3. \square

It is worth pointing out that, in general, it is not true that every atomic, measurable relation algebra has a scaffold.

Fix a semi-scaffold

$$a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$$

in \mathfrak{A} , and consider an arbitrary atom $b = a_{xy}$ in this semi-scaffold. The right and left stabilizers H_b and K_b are normal subgroups of the groups G_x and G_y of permutations of x and y respectively, by Corollary 4.16. Write

$$H_{xy} = H_b \quad \text{and} \quad K_{xy} = K_b.$$

The quotient groups G_x/H_{xy} and G_y/K_{xy} are isomorphic, and in fact there are coset systems

$$\langle H_{xy,\xi} : \xi < \kappa_{xy} \rangle \quad \text{and} \quad \langle K_{xy,\xi} : \xi < \kappa_{xy} \rangle$$

of H_{xy} and K_{xy} in G_x and G_y respectively such that

$$H_{xy,\xi} ; a_{xy} = a_{xy} ; K_{xy,\xi},$$

and the function φ_{xy} from G_x/H_{xy} to G_y/K_{xy} defined by

$$\varphi_{xy}(H_{xy,\xi}) = K_{xy,\xi}$$

for each ξ is an isomorphism, by Isomorphism Theorem 5.4 and the subsequent remarks. Without loss of generality, it may be assumed that

$$H_{xy,0} = H_{xy} \quad \text{and} \quad K_{xy,0} = K_{xy}.$$

The left stabilizer $H_{b\sim}$ coincides with K_b , the right stabilizer $K_{b\sim}$ coincides with H_b , and the quotient isomorphism $\varphi_{b\sim}$ coincides with φ_b^{-1} , by Converse Theorem 5.11 (with b in place of a). Consequently, since

$$a_{yx} = a_{xy}^\sim = b^\sim,$$

by semi-scaffold condition (ii) and the definition of b , we have

$$H_{yx} = K_{xy} \quad \text{and} \quad K_{yx} = H_{xy}.$$

Apply Convention 5.12 to choose coset systems

$$\langle H_{yx,\xi} : \xi < \kappa_{xy} \rangle \quad \text{and} \quad \langle K_{yx,\xi} : \xi < \kappa_{xy} \rangle$$

of H_{yx} and K_{yx} in G_y and G_x respectively such that

$$H_{yx,\xi} = K_{xy,\xi} \quad \text{and} \quad K_{yx,\xi} = H_{xy,\xi}$$

for each $\xi < \kappa_{xy}$, and the function φ_{yx} from G_y/H_{yx} to G_x/K_{yx} that is the inverse of φ_{xy} is determined by

$$\varphi_{yx}(H_{yx,\xi}) = K_{xy,\xi}.$$

For each $\alpha < \kappa_{xy}$, the element

$$a_{xy,\alpha} = H_{xy,\alpha} ; a_{xy} = a_{xy} ; K_{xy,\alpha}$$

is well defined, by Corollary 3.10, and the system of elements $\langle a_{xy,\alpha} : \alpha < \kappa_{xy} \rangle$ is a partition of $x ; 1 ; y$ into atoms, by Atomic Partition Lemma 4.11. The following lemma summarizes these observations.

Lemma 6.5. *Let (x, y) be a pair in \mathcal{E} . The system of elements $\langle a_{xy,\alpha} : \alpha < \kappa_{xy} \rangle$ is an atomic partition of $x ; 1 ; y$. Consequently, an element below $x ; 1 ; y$ is an atom if and only if it has the form $a_{xy,\alpha}$ for a (unique) $\alpha < \kappa_{xy}$.*

The preceding lemma is local in character, referring to atoms below a given rectangle with measurable sides. The lemma implies a corresponding global result.

Lemma 6.6 (Semi-scaffold Partition Lemma). *The system of elements*

$$\langle a_{xy,\alpha} : (x, y) \in \mathcal{E} \text{ and } \alpha < \kappa_{xy} \rangle$$

is an atomic partition of the unit. Consequently, an element in the algebra is an atom if and only if it has the form $a_{xy,\alpha}$ for some (unique) pair (x, y) in \mathcal{E} and a (unique) $\alpha < \kappa_{xy}$.

Proof. The given relation algebra is assumed to be measurable, so

$$(1) \quad 1' = \sum I,$$

by the definition of measurability. Use the identity laws, (1), complete distributivity, and definition of \mathcal{E} as the set of pairs (x, y) such that $x ; 1 ; y \neq 0$ to obtain

$$(2) \quad 1 = 1' ; 1 ; 1' = (\sum I) ; 1 ; (\sum I) \\ = \sum \{x ; 1 ; y : x, y \in I\} = \sum \{x ; 1 ; y : (x, y) \in \mathcal{E}\}.$$

For each pair (x, y) in \mathcal{E} , we have

$$(3) \quad x ; 1 ; y = \sum \{a_{xy, \alpha} : \alpha < \kappa_{xy}\},$$

by Lemma 6.5. Combine (2) and (3) to arrive at

$$1 = \sum \{a_{xy, \alpha} : (x, y) \in \mathcal{E} \text{ and } \alpha < \kappa_{xy}\}.$$

It remains to prove that two atoms $a_{xy, \alpha}$ and $a_{uv, \beta}$ are disjoint if $x \neq u$, or $y \neq v$, or $x = y$ and $u = v$, but $\alpha \neq \beta$. In the first case, we have

$$(4) \quad a_{xy, \alpha} \cdot a_{uv, \beta} \leq (x ; 1 ; y) \cdot (u ; 1 ; v) = (x \cdot u) ; 1 ; (y \cdot v) = 0 ; 1 ; (y \cdot v) = 0,$$

by monotony, the assumption that $\langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ is a semi-scaffold, Lemma 2.5(ii), the assumption that x and u are distinct atoms, and Lemma 2.2(i). A similar argument applies in the second case, when $y \neq v$. In the third case, the desired disjointness follows directly from Lemma 6.5. \square

The next lemma specifies the elements $a_{xy, \alpha}$ that are subidentity atoms

Lemma 6.7 (Semi-scaffold Identity Lemma). *An element $a_{xy, \alpha}$ is a subidentity atom if and only if $x = y$ and $\alpha = 0$. Consequently,*

$$1' = \sum \{a_{xx, 0} : x \in I\}.$$

Proof. The rectangles $x ; 1 ; y$ have a non-zero meet with the identity element $1'$ if and only if $x = y$, and in this case that meet is x , by Lemma 2.5(i) and the assumption that x and y are atoms. This means that the atom $a_{xy, \alpha}$, which is below the rectangle $x ; 1 ; y$, by Lemma 6.5, can lie below $1'$ only when $x = y$ and

$$a_{xx, \alpha} = x = a_{xx}.$$

For this last equality to hold, the index α must be 0, by the definition of $a_{xy, \alpha}$ and the assumption about the indexing of the coset systems. This proves the first assertion of the lemma. The second follows from the first and the assumption that the relation algebra is measurable. \square

The next lemma determines the converse of each atom $a_{xy, \alpha}$.

Lemma 6.8 (Semi-scaffold Converse Lemma). *For each pair (x, y) in \mathcal{E} , and each $\alpha < \kappa_{xy}$, we have $a_{xy, \alpha}^{\sim} = a_{yx, \beta}$, where $H_{xy, \alpha}^{\sim} = H_{xy, \beta}$.*

Proof. Assume

$$(1) \quad H_{xy, \alpha}^{\sim} = H_{xy, \beta}.$$

Use the definition of $a_{xy, \alpha}$, the second involution law, semi-scaffold condition (iii) and (1), the assumption about the system of cosets $K_{yx, \beta}$, described after Lemma 6.4

and based on Convention 5.12, and Convention 5.5 and the definition of $a_{yx,\beta}$ to arrive at

$$\begin{aligned} (a_{xy,\alpha})^\sim &= (H_{xy,\alpha}; a_{xy})^\sim = a_{xy}^\sim; H_{xy,\alpha}^\sim = a_{yx}; H_{xy,\beta} \\ &= a_{yx}; K_{yx,\beta} = H_{yx,\beta}; a_{yx} = a_{yx,\beta}. \end{aligned}$$

□

The next step is to determine the relative product of two atoms $a_{xy,\alpha}$ and $a_{wz,\beta}$. Begin with the trivial case.

Lemma 6.9. *Let (x, y) and (w, z) be pairs in \mathcal{E} , and $\alpha < \kappa_{xy}$ and $\beta < \kappa_{wz}$. If $y \neq w$, then $a_{xy,\alpha}; a_{wz,\beta} = 0$.*

Proof. If y and w are not equal, then they are disjoint, because they are atoms, and therefore

$$a_{xy,\alpha}; a_{wz,\beta} \leq (x; 1; y); (w; 1; z) = 0,$$

by semi-scaffold condition (i), monotony, and Lemma 2.5(v). □

The non-trivial case of relative multiplication is more interesting and more involved.

Lemma 6.10 (Semi-scaffold Relative Product Lemma). *Let (x, y) and (y, z) be pairs in \mathcal{E} , and $\alpha < \kappa_{xy}$ and $\beta < \kappa_{yz}$. If $\xi < \kappa_{xz}$ is any index such that*

$$H_{xz,\xi}; a_{xz} \leq a_{xy}; a_{yz},$$

then

$$a_{xy,\alpha}; a_{yz,\beta} = \sum \{a_{xz,\gamma}; \gamma < \kappa_{xz} \text{ and } H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]; H_{xz,\xi}\}.$$

Proof. The relative product $a_{xy}; a_{yz}$ is a regular element below $x; 1; z$ with normal stabilizers, by Relative Product Theorem 5.14, and the left stabilizer is the product group $H_{xy}; H_{xz}$, by Image Theorem 5.18 (with (x, y) , (y, z) , and (x, z) in place of a , b , and c respectively). The isomorphism φ_{xy} from G_x/H_{xy} to G_y/K_{xy} induces an isomorphism $\hat{\varphi}_{xy}$ from

$$G_x/(H_{xy}; H_{xz}) \quad \text{to} \quad G_y/(K_{xy}; H_{yz}),$$

by Image Theorem 5.18. To simplify notation, write

$$M = H_{xy}; H_{xz} \quad \text{and} \quad P = K_{xy}; H_{yz},$$

so that M is the left stabilizer of $a_{xy}; a_{yz}$, and $\hat{\varphi}_{xy}$ maps G_x/M isomorphically to G_y/P . Let

$$\langle M_\eta; \eta < \lambda \rangle \quad \text{and} \quad \langle P_\eta; \eta < \lambda \rangle$$

be coset systems of M and P in G_x and G_y respectively such that

$$\varphi_{xy}[M_\eta] = P_\eta$$

for each $\eta < \lambda$. The coset product $K_{xy,\alpha}; H_{yz,\beta}$ is a coset of P , so it must coincide with P_ρ for some $\rho < \lambda$, and therefore

$$(1) \quad M_\rho = \varphi_{xy}^{-1}[P_\rho] = \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}].$$

The principal step in the proof is showing that

$$(2) \quad M_\rho; a_{xy}; a_{yz} = a_{xy,\alpha}; a_{yz,\beta}.$$

As H_{xy} is a subgroup of M , there must be a partition $\langle \Gamma_\eta : \eta < \lambda \rangle$ of the index set κ_{xy} such that

$$(3) \quad M_\eta = \bigcup_{\zeta \in \Gamma_\eta} H_{xy, \zeta}$$

for each η . Taking ρ for η gives

$$(4) \quad \begin{aligned} K_{xy, \alpha}; H_{yz, \beta} = \varphi_{xy}[M_\rho] &= \varphi_{xy}[\bigcup_{\zeta \in \Gamma_\rho} H_{xy, \zeta}] \\ &= \bigcup_{\zeta \in \Gamma_\rho} \varphi_{xy}(H_{xy, \zeta}) = \bigcup_{\zeta \in \Gamma_\rho} K_{xy, \zeta}, \end{aligned}$$

by (1), (3), the distributivity of function images over unions, and the definition of φ_{xy} . Recall from the remarks preceding Lemma 6.5 that

$$(5) \quad H_{xy, \zeta}; a_{xy} = a_{xy}; K_{xy, \zeta}.$$

for each $\zeta < \kappa_{xy}$. Compute:

$$\begin{aligned} M_\rho; a_{xy}; a_{yz} &= (\bigcup_{\zeta \in \Gamma_\rho} H_{xy, \zeta}); a_{xy}; a_{yz} = \sum_{\zeta \in \Gamma_\rho} H_{xy, \zeta}; a_{xy}; a_{yz} \\ &= \sum_{\zeta \in \Gamma_\rho} a_{xy}; K_{xy, \zeta}; a_{yz} = a_{xy}; (\bigcup_{\zeta \in \Gamma_\rho} K_{xy, \zeta}); a_{yz} \\ &= a_{xy}; (K_{xy, \alpha}; H_{yz, \beta}); a_{yz} = (a_{xy}; K_{xy, \alpha}); (H_{yz, \beta}; a_{yz}) \\ &= a_{xy, \alpha}; a_{yz, \beta}, \end{aligned}$$

by (3), complete distributivity, (5), complete distributivity, (4), associativity, and the definitions of $a_{xy, \alpha}$ and $a_{yz, \beta}$. This proves (2).

The group H_{xz} is included in M , by the definition of M , so there is a partition $\langle \Delta_\eta : \eta < \lambda \rangle$ of κ_{xz} such that

$$(6) \quad M_\eta = \bigcup_{\gamma \in \Delta_\eta} H_{xz, \gamma}$$

for each $\eta < \lambda$. Let $\xi < \kappa_{xz}$ be an index such that

$$(7) \quad H_{xz, \xi}; a_{xz} \leq a_{xy}; a_{yz}.$$

There is a unique index $\sigma < \mu$ such that

$$(8) \quad M_\sigma = M_\rho; H_{xz, \xi},$$

and for this index we have

$$\bigcup_{\gamma \in \Delta_\sigma} H_{xz, \gamma} = M_\sigma = M_\rho; H_{xz, \xi} = \varphi_{xy}^{-1}[K_{xy, \alpha}; K_{yz, \beta}]; H_{xz, \xi}$$

by (6) (with σ in place of η), (8), (7), and (1). Thus, Δ_σ is precisely the set of indices γ such that

$$(9) \quad H_{xz, \gamma} \subseteq \varphi_{xy}^{-1}(K_{xy, \alpha}; K_{yz, \beta}); H_{xz, \xi} \quad \text{if and only if} \quad \gamma \in \Delta_\sigma.$$

Use assumption (7) for the first and only time, Partition Lemma 4.12 (with M , ρ , $a_{xy}; a_{yz}$, and $H_{xz, \xi}; a_{xz}$ in place of H_b , η , b , and a respectively), and (8), to arrive at

$$(10) \quad M_\rho; a_{xy}; a_{yz} = \sum M_\rho; H_{xz, \xi}; a_{xz} = \sum M_\sigma; a_{xz}.$$

Conclude that

$$(11) \quad \begin{aligned} a_{xy, \alpha}; a_{yz, \beta} &= M_\rho; a_{xy}; a_{yz} = \sum M_\sigma; a_{xz} \\ &= \sum (\bigcup_{\gamma \in \Delta_\sigma} H_{xz, \gamma}); a_{xz} = \sum_{\gamma \in \Delta_\sigma} H_{xz, \gamma}; a_{xz} = \sum_{\gamma \in \Delta_\sigma} a_{xz, \gamma}, \end{aligned}$$

by (2), (10), (6) (with σ in place of η), complete distributivity, and the definition of $a_{xz, \gamma}$. In view of (9), the equality of the first and last terms in (11) is just what was to be shown. \square

The formula in Lemma 6.10 for computing the relative product $a_{xy,\alpha}; a_{yz,\beta}$ takes on a more familiar form when the system a is actually a scaffold, and not just a semi-scaffold.

Lemma 6.11 (Scaffold Relative Product Lemma). *Let (x, y) and (y, z) be pairs in \mathcal{E} , and $\alpha < \kappa_{xy}$ and $\beta < \kappa_{yz}$. If a is a scaffold, then*

$$a_{xy,\alpha}; a_{yz,\beta} = \sum \{a_{xz,\gamma} : \gamma < \kappa_{xz} \text{ and } H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]\}.$$

Proof. The assumption that a is a scaffold implies that $a_{xz} \leq a_{xy}; a_{yz}$. Consequently, the index $\xi < \kappa_{xz}$ in the statement of Lemma 6.10 may be taken to be 0. The term $\varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]$ in the formula for computing the relative product $a_{xy,\alpha}; a_{yz,\beta}$ is a coset of the product group

$$M = H_{xy}; H_{xz},$$

which is the identity element of the quotient group G_z/M . Forming the relative product of this term on the right by the subset $H_{xz,0} = H_{xz}$ of M does not change it:

$$\varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]; H_{xz,0} = \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}].$$

Thus, the formula for computing the relative product $a_{xy,\alpha}; a_{yz,\beta}$ that is given in Lemma 6.10 reduces in this case to the formula that is given in the statement of the present lemma. \square

7. REPRESENTATION THEOREMS

This section contains the main result of the paper, a representation theorem for atomic, measurable relation algebras. Continue with the assumption that \mathfrak{A} is a complete and atomic, measurable relation algebra. Here is a summary of what has been accomplished so far in the analysis of the structure of \mathfrak{A} .

In Definition 3.4, a system of mutually disjoint groups

$$G = \langle G_x : x \in I \rangle$$

indexed by the set I of measurable atoms in \mathfrak{A} is defined, where G_x is the group of permutations in \mathfrak{A} of the measurable atom x (see Lemmas 3.3 and 3.6). After Definition 6.1, an equivalence relation \mathcal{E} is defined on the set I by putting a pair (x, y) in \mathcal{E} just in case the rectangle $x; 1; y$ is not zero. It is shown in Semi-scaffold Existence Lemma 6.4 that \mathfrak{A} has a semi-scaffold

$$a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle.$$

Fix such a semi-scaffold a for the remainder of the discussion.

In terms of a , for each pair (x, y) in \mathcal{E} , normal subgroups H_{xy} and K_{xy} of G_x and G_y respectively are defined after Semi-scaffold Existence Lemma 6.4 as the left and right stabilizers of the atom a_{xy} . With the help of Isomorphism Theorem 5.4, it is shown that there is a canonical quotient isomorphism φ_{xy} from G_x/H_{xy} to G_y/K_{xy} . In fact, there are coset systems

$$(1) \quad \langle H_{xy,\xi} : \xi < \kappa_{xy} \rangle \quad \text{and} \quad \langle K_{xy,\xi} : \xi < \kappa_{xy} \rangle$$

such that

$$H_{xy,\xi} ; a_{xy} = a_{xy} ; K_{xy,\xi}$$

for each ξ , and the isomorphism φ_{xy} maps $H_{xy,\xi}$ to $K_{xy,\xi}$ for each ξ . The system of isomorphisms

$$\varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle$$

has the following properties. First, for each x in I , the atom a_{xx} coincides with x by semi-scaffold condition (ii), so the isomorphism φ_{xx} is the identity automorphism of $G_x/\{x\}$, by Identity Theorem 5.10. Second, for each pair (x, y) in \mathcal{E} , the atom a_{yx} coincides with the converse a_{xy}^\sim , by semi-scaffold condition (iii), so the isomorphism φ_{yx} is the inverse of the isomorphism φ_{xy} , by Converse Theorem 5.11. In fact,

$$H_{yx} = K_{xy} \quad \text{and} \quad K_{yx} = H_{xy},$$

and we may choose cosets systems

$$\langle H_{yx,\xi} : \xi < \kappa_{xy} \rangle \quad \text{and} \quad \langle K_{yx,\xi} : \xi < \kappa_{xy} \rangle$$

of H_{yx} and K_{yx} in G_y and G_x respectively such that

$$(2) \quad H_{yx,\xi} = K_{xy,\xi} \quad \text{and} \quad K_{yx,\xi} = H_{xy,\xi},$$

so that φ_{yx} maps $K_{xy,\xi}$ to $H_{yx,\xi}$ for each ξ .

Third, for every triple (x, y, z) in the set

$$\mathcal{E}_3 = \{(x, y, z) : (x, y) \text{ and } (y, z) \in \mathcal{E}\},$$

the atoms a_{xy} , a_{yz} , and a_{xz} satisfy the inequalities

$$a_{xy} \leq x ; 1 ; y, \quad a_{yz} \leq y ; 1 ; z, \quad \text{and} \quad a_{xz} \leq x ; 1 ; z,$$

by semi-scaffold condition (i), and therefore

$$\varphi_{xy}[H_{xy} ; H_{xz}] = K_{xy} ; H_{yz},$$

by Image Theorem 5.18. The preceding observations combine to show that conditions (i)–(iii) in Definition 5.3 of a coset semi-frame are satisfied.

Turn now to the task of defining a coset system

$$(3) \quad C = \langle C_{xyz} : (x, y, z) \in \mathcal{E}_3 \rangle$$

such that condition (iv) in Definition 5.3 is satisfied. For each triple (x, y, z) in \mathcal{E}_3 the relative product $a_{xy} ; a_{yz}$ is a regular element below $x ; 1 ; z$ with normal stabilizers, by Relative Product Theorem 5.14. In particular, this relative product is not zero, by Lemma 4.4. The atom a_{xz} is also below $x ; 1 ; z$, by semi-scaffold condition (i), so there must be a coset $H_{xz,\zeta}$ of H_{xz} (with $\zeta < \kappa_{xz}$) such that

$$(4) \quad a_{xz,\zeta} = H_{xz,\zeta} ; a_{xz} \leq a_{xy} ; a_{yz},$$

by Lemma 6.5 (with z in place of y). Notice in passing that Lemma 6.5 is an easy consequence of Atomic Partition Lemma 4.11. Choose any such index $\zeta < \kappa_{xz}$, and write

$$(5) \quad C_{xyz} = H_{xy} ; H_{xz,\zeta}.$$

Observe that C_{xyz} is a coset of the product group $H_{xy} ; H_{xz}$ in G_x . The coset $H_{xz,\zeta}$ determines an inner automorphism τ of G_z/H_{xz} that is defined by

$$\tau(H_{xz,\eta}) = H_{xz,\zeta}^\sim ; H_{xz,\eta} ; H_{xz,\zeta}$$

for each $\eta < \kappa_{xz}$. In turn, τ induces an inner automorphism $\hat{\tau}$ of $G_z/(H_{xy}; H_{xz})$ that is defined as follows. Write a given coset $H_{xy,\alpha}; H_{xz,\beta}$ of $H_{xy}; H_{xz}$ as a union

$$H_{xy,\alpha}; H_{xz,\beta} = \bigcup \{H_{xz,\eta} : \eta \in \Gamma\}$$

of cosets of H_{xz} , where Γ is some subset of the index set κ_{xz} , and define $\hat{\tau}$ by

$$(6) \quad \hat{\tau}(H_{xy,\alpha}; H_{xz,\beta}) = \bigcup \{\tau(H_{xz,\eta}) : \eta \in \Gamma\} = \bigcup \{H_{xz,\zeta}^-; H_{xz,\eta}; H_{xz,\zeta} : \eta \in \Gamma\}.$$

The definition of C_{xyz} in (5), the second involution law, complete distributivity, and the fact that H_{xz} is the identity coset of the quotient group G_x/H_{xz} together imply that

$$(7) \quad \begin{aligned} C_{xyz}^-; (H_{xy,\alpha}; H_{xz,\beta}); C_{xyz} \\ &= (H_{xz}; H_{xz,\zeta})^-; (\bigcup \{H_{xz,\eta} : \eta \in \Gamma\}); (H_{xz}; H_{xz,\zeta}) \\ &= \bigcup \{H_{xz,\zeta}^-; H_{xz}^-; H_{xz,\eta}; H_{xz}; H_{xz,\zeta} : \eta \in \Gamma\} \\ &= \bigcup \{H_{xz,\zeta}^-; H_{xz,\eta}; H_{xz,\zeta} : \eta \in \Gamma\}. \end{aligned}$$

Compare (6) with (7) to see that the inner automorphism $\hat{\tau}$ of $G_x/(H_{xy}; H_{xz})$ induced by τ , that is to say, induced by the coset $H_{xz,\zeta}$, coincides with the inner automorphism determined by C_{xyz} .

The index ζ was chosen so that (4) is satisfied. Apply Relative Product Theorem 5.15 (with ζ in place of ξ) to conclude that

$$\hat{\varphi}_{xy} | \hat{\varphi}_{yz} = \hat{\tau} | \hat{\varphi}_{xz}.$$

The remarks of the previous paragraph show that $\hat{\tau}$ coincides with the inner automorphism of $G_x/(H_{xy}; H_{xz})$ determined by the coset C_{xyz} . Consequently, semi-frame condition (iv) is satisfied.

Take C to be the system of cosets defined in (3) and (5), and form the group triple

$$\mathcal{F} = (G, \varphi, C).$$

The following theorem about \mathcal{F} has been proved.

Theorem 7.1 (Semi-frame Theorem). *The group triple \mathcal{F} is a coset semi-frame.*

We shall refer to \mathcal{F} as the semi-frame *associated with* the semi-scaffold a .

It may appear as if the semi-frame depends not only on the particular semi-scaffold a that has been selected, but also on the particular coset that is chosen to satisfy (4). However, it is not difficult to see that this is in fact not the case. If $H_{xz,\zeta'}$ is any other coset such that (4) (with ζ' in place of ζ) holds, then

$$H_{xy}; H_{xz,\zeta'} = H_{xy}; H_{xz,\zeta} = C_{xyz}.$$

In more detail, the product subgroup $H_{xy}; H_{xz}$ is the left stabilizer of the relative product $a_{xy}; a_{yz}$, by Relative Product Theorem 5.14 and Image Theorem 5.18. In particular, it leaves this relative product fixed under relative multiplication on the left. Use this fact (at the end of the computation), together with the identity property of H_{xz} in the quotient group G_x/H_{xz} , (4), and Partition Lemma 4.12 (with

$$H_{xz,\zeta}; a_{xz}, \quad a_{xy}; a_{yz}, \quad \text{and} \quad H_{xy}; H_{xz}$$

in place of a , b , and $H_{b,\eta}$ respectively), to obtain

$$\begin{aligned} \sum H_{xy} ; H_{xz,\zeta} ; a_{xz} &= \sum H_{xy} ; H_{xz} ; H_{xz,\zeta} ; a_{xz} \\ &= H_{xy} ; H_{xz} ; a_{xy} ; a_{yz} = a_{xy} ; a_{yz}. \end{aligned}$$

A similar argument shows that

$$\sum H_{xy} ; H_{xz,\zeta'} ; a_{xz} = a_{xy} ; a_{yz},$$

so

$$\sum H_{xy} ; H_{xz,\zeta} ; a_{xz} = \sum H_{xy} ; H_{xz,\zeta'} ; a_{xz}.$$

Use Corollary 4.10 to conclude that the two cosets $H_{xy} ; H_{xz,\zeta}$ and $H_{xy} ; H_{xz,\zeta'}$ are equal.

In terms of the semi-frame \mathcal{F} , a complete and atomic Boolean algebra with additional completely distributive operations, that is to say, a complete and atomic Boolean algebra with complete operators

$$\mathfrak{C}[\mathcal{F}] = \langle C[\mathcal{F}], \cup, \sim, \otimes, ^{-1}, id_U \rangle$$

of the same similarity type as relation algebras, can be defined. The atoms of this algebra are the binary relations $R_{xy,\alpha}$ that are the subsets of the Cartesian product $G_x \times G_y$ defined by

$$R_{xy,\alpha} = \bigcup \{ H_{xy,\xi} \times (K_{xy,\xi} ; K_{xy,\alpha}) : \xi < \kappa_{xy} \}$$

for each pair (x, y) in \mathcal{E} and each $\alpha < \kappa_{xy}$. In particular, for $x = y$, the left and right stabilizers are the trivial subgroups

$$H_{xy} = \{x\} \quad \text{and} \quad K_{xy} = \{y\},$$

and the cosets are the singletons of elements of the respective groups, by semi-frame condition (i), so that the definition of $R_{xx,\alpha}$ assumes the form

$$R_{xx,\alpha} = \{(g, g ; g_\alpha) : g \in G_x\},$$

where $K_{xx,\alpha} = \{g_\alpha\}$. When $\alpha = 0$, the coset $K_{xx,\alpha}$ coincides with the trivial subgroup $K_0 = \{x\}$, by convention, so that $g_0 = x$, and therefore $R_{xx,0}$ is the identity relation on the set G_x .

The elements of the algebra are arbitrary unions of sets of atoms, so the universe of the algebra is a set of binary relations on the base set

$$U = \bigcup \{G_x : x \in I\}.$$

The Boolean operations of the algebra are the binary set-theoretic operation \cup of forming unions of binary relations, and the unary set-theoretic operation \sim of forming complements of binary relations with respect to the unit, or universal, relation $U \times U$, which is the union of the set of atoms. The distinguished constant id_U is the identity relation on U , and

$$id_U = \bigcup \{R_{xx,0} : x \in I\}.$$

The operation $^{-1}$ is the unary set-theoretic operation on binary relations of forming the converse, or inverse, of a relation. Semi-frame condition (ii) in Definition 5.3 implies that this operation is determined on atoms by

$$R_{xy,\alpha}^{-1} = R_{yx,\beta}, \quad \text{where} \quad H_{xy,\alpha}^\sim = H_{xy,\beta},$$

that is to say, where $H_{xy,\beta}$ is the coset inverse of $H_{xy,\alpha}$ in the quotient group G_x/H_{xy} , and the operation is extended to all elements in the algebra by making it completely distributive over unions.

The binary operation \otimes is defined on atoms as follows. For pairs (x, y) and (w, z) in \mathcal{E} with $y \neq w$,

$$R_{xy,\alpha} \otimes R_{wz,\beta} = \emptyset$$

for all $\alpha < \kappa_{xy}$ and $\beta < \kappa_{wz}$, and for pairs (x, y) and (y, z) in \mathcal{E} ,

$$R_{xy,\alpha} \otimes R_{yz,\beta} = \bigcup \{ R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha} ; H_{yz,\beta}] ; C_{xyz} \}$$

for all $\alpha < \kappa_{xy}$ and all $\beta < \kappa_{yz}$. The operation is extended to arbitrary pairs of elements in $\mathfrak{C}[\mathcal{F}]$ by making it be completely distributive over arbitrary unions.

The algebra $\mathfrak{C}[\mathcal{F}]$ turns out to be a relation algebra if and only if certain conditions called the *coset conditions* are satisfied, and in this case $\mathfrak{C}[\mathcal{F}]$ is called a *coset relation algebra*. The coset conditions do not play a role in the discussion below, so we do not go into them further (see [1]).

In the proof of the representation theorem for atomic, measurable relation algebras, we shall use a form of the Atomic Isomorphism Theorem (see [3]). The hypothesis of this form of the theorem is that two complete and atomic Boolean algebras with completely distributive operators are given, say \mathfrak{A} and \mathfrak{B} , and say of the same similarity type as relation algebras, together with a bijection ϑ from the set of atoms in \mathfrak{A} to the set of atoms in \mathfrak{B} . Using relation algebraic notation for the operations, the conclusion of the theorem may be formulated as follows. The bijection ϑ can be extended to an isomorphism from \mathfrak{A} to \mathfrak{B} if and only if ϑ preserves the Peircean operations on atoms in the sense that

$$\begin{aligned} c \leq a ; b & \quad \text{if and only if} & \quad \vartheta(c) \leq \vartheta(a) ; \vartheta(b), \\ c \leq a^\smile & \quad \text{if and only if} & \quad \vartheta(c) \leq \vartheta(a)^\smile, \\ c \leq 1' & \quad \text{if and only if} & \quad \vartheta(c) \leq 1', \end{aligned}$$

for all atoms a , b , and c in \mathfrak{A} , where the operations on the left (including the operation $1'$ of rank 0) are those of \mathfrak{A} , and the ones on the right are those of \mathfrak{B} . If these conditions are satisfied, then the isomorphism from \mathfrak{A} to \mathfrak{B} is the function ψ defined by

$$\psi(r) = \sum \{ \vartheta(a) : a \in X \}$$

for every element r in \mathfrak{A} , where X is the set of atoms in \mathfrak{A} that are below r .

The next theorem says that coset relation algebras are essentially the only possible examples of atomic, measurable relation algebras.

Theorem 7.2 (Representation Theorem). *Every atomic, measurable relation algebra is essentially isomorphic to a coset relation algebra.*

Proof. Start with an atomic, measurable relation algebra \mathfrak{B} , and pass to its completion \mathfrak{A} , that is to say, pass to its minimal complete extension. The completion \mathfrak{A} is well known to be a complete and atomic relation algebra, and its atoms are the same as the atoms in \mathfrak{B} (see [11]). It follows that each subidentity atom x is measurable not only in \mathfrak{B} , but also in \mathfrak{A} , because the same atoms are below the rectangle $x ; 1 ; x$ in both \mathfrak{B} and \mathfrak{A} , and consequently \mathfrak{A} is a complete and atomic, measurable relation algebra.

Let I be the set of measurable atoms in \mathfrak{A} , and \mathcal{E} the equivalence relation defined on I by putting (x, y) in \mathcal{E} if and only if $x ; 1 ; y \neq 0$. The algebra \mathfrak{A} has a semi-scaffold, by Semi-scaffold Existence Lemma 6.4. Fix such a semi-scaffold

$$a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle,$$

and let

$$\mathcal{F} = (G, \varphi, C)$$

be the coset semi-frame associated with a , where

$$G = \langle G_x : x \in I \rangle, \quad \varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle, \quad C = \langle C_{xyz} : (x, y, z) \in \mathcal{E}_3 \rangle$$

are as defined at the beginning of the section. (Here, Semi-frame Theorem 7.1 is being used.) The goal is to show that \mathfrak{A} is isomorphic to $\mathfrak{C}[\mathcal{F}]$. It then follows that $\mathfrak{C}[\mathcal{F}]$ is a relation algebra, and therefore it automatically satisfies the coset conditions. Conclusion: \mathfrak{A} is isomorphic to a coset relation algebra, so \mathfrak{B} is essentially isomorphic to a coset relation algebra.

The distinct atoms in \mathfrak{A} are the elements $a_{xy, \alpha}$ defined before Lemma 6.5, by Semi-scaffold Partition Lemma 6.6. The distinct atoms in $\mathfrak{C}[\mathcal{F}]$ are the binary relations $R_{xy, \alpha}$ defined after Semi-frame Theorem 7.1. Let ϑ be the bijection from the set of atoms in \mathfrak{A} to the set of atoms in $\mathfrak{C}[\mathcal{F}]$ that is defined by

$$(1) \quad \vartheta(a_{xy, \alpha}) = R_{xy, \alpha}$$

for every pair (x, y) in \mathcal{E} and every $\alpha < \kappa_{xy}$. It must be shown that ϑ preserves the Peircean operations on atoms in the sense of the Atomic Isomorphism Theorem.

Fix three arbitrary atoms in \mathfrak{A} and the corresponding images, under ϑ , of these three atoms in $\mathfrak{C}[\mathcal{F}]$, say

$$a_{xy, \alpha}, \quad a_{wz, \beta}, \quad a_{uv, \gamma} \quad \text{and} \quad R_{xy, \alpha}, \quad R_{wy, \beta}, \quad R_{uv, \gamma}$$

respectively. Treat first the case of the operation of relative multiplication. In view of (1), it is to be shown that

$$(2) \quad a_{uv, \gamma} \leq a_{xy, \alpha} ; a_{wz, \beta} \quad \text{if and only if} \quad R_{uv, \gamma} \subseteq R_{xy, \alpha} \otimes R_{wz, \beta}.$$

Lemma 6.9 and Semi-scaffold Relative Product Lemma 6.10 imply that

$$(3) \quad a_{uv, \gamma} \leq a_{xy, \alpha} ; a_{wz, \beta} \quad \text{if and only if} \quad y = w, \quad u = x, \quad v = z,$$

and the coset $H_{xz, \gamma}$ determined by the index γ satisfies the inclusion

$$(4) \quad H_{xz, \gamma} \subseteq \varphi_{xy}^{-1}[K_{xy, \alpha} ; H_{yz, \beta}] ; H_{xz, \zeta},$$

where ζ is the index chosen for the triple (x, y, z) so that

$$H_{xz, \zeta} ; a_{xz} \leq a_{xy} ; a_{yz}.$$

The definition of the operation \otimes implies that

$$(5) \quad R_{uv, \gamma} \subseteq R_{xy, \alpha} \otimes R_{wz, \beta} \quad \text{if and only if} \quad y = w, \quad u = x, \quad v = z,$$

and the coset $H_{xz, \gamma}$ determined by the index γ satisfies the inclusion

$$(6) \quad H_{xz, \gamma} \subseteq \varphi_{xy}^{-1}[K_{xy, \alpha} ; H_{yz, \beta}] ; C_{xyz},$$

where C_{xyz} is the coset of the product group $H_{xy} ; H_{xz}$ that is defined by

$$(7) \quad C_{xyz} = H_{xy} ; H_{xz, \zeta}.$$

Observe that the inclusion in (6) is equivalent to the one in (4), because the right sides of these two inclusions are equal. In more detail,

$$\begin{aligned} \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]; C_{xyz} &= \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]; H_{xy}; H_{xz,\zeta} \\ &= \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]; H_{xy}; H_{xz}; H_{xz,\zeta} = \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]; H_{xz,\zeta}, \end{aligned}$$

by (7), the identity property of the coset H_{xz} in the quotient group G_x/H_{xz} , and the facts that $\varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}]$ is a coset of the product group $H_{xy}; H_{xz}$, and this product group is the identity element in the quotient group $G_x/(H_{xy}; H_{xz})$. It follows that the condition in (4) may be replaced by the one in (6), so that the inequality on the left side of (3) and the inclusion on the left side of (5) are both equivalent to the same condition, and therefore they are equivalent to each other. This establishes (2).

Turn next to the operation of converse, with the goal of showing that

$$(9) \quad a_{uv,\gamma} \leq a_{xy,\alpha}^{\sim} \quad \text{if and only if} \quad R_{uv,\gamma} \subseteq R_{xy,\alpha}^{-1}.$$

Semi-scaffold Converse Lemma 6.8 implies that

$$(10) \quad a_{uv,\gamma} \leq a_{xy,\alpha}^{\sim} \quad \text{if and only if} \quad u = y, v = x, \text{ and } H_{xy,\alpha}^{\sim} = H_{xy,\gamma},$$

and if the conditions on the right side of this equivalence are satisfied, then equality actually holds on the left side. Semi-frame condition (ii) implies that

$$(11) \quad R_{uv,\gamma} \subseteq R_{xy,\alpha}^{-1} \quad \text{if and only if} \quad u = y, v = x, \text{ and } H_{xy,\alpha}^{\sim} = H_{xy,\gamma},$$

and if the conditions on the right side of this equivalence are satisfied, then equality actually holds. The conditions on the right sides of (10) and (11) are the same, so the inequalities on the left sides must be equivalent. This establishes (9).

Turn finally to the identity element. It is to be shown that that

$$(12) \quad a_{uv,\gamma} \leq 1' \quad \text{if and only if} \quad R_{uv,\gamma} \subseteq id_U.$$

Semi-scaffold Identity Lemma 6.7 implies that

$$(13) \quad a_{uv,\gamma} \leq 1' \quad \text{if and only if} \quad u = v \text{ and } \gamma = 0.$$

The definition of the relation $R_{uv,\gamma}$ and semi-frame condition (i) imply that

$$(14) \quad R_{uv,\gamma} \subseteq id_U \quad \text{if and only if} \quad u = v \text{ and } \gamma = 0.$$

As before, the conditions on the right sides of (13) and (14) are the same, so the inequalities on the left sides must be equivalent. This proves (12).

It has been shown that the bijection ϑ satisfies the conditions of the Atomic Isomorphism Theorem. Apply that theorem to conclude that ϑ can be extended to an isomorphism from \mathfrak{A} to $\mathfrak{C}[\mathcal{F}]$. \square

As was mentioned after Definition 5.3, not every atomic, measurable relation algebra has a scaffold, but if there is a scaffold, then a stronger result than Representation Theorem 7.2 is true. The existence of a scaffold means that for every triple (x, y, z) in \mathcal{E}_3 with $x < y < z$, it is always possible to choose the atoms a_{xy} , a_{yz} and a_{xz} so that the inequality

$$a_{xz} \leq a_{xy}; a_{yz}$$

holds. Consequently, the coset $H_{xy,\zeta}$ of H_{xz} that is chosen to translate a_{xz} to a position below a_{xy} ; a_{xz} in the sense that

$$H_{xz,\zeta}; a_{xz} \leq a_{xy}; a_{yz}$$

may always be taken to be the identity coset H_{xz} , or put another way, one may always choose $\zeta = 0$. The inner automorphism τ of the the quotient group G_x/H_{xz} that is determined by this coset is then the identity automorphism of this group, and therefore the inner automorphism $\hat{\tau}$ of the quotient group $G_x/(H_{xz}; H_{xz})$ that is induced by τ is identity automorphism of its group. As a result, semi-frame condition (iv) assumes the form

$$(1) \quad \hat{\varphi}_{xy} | \hat{\varphi}_{yz} = \hat{\varphi}_{xz}.$$

A semi-frame satisfying this condition instead of semi-frame condition (iv) is called a *frame*.

Under these conditions, the shifting coset C_{xyz} that is defined in terms of the coset $H_{xz,\zeta}$ becomes the identity coset

$$C_{xyz} = H_{xy}; H_{xz}$$

of the quotient group $G_x/(H_{xy}; H_{xz})$, and the definition of the operator \otimes between the atomic relations $R_{xy,\alpha}$ and $R_{yz,\beta}$ assumes the form

$$(2) \quad R_{xy,\alpha} \otimes R_{yz,\beta} = \bigcup \{ R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}] \}.$$

It was shown in Composition Theorem 3.7 of [2] that under the hypothesis of (1) and semi-frame condition (iii), the operation of relational composition between the atomic relations $R_{xy,\alpha}$ and $R_{yz,\beta}$ satisfies the equation

$$(3) \quad R_{xy,\alpha} | R_{yz,\beta} = \bigcup \{ R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha}; H_{yz,\beta}] \}.$$

The right sides of (2) and (3) are the same, so the left sides must be equal. Conclusion: when an atomic, measurable relation algebra has a scaffold, the equation

$$R_{xy,\alpha} \otimes R_{yz,\beta} = R_{xy,\alpha} | R_{yz,\beta}$$

holds, so that the defined operation \otimes coincides with the set-theoretic operation of relational composition.

In this case, the coset relation algebra $\mathfrak{C}[\mathcal{F}]$ is a set relation algebra, and actually a subalgebra of the full set relation algebra with base set and unit

$$U = \bigcup \{ G_x : x \in I \} \quad \text{and} \quad E = \bigcup \{ G_x \times G_y : (x, y) \in \mathcal{E} \}$$

respectively. This algebra is called the *group relation algebra on the frame \mathcal{F}* in [2], and is denoted by $\mathfrak{G}[\mathcal{F}]$.

The system of cosets

$$C = \langle C_{xyz} : (x, y, z) \in \mathcal{E}_3 \rangle$$

is entirely unnecessary in this case, as are the inner automorphisms τ determined by these cosets, so instead of considering group triples $\mathcal{F} = (G, \varphi, C)$ satisfying semi-frame conditions (i)–(iv) in Definition 5.3, it suffices to consider group pairs $\mathcal{F} = (G, \varphi)$ satisfying the four frame conditions, namely conditions (i)–(iii) from Definition 5.3 and condition (1) above. This is the approach that is taken in [2].

The group pair constructed from a scaffold $a = \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ in a measurable relation algebra is called the *group pair associated with a* .

Theorem 7.3. (Frame Theorem) *The group pair associated with a scaffold in a measurable relation algebra is always a frame.*

Notice that the hypothesis on the measurable relation algebra of being atomic is unnecessary when there is a scaffold (or even a semi-scaffold). The existence of a scaffold always implies that the measurable relation algebra under consideration is atomic, by the argument of Semi-scaffold Partition Lemma 6.6.

The preceding observations and remarks, including Frame Theorem 7.3, show that the existence of a scaffold in a measurable relation algebra implies that the semi-frame \mathcal{F} associated with the given scaffold is actually a frame, and consequently the coset relation algebra $\mathfrak{C}[\mathcal{F}]$ constructed from \mathcal{F} is actually the group relation algebra $\mathfrak{G}[\mathcal{F}]$, which is of course a set relation algebra. Representation Theorem 7.2 therefore assumes the the following stronger form.

Theorem 7.4 (Scaffold Representation Theorem). *Every measurable relation algebra with a scaffold is essentially isomorphic to a group relation algebra.*

The preceding theorem implies that a measurable relation algebra with a scaffold is representable as a set relation algebra in a stronger sense than is usually intended. To explain this stronger sense, consider an arbitrary relation algebra \mathfrak{B} and its completion \mathfrak{A} . An isomorphism ϑ from \mathfrak{A} to a complete set relation algebra \mathfrak{C} —that is to say, to a set relation algebra \mathfrak{C} in which the union of every set of relations in the algebra is again a relation in the algebra—must preserve all existing suprema as unions in the sense that, for every subset X of \mathfrak{A} with $a = \sum X$, we have

$$\vartheta(a) = \vartheta(\sum X) = \bigcup\{\vartheta(b) : b \in X\}.$$

The reason is that isomorphisms preserve suprema, and the supremum of each subset Y of \mathfrak{C} is, by assumption, the union of the relations in Y . The restriction of ϑ to \mathfrak{B} is an embedding of \mathfrak{B} into \mathfrak{C} and therefore a representation of \mathfrak{B} as a set relation algebra, but it also inherits from ϑ the stronger property of preserving all existing suprema as unions. Indeed, if X is any subset of \mathfrak{B} such that the supremum $a = \sum X$ exists in \mathfrak{B} , then a remains the supremum of X in \mathfrak{A} , because \mathfrak{A} is the completion of \mathfrak{B} , and therefore ϑ preserves this supremum as a union. Representations that preserve all existing suprema as unions are called *complete representations*, and a relation algebra with a complete representation is said to be *completely representable*.

Corollary 7.5. *Every measurable relation algebra with a scaffold is completely representable.*

It turns out that a kind of converse to Corollary 7.5 is true.

Theorem 7.6. *Every measurable relation algebra that is completely representable has a scaffold.*

Proof. Let \mathfrak{A} be a measurable relation algebra, and assume that ϑ is a complete representation of \mathfrak{A} . Thus, ϑ is an embedding of \mathfrak{A} into the set relation algebra of all subrelations of some equivalence relation E on a base set U , so that

$$(1) \quad \vartheta(1) = E,$$

and ϑ preserves all existing suprema in \mathfrak{A} as unions. A completely representable relation algebra is always atomic (see [5]), so \mathfrak{A} must be atomic. Take X to be the set of atoms in \mathfrak{A} . Each element d in \mathfrak{A} is the sum of the atoms below it,

$$d = \sum\{a \in X : a \leq d\},$$

and therefore

$$(2) \quad \vartheta(d) = \bigcup \{ \vartheta(a) : a \in X \text{ and } a \leq d \},$$

by the assumption that the representation ϑ is complete.

The set \mathcal{E} of pairs (x, y) of measurable atoms such that $x;1;y \neq 0$, or equivalently, such that

$$1; x; 1 = 1; y; 1,$$

is an equivalence relation on the set of measurable atoms. Let $\langle w_\xi : \xi \leq \kappa \rangle$ be a system of representatives for the equivalence classes, so that each equivalence class of \mathcal{E} contains exactly one w_ξ .

The relations $\vartheta(w_\xi)$ are non-empty, because the elements w_ξ are atoms, and they are included in the equivalence relation E , which is the unit of the representing algebra. It follows that each relation of the form

$$(4) \quad E | \vartheta(w_\xi) | E$$

is a non-empty union of components of E , that is to say, it is a non-empty union of relations of the form $V \times V$, where each V is an equivalence class of E . In more detail, if (p, q) is a pair in $\vartheta(w_\xi)$, then p and q are in the same equivalence class of E , call it V . If r and s are any other elements in V , then the pairs (r, p) and (q, s) are both in E , and therefore the pair (r, s) must be in (4), by the definition of relational composition. Thus, every pair in $V \times V$ belongs to (4). For each representative w_ξ , choose an equivalence class V_ξ of E such that

$$(5) \quad V_\xi \times V_\xi \subseteq E | \vartheta(w_\xi) | E.$$

Consider now an arbitrary measurable atom x . Let w_ξ be the representative of x , so that

$$(6) \quad 1; x; 1 = 1; w_\xi; 1,$$

by the definition of \mathcal{E} . Observe that

$$\begin{aligned} E | \vartheta(x) | E &= \vartheta(1) | \vartheta(x) | \vartheta(1) = \vartheta(1; x; 1) = \vartheta(1; w_\xi; 1) \\ &= \vartheta(1) | \vartheta(w_\xi) | \vartheta(1) = E | \vartheta(w_\xi) | E, \end{aligned}$$

by (1), the representation properties of ϑ , and (6). It follows from this computation, the choice of w_ξ , and (5) that

$$(7) \quad V_\xi \times V_\xi \subseteq E | \vartheta(x) | E.$$

Since x is assumed to be a measurable atom in \mathfrak{A} , we have $0 \neq x \leq 1'$, and therefore

$$\emptyset \neq \vartheta(x) \subseteq \vartheta(1') = id_U,$$

by the representation properties of ϑ . Use this observation and (7) to choose an element p_x in V_ξ with the property that the pair (p_x, p_x) belongs to the relation $\vartheta(x)$.

Consider next an arbitrary pair (x, y) in \mathcal{E} . The elements x and y are, by definition, in the same equivalence class of \mathcal{E} , and therefore they have the same representative, say w_ξ . The elements p_x and p_y are both chosen to be in V_ξ , so the pair (p_x, p_y) belongs to the component $V_\xi \times V_\xi$, and therefore also to the unit relation E . The pairs (p_x, p_x) and (p_y, p_y) are chosen to be in $\vartheta(x)$ and $\vartheta(y)$ respectively, so the pair (p_x, p_y) belongs to the relation

$$\vartheta(x) | E | \vartheta(y),$$

by the definition of relational composition. Since

$$\vartheta(x; 1; y) = \vartheta(x) | \vartheta(1) | \vartheta(y) = \vartheta(x) | E | \vartheta(y),$$

by the representation properties of ϑ and (1), it follows that

$$(8) \quad (p_x, p_y) \in \vartheta(x; 1; y).$$

We now come to the heart of the argument. The element $x; 1; y$ is non-zero, by the definition of \mathcal{E} , so it must be the sum of a non-empty set of atoms. In view of (8) and (2) (with $x; 1; y$ in place of d), there must be an atom below $x; 1; y$ whose image under ϑ contains the pair (p_x, p_y) . Moreover, this atom is unique, because (2) is a disjoint union of images of mutually distinct atoms. Call the atom a_{xy} . In other words, a_{xy} is the unique atom in \mathfrak{A} with the property that

$$(9) \quad (p_x, p_y) \in \vartheta(a_{xy}).$$

We shall show that the system

$$(10) \quad \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$$

is a scaffold.

First of all, x is an atom below $x; 1; x$, and the pair (p_x, p_x) belongs to $\vartheta(x)$, by the choice of the element p_x . On the hand, a_{xx} is defined to be the unique atom below $x; 1; x$ with the property that its image under ϑ contains the pair (p_x, p_x) . Consequently,

$$(11) \quad a_{xx} = x.$$

Second, a_{xy} is an atom below $x; 1; y$, so its converse a_{xy}^\sim is an atom below $y; 1; x$, by Lemmas 2.1(vi) and 2.5(iii), and monotony. Moreover, the pair (p_y, p_x) belongs to the image $\vartheta(a_{xy}^\sim)$, because the pair (p_x, p_y) belongs to $\vartheta(a_{xy})$, by definition, and

$$\vartheta(a_{xy}^\sim) = \vartheta(a_{xy})^{-1},$$

by the representation properties of ϑ . On the other hand, the element a_{yx} is defined to be the unique atom below $y; 1; x$ whose image under ϑ contains the pair (p_y, p_x) , so it follows that

$$(12) \quad a_{yx} = a_{xy}^\sim.$$

Finally, the pairs (p_x, p_y) and (p_y, p_z) are in $\vartheta(a_{xy})$ and $\vartheta(a_{yz})$ respectively, by (9), so the pair (p_x, p_z) is in the relational composition $\vartheta(a_{xy}) | \vartheta(a_{yz})$, by the definition of relational composition. Since

$$\vartheta(a_{xy}) | \vartheta(a_{yz}) = \vartheta(a_{xy}; a_{yz}),$$

it follows that the pair (p_x, p_z) belongs to the relation $\vartheta(a_{xy}; a_{yz})$. The same pair also belongs to the relation $\vartheta(a_{xz})$, by (9) so the two relations have a non-empty intersection. Use the representation properties of ϑ to see that the elements $a_{xy}; a_{yz}$ and a_{xz} cannot be disjoint. Since a_{xz} is an atom, it follows that

$$(13) \quad a_{xz} \leq a_{xy}; a_{yz}.$$

Equations (11)–(13) are just the conditions required for (10) to be a scaffold. \square

When the unit element E of the representation coincides with the universal relation $U \times U$, the steps leading up to (8) in the preceding proof are unnecessary. In this case, the equivalence relation \mathcal{E} consists of all pairs of measurable atoms. For each measurable atom x , choose an element p_x in U such that the pair (p_x, p_x)

is in $\vartheta(x)$. Certainly, all of the pairs (p_x, p_y) are in $U \times U$, that is, in $\vartheta(1)$. Because the pairs (p_x, p_x) and (p_y, p_y) are in $\vartheta(x)$ and $\vartheta(y)$ respectively, the pair (p_x, p_y) must be in

$$\vartheta(x) | \vartheta(1) | \vartheta(y).$$

Thus, we obtain (8) directly. The remainder of the proof is just as before.

The next corollary is a direct consequence of the preceding theorem and Scaffold Isomorphism Theorem 7.4.

Corollary 7.7. *Every measurable relation algebra that is completely representable is essentially isomorphic to a group relation algebra.*

A finite relation algebra is necessarily complete, and any representation of it is necessarily a complete representation. Thus, the preceding corollary can be given a strong formulation when the algebra in question is finite.

Corollary 7.8. *Every measurable relation algebra that is finite and representable is isomorphic to a group relation algebra.*

8. FINITELY MEASURABLE RELATION ALGEBRAS

Atomic Partition Lemma 4.11 raises several questions. For example, under what conditions will there be at least one atom below a given rectangle $x ; 1 ; y$ when the sides x and y are measurable atoms? Speaking more broadly, under what conditions will a measurable relation algebra automatically be atomic? It turns out that finite measurability implies atomicity. In other words, if the group G_x is finite for every measurable atom x in a measurable relation algebra, then the algebra is automatically atomic. Keep in mind that finitely measurable relation algebras may be infinite in size, because there may be infinitely many measurable atoms in the algebra.

The key observation is contained in the following lemma, which could have been proved immediately after Product Lemma 4.13. Recall that the set X_a is a union of cosets of H_a . If it is the union of only finitely many such cosets, then we shall say that H_a has *finite index* in X_a . This terminology parallels that of group theory, where one speaks of the index of a subgroup in a group.

Lemma 8.1. *Let x and y be measurable atoms, and $0 < a \leq x ; 1 ; y$. If H_a has finite index in X_a , then there is left-regular element below a .*

Proof. Consider the collection of subsets Z of X_a with the following properties. First, x is in Z . Second,

$$\prod_{f \in Z} f ; a \neq 0.$$

Because $f ; a = g ; a$ if and only if f and g are in the same coset of H_a , by Corollary 3.10, it may also be assumed that Z is a union of cosets of H_a .

There are only finitely many cosets of H_a included in X_a , by assumption, and each Z under consideration is assumed to be a union of some of these cosets, so there are only finitely many possible choices for Z . Moreover, there certainly exist sets Z with the required properties. For instance, the left stabilizer H_a is such a set. In more detail, the identity element x is in H_a , because H_a is a subgroup of G_x . Also,

$$\prod_{f \in H_a} f ; a = a \neq 0,$$

because H_a is the left stabilizer of a . Finally, H_a is the union of the singleton coset $\{H_a\}$.

As there are only finitely many choices for the set Z , it is possible to choose one of maximal cardinality. Let Z be such a maximal choice, and write

$$(1) \quad b = \prod_{f \in Z} f ; a,$$

with the goal of showing that b is a left-regular element below a . First of all, b is not 0, by the second property. Also, x is in Z , by the first property, so $x ; a$ is one of the elements in the product (1). Lemma 2.3(iii) implies that $x ; a = a$, so a must be one of the elements in in the product (1), and therefore b must be below a , by Boolean algebra. It remains to show that b is left-regular. This amounts to proving that

$$(2) \quad X_b = H_b,$$

by Corollary 4.3.

The inclusion from right to left in (2) is a consequence of Lemma 4.5(ii). To establish the reverse inclusion, consider an element g in X_b , with the goal of showing that g is in H_b . Observe that (1) is essentially a finite product. Use (1) and the distributive law for functions to obtain

$$(3) \quad g ; b = g ; (\prod_{f \in Z} f ; a) = \prod_{f \in Z} g ; f ; a.$$

The product $(g ; b) \cdot b$ is non-zero, by Lemma 4.5(i) (with g and b in place of f and a respectively), because g is assumed to belong to X_b . Consequently,

$$(4) \quad (\prod_{f \in Z} g ; f ; a) \cdot (\prod_{f \in Z} f ; a) = (g ; b) \cdot b \neq 0,$$

by (3) and (1). Use (3), (4), and the assumed maximality of the cardinality of Z to conclude that every element in the set

$$(5) \quad g ; Z = \{g ; f : f \in Z\}$$

must also belong to Z , that is to say,

$$(6) \quad g ; Z \subseteq Z.$$

For each coset H_ξ of H_a that is included in Z , the coset $g ; H_\xi$ is included in $g ; Z$, by (5), and these cosets are distinct for distinct ξ , by the cancellation law for groups. Consequently, the sets $g ; Z$ and Z are unions of the same finite number of cosets of H_a . Use this observation and (6) to arrive at $g ; Z = Z$, from which it follows that

$$g ; b = \prod_{f \in Z} g ; f ; a = \prod_{f \in Z} f ; a = b,$$

by (3) and (1). This proves that g is in the left stabilizer H_b , so (2) holds and therefore b is left-regular. \square

Lemma 8.2. *For any measurable atoms x and y , and any non-zero element $a \leq x ; 1 ; y$, if the set X_a is finite, then there is an atom below a .*

Proof. Observe first that if $b \leq a$, then $X_b \subseteq X_a$. Indeed, if $b \leq a$, then the monotony laws imply that $b^\smile \leq a^\smile$ and therefore that $b ; b^\smile \leq a ; a^\smile$. Consequently,

$$\sum X_b = b ; b^\smile \leq a ; a^\smile = \sum X_a,$$

by definition of the sets X_a and X_b . As these are sets of atoms, it follows that X_b is included in X_a .

Let W be the set of left-regular elements below a . For every element b in W ,

$$H_b = X_b \subseteq X_a,$$

by Corollary 4.3 and the initial observation of the proof, so the stabilizer of every element in W is finite. Moreover, the set W is not empty, by Lemma 8.1. It is therefore possible to choose an element b in W such that its left stabilizer H_b has minimal finite cardinality among the left stabilizers of elements in W . Clearly,

$$(1) \quad 0 < b \leq a,$$

by Lemma 4.4 and the definition of W .

The argument that b must be an atom proceeds by contraposition. If b is not an atom, then there is a non-zero element c that is strictly below b , by Boolean algebra and the first inequality in (1). Obviously, c is also below a , by the second inequality in (1), so the set X_c is included in the set X_a , by the initial observation of the proof (with c in place of b). In particular, the set X_c must be finite, since this is true of X_a . Apply Lemma 8.1 (with c in place of a) to obtain a left-regular element d below c . It follows from the definition of the set W that d belongs to this set. Both b and d are left-regular elements, and

$$0 < d = b \cdot d \leq c < b,$$

by the choices of the elements c and d , together with Lemma 4.4. These inequalities and Lemma 4.14 imply that the left stabilizer H_d must be a proper subgroup of the finite left stabilizer H_b . Consequently, the cardinality of H_b cannot be minimal among the left stabilizers of elements b in W . \square

Theorem 8.3. *Every finitely measurable relation algebra is atomic.*

Proof. Let a be an arbitrary non-zero element in the relation algebra, with the goal of showing that there is an atom below a . Consider the set I of measurable atoms. Each element x in I is finitely measurable, by assumption, so the corresponding group G_x is finite, by the definition of finite measurability. The identity element $1'$ is the sum of the set I , by the assumption of measurability, so

$$(1) \quad 1 = 1' ; 1 ; 1' = (\sum I) ; 1 ; (\sum I) = \sum \{x ; 1 ; y : x, y \in I\},$$

by the complete distributivity of relative multiplication. It follows from (1) and Boolean algebra that

$$(2) \quad a = a \cdot 1 = \sum \{a \cdot (x ; 1 ; y) : x, y \in I\}.$$

The element a is assumed to be different from zero, so (2) implies that there must be measurable atoms x and y in I such that the element

$$d = a \cdot (x ; 1 ; y)$$

is not 0. The set X_d is a subset of the finite group G_x and is therefore itself a finite set. Apply Lemma 8.2 (with d in place of a) to obtain an atom b below d . Clearly, b is also an atom below a . \square

9. A CHARACTERIZATION OF REGULAR ELEMENTS

The next theorem and its corollary characterize regular elements in several illuminating ways. One consequence of the corollary is that, in the presence of atoms, left-regularity implies right-regularity and conversely. In other words, left-regular (or right-regular) elements are regular. Moreover, regular elements are precisely the elements that can be written in the form $\sum M; a$ for some atom a below $x; 1; y$ and some subgroup M of G_x that includes H_a . Equivalently, regular elements are precisely the elements that can be written in the form $\sum M_\xi; a$ for some atom a and some coset M_ξ of a subgroup M of G_x that includes H_a .

Theorem 9.1 (Regular Characterization Theorem). *Let x and y be measurable atoms, and $a \leq x; 1; y$ a regular element with normal stabilizers. For each element $b \leq x; 1; y$ with $H_a \subseteq H_b$, the following conditions are equivalent.*

- (i) b is left-regular.
- (ii) b is regular.
- (iii) $b = \sum M_\zeta; a$ for some coset M_ζ of a subgroup M of G_x such that $H_a \subseteq M$.
- (iv) $b = \sum M; c$ for some subgroup M of G_x such that $H_a \subseteq M$ and some (any) left translation c of a that is below b .
- (v) $b = \sum H_{b,\rho}; a$ for some coset $H_{b,\rho}$ of H_b .
- (vi) $b = \sum H_b; c$ for some (any) left translation c of a that is below b .

Proof. Obviously, (ii) implies (i). To establish the implication from (i) to (vi), assume that b is left-regular. The assumptions that a is regular with normal stabilizers, and that $H_a \subseteq H_b$, mean that the implication from (i) to (iii) in Translation Lemma 4.17 may be applied to obtain a left translation c of a that is below b . Any such left translation c is a left-regular element with the same normal stabilizer H_a as a , by Translation Lemmas 4.15 and 4.17, and the assumptions on a . Apply Partition Lemma 4.12 (with c and H_b in place of a and $H_{b,\eta}$ respectively), and use the definition of H_b as the stabilizer of b , to conclude that

$$\sum H_b; c = H_b; b = b,$$

as desired.

To prove that (vi) implies (v), assume that

$$(1) \quad c = H_{a,\rho}; a \quad \text{and} \quad b = \sum H_b; c$$

for some coset $H_{a,\rho}$ of H_a . The set defined by

$$(2) \quad H_{b,\rho} = H_b; H_{a,\rho}$$

is a coset of H_b , because the assumption $H_a \subseteq H_b$ implies that every element in $H_{a,\rho}$ gives rise to the same coset of H_b . Use (1) and (2) to arrive at

$$b = \sum H_b; c = \sum H_b; H_{a,\rho}; a = \sum H_{b,\rho}; a.$$

This proves (v).

To see that (v) implies (iii), take

$$M = H_b \quad \text{and} \quad M_\zeta = H_{b,\rho},$$

and use the assumption that $H_a \subseteq H_b$.

To establish the implications from (iii) to (iv) and from (iv) to (ii), it is helpful to introduce some notation. Let M be a subgroup of G_x that includes H_a . Fix left

coset systems

$$\langle H_\xi : \xi < \kappa \rangle \quad \text{and} \quad \langle M_\eta : \eta < \lambda \rangle$$

for H_a and M respectively in G_x . The assumption $H_a \subseteq M$ implies the existence of a partition $\langle \Gamma_\eta : \eta < \lambda \rangle$ of the index set κ such that

$$(3) \quad M_\eta = \bigcup_{\xi \in \Gamma_\eta} H_\xi$$

for each $\eta < \lambda$.

We now take up the implication from (iii) to (iv). If (iii) holds for the subgroup M , then

$$(4) \quad b = \sum M_\zeta ; a = \sum (\bigcup_{\xi \in \Gamma_\zeta} H_\xi) ; a = \sum_{\xi \in \Gamma_\zeta} H_\xi ; a$$

by (3) (with ζ in place of η) and complete distributivity. The equality of the first and last terms in (4), and Partition Lemma 4.9, together imply that the elements $H_\xi ; a$ with ξ in Γ_ζ are precisely the left translations of a that are below b . Take c to be any one of these translations, say $c = H_\xi ; a$. Certainly c is below b , by (4) and the definition of c . The assumption in (3) (with ζ in place of η) implies that every element in H_ξ gives rise to the same coset of M as every element in M_ζ , and of course that coset is M_ζ . Consequently,

$$(5) \quad M ; H_\xi = M ; M_\zeta = M_\zeta.$$

Use (4) and (5) to conclude that

$$b = \sum M_\zeta ; a = \sum M ; H_\xi ; a = \sum M ; c,$$

as desired.

Turn, finally, to the implication from (iv) to (ii). The element a is assumed to be regular with normal stabilizers, and c is assumed to be a left translation of a , so c must be a regular element with the same normal stabilizers as a , by Translation Lemma 4.15(i) and its right-regular version. In particular,

$$(6) \quad H_c = H_a.$$

Isomorphism Theorem 5.4 guarantees the existence of an isomorphism φ_c from G_x/H_c to G_y/K_c with the property that, writing $K_\xi = \varphi_c(H_\xi)$, we have

$$(7) \quad H_\xi ; c = c ; K_\xi,$$

for each $\xi < \kappa$. Take $\eta = 0$ in (3), and use the convention $M_0 = M$, to obtain

$$(8) \quad M = \bigcup_{\xi \in \Gamma_0} H_\xi.$$

Put

$$(9) \quad N = \bigcup_{\xi \in \Gamma_0} K_\xi.$$

The subgroup M of G_x is assumed in (iv) to include the subgroup H_a , so it also includes H_c , by (6). Use group theory, the definition of a quotient set and (8) to see that the quotient

$$(10) \quad M/H_c = \{f/H_c : f \in M\} = \{H_\xi : \xi \in \Gamma_0\}$$

is a subgroup of G_x/H_c . The image of this subgroup under the quotient isomorphism φ_c is the set

$$(11) \quad \begin{aligned} \varphi_c(M/H_c) &= \{\varphi_c(H_\xi) : \xi \in \Gamma_0\} = \{K_\xi : \xi \in \Gamma_0\} \\ &= \{g/K_c : g \in N\} = N/K_c, \end{aligned}$$

by the definition of an image set, (10), the definition of φ_c , (9), and the definition of the set N/K_c . Isomorphisms preserve the property of being a subgroup, so the equality of the first and last terms in (11) implies that N/K_c is a subgroup of G_y/K_c . It follows from group theory that N must be a subgroup of G_y that includes K_c .

Use (8), complete distributivity, (7), complete distributivity, and (9) to obtain

$$(12) \quad \begin{aligned} \sum M ; c &= \sum (\bigcup_{\xi \in \Gamma_0} H_\xi) ; c = \sum_{\xi \in \Gamma_0} H_\xi ; c \\ &= \sum_{\xi \in \Gamma_0} c ; K_\xi = \sum c ; (\bigcup_{\xi \in \Gamma_0} K_\xi) = \sum c ; N. \end{aligned}$$

Use the assumption in (iv), the second involution law, complete distributivity, the associative law, the regularity of c , and the assumption that M is a subgroup of G_x (and hence closed under formation of converses and relative products), and $H_c \subseteq M$ to get

$$(13) \quad \begin{aligned} b ; b^- &= (\sum M ; c) ; (\sum M ; c)^- = (\sum M ; c) ; (\sum c^- ; M^-) \\ &= \sum M ; c ; c^- ; M^- = \sum M ; H_c ; M^- = \sum M. \end{aligned}$$

A similar computation using (iv), the second involution law, complete distributivity, the associative law, the group properties of M , (12), the regularity of c , and the fact proved above that N is a subgroup of G_y that includes K_c , yields

$$(14) \quad \begin{aligned} b^- ; b &= (\sum M ; c)^- ; (\sum M ; c) = (\sum c^- ; M^-) ; (\sum M ; c) \\ &= \sum c^- ; M^- ; M ; c = \sum c^- ; M ; c \\ &= \sum c^- ; c ; N = \sum K_c ; N = \sum N. \end{aligned}$$

In view of (13), (14), and the definition of regularity, the proof that b is regular will be complete once it is shown that

$$(15) \quad M = H_b \quad \text{and} \quad N = K_b.$$

If f is in M , then $f ; M = M$, since M is a group, and therefore

$$f ; b = f ; (\sum M ; c) = \sum f ; M ; c = \sum M ; c = b,$$

by (iv) and complete distributivity. Thus, every element in M belongs to the left stabilizer of b , so $M \subseteq H_b$. To establish the reverse inclusion, observe that

$$(16) \quad \sum H_b \leq \sum X_b = b ; b^- = \sum M,$$

by Lemma 4.5(ii) and monotony, the definition of the set X_b , and (13). The sets being summed on the left and on the right are sets of atoms, so the inequality in (16) implies the inclusion $H_b \subseteq M$.

The proof of the second equation in (15) is entirely analogous, but uses (14) instead of (13). \square

Corollary 9.2. *Let x and y be measurable atoms. If there is an atom below $x ; 1 ; y$, then for each element $b \leq x ; 1 ; y$ the following conditions are equivalent.*

- (i) b is left-regular.
- (ii) b is regular.
- (iii) $b = \sum M_\zeta ; a$ for some atom $a \leq x ; 1 ; y$ and some left coset M_ζ of a subgroup M of G_x such that $H_a \subseteq M$.
- (iv) $b = \sum M ; a$ for some atom $a \leq x ; 1 ; y$ and some subgroup M of G_x such that $H_a \subseteq M$.

- (v) $b = \sum H_{b,\rho}$; a for some atom $a \leq x; 1; y$ and some coset $H_{b,\rho}$ of H_b .
 (vi) $b = \sum H_b$; a for some atom $a \leq x; 1; y$.

Proof. It may be assumed that b is non-zero, since this is implied by each of the conditions (i)–(vi). The hypothesis that there is an atom below the rectangle $x; 1; y$ implies that this rectangle is a sum of atoms, by Atomic Partition Lemma 4.11. In particular, there must be an atom a below b , since

$$0 < b \leq x; 1; y.$$

Of course, $H_a \subseteq H_b$, by Lemma 4.8. Atoms are regular elements with normal stabilizers, by Corollaries 4.7 and 4.16, so the hypotheses of Theorem 9.1 are satisfied. The equivalence of statements (i)–(vi) in that theorem, and the fact that any translation of an atom is itself an atom, by Lemma 4.11, immediately yield the corollary. \square

Corollary 9.3. *Let x and y be measurable atoms, and $a \leq x; 1; y$ a regular element with normal stabilizers. If $b = \sum M_\zeta$; a for some coset M_ζ of a subgroup M of G_x such that $H_a \subseteq M$, then M is the left stabilizer of b .*

Proof. If $b = \sum M_\zeta$; a , then for some (any) translation c of a that is below b , we have

$$b = \sum M; c \quad \text{and} \quad b = \sum H_b; c,$$

by the implications in Theorem 9.1 from (iii) to (iv) and from (iii) to (vi) respectively. Therefore, $M = H_b$, by Corollary 4.10 (with M , H_b , and c in place of X , Y , and a respectively). \square

The two previous corollaries show that, in general, one cannot expect the stabilizers of regular elements to be *normal* subgroups. To see this, suppose that a is an atom below $x; 1; y$. Take M to be any subgroup of G_x that includes H_a but is not normal. The element $b = \sum M; a$ is regular, by Corollary 9.2, and its left stabilizer is the non-normal subgroup M , by Corollary 9.3.

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