On the Number of Solutions of Exponential Congruences

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Abstract

For a prime p and an integer $a \in \mathbb{Z}$ we obtain nontrivial upper bounds on the number of solutions to the congruence $x^x \equiv a \pmod p$, $1 \le x \le p-1$. We use these estimates to estimate the number of solutions to the congruence $x^x \equiv y^y \pmod p$, $1 \le x, y \le p-1$, which is of cryptographic relevance.

1 Introduction

For a prime p and an integer $a \in \mathbb{Z}$ we denote by $N(p;a)$ the number of solutions to the congruence

$$x^x \equiv a \pmod p, \qquad 1 \le x \le p-1. \tag{1}$$

Obviously only the case of $\gcd(a,p) = 1$ is of interest.

We note that other than the result Crocker [3] showing that there are at least $\lfloor \sqrt{(p-1)/2} \rfloor$ incongruent values of $x^x \pmod p$ when $1 \le x \le p-1$ and our estimates, little appears to be known about the solutions to (1). The function $x \mapsto x^x \pmod p$, is also used in some cryptographic protocols (see [9, Sections 11.70 and 11.71]), so certainly deserves further investigation, see also [8] for various conjectures concerning this function.

Here we suggest several approaches to studying this congruence and derive some upper bounds for $N(p;a)$.

Our first bound is nontrivial if a is of small multiplicative order, which in the particular case when $a = 1$, takes the form $N(p;a) \le p^{1/3+o(1)}$ as $p \to \infty$. The second bound is nontrivial if a is of large multiplicative order, which in the particular case when a is a primitive root modulo p, takes the form $N(p;a) \le p^{11/12+o(1)}$ as $p \to \infty$.

Furthermore, both bounds combined imply that as $p \to \infty$, we have the uniform estimate

$$N(p;a) \le p^{12/13+o(1)}. \tag{2}$$

Finally, we estimate the number of solutions $M(p)$ to the symmetric congruence

$$x^x \equiv y^y \pmod p, \qquad 1 \le x, y \le p-1, \tag{3}$$

which has been considered by Holden & Moree [8] in their study of short cycles in the iterations of the discrete logarithm modulo p, see also [6, 7]. However, no nontrivial estimate of $M(p)$ has been known prior to this work. Clearly

$$M(p) = \sum_{a=1}^{p-1} N(p;a)^2. \tag{4}$$

Thus using the bound (2) and the identity

$$\sum_{a=1}^{p-1} N(p;a) = p-1, \tag{5}$$

we immediately derive

$$M(p) \le p^{25/13+o(1)}. \tag{6}$$

However here we obtain a slightly stronger bound, namely

$$M(p) \le p^{48/25+o(1)}.$$

Surprisingly enough, besides elementary number theory arguments, the bounds derived here rely on some results and arguments from additive combinatorics, in particular on results of Garaev [4].

For an integer $m \ge 1$ we use $\mathbb{Z}_m$ to denote the residue ring modulo m and we use $\mathbb{Z}_m^*$ to denote the unit group of $\mathbb{Z}_m$.

Note that without the condition $1 \le x \le p-1$ (needed in the cryptographic application) there are always many solutions. Let g be a primitive root modulo p. For any element $a \in \mathbb{Z}_p^*$ (and so for any integer $a \not\equiv 0 \pmod p$) we use $\operatorname{ind} a$ for its discrete logarithm modulo p, that is, the unique residue class $v \pmod{p-1}$ with

$$g^v \equiv a \pmod p.$$

Now, if for a primitive root g we have

$$x \equiv p \operatorname{ind} a - (p-1)g \pmod{p(p-1)},$$

then

$$x^x \equiv g^{p\ \operatorname{ind} a-(p-1)g} \equiv (g^p)^{\operatorname{ind} a} \cdot (g^{-g})^{p-1} \equiv a \pmod p.$$

2 Elements of Small Order

We need to recall some notions and results from additive combinatorics.

For a prime p and a set $\mathcal{A} \subseteq \mathbb{Z}_p^*$ we define the sets

$$\mathcal{A}+\mathcal{A} = \{a_1+a_2 \ : \ a_1,a_2 \in \mathcal{A}\}, \quad \mathcal{A}\cdot\mathcal{A} = \{a_1a_2 \ : \ a_1,a_2 \in \mathcal{A}\}.$$

Our bound on $N(p,a)$ makes use of the following estimate of Garaev [4, Theorem 1].

Lemma 1 *For any set $\mathcal{A} \subseteq \mathbb{Z}_p^*$,*

$$\#(\mathcal{A}+\mathcal{A}) \cdot \#(\mathcal{A}\cdot\mathcal{A}) \gg \min\left\{p\#\mathcal{A}, \frac{(\#\mathcal{A})^4}{p}\right\}.$$

Let $\operatorname{ord} a$ denote the multiplicative order of $a \in \mathbb{Z}_p^*$.

Theorem 2 *Uniformly over $t \mid p-1$, we have, as $p \to \infty$,*

$$\sum_{\substack{a \in \mathbb{Z}_p^* \\ \operatorname{ord} a \mid t}} N(p;a) \le \max\{t, p^{1/2}t^{1/4}\} p^{o(1)}.$$

Proof. Fix a primitive root g mod p. The union of non-zero residue classes a with $\operatorname{ord} a \mid t$ of all the solutions to (1) is precisely the set of solutions to

$$x^{tx} \equiv 1 \pmod p, \qquad 1 \le x \le p-1. \tag{7}$$

This congruence is equivalent to

$$tx \operatorname{ind} x \equiv 0 \pmod{p-1},$$

or if we put

$$T = \frac{p-1}{t}$$

to

$$x \operatorname{ind} x \equiv 0 \pmod T,$$

or after fixing $d \mid T$ and considering only the solutions to (7) with

$$\gcd(x, T) = d,$$

they can be written as $x = dy$ and satisfy

$$\operatorname{ind}(dy) \equiv 0 \pmod{T_d}, \qquad 1 \le y \le D, \qquad \gcd(y, T_d) = 1. \tag{8}$$

where

$$T_d = \frac{T}{d} \qquad \text{and} \qquad D = \frac{p-1}{d}.$$

Let us denote by $\mathcal{Y}_d$ the set of integers y satisfying (8), and by $\mathcal{W}_d$ the set of the residue classes mod p represented by the elements of $\mathcal{Y}_d$. Obviously $\#\mathcal{Y}_d = \#\mathcal{W}_d$, and we have

$$\sum_{\substack{a \in \mathbb{Z}_p^* \\ \operatorname{ord} a \mid t}} N(p;a) = \sum_{d \mid T} \#\mathcal{Y}_d = \sum_{d \mid T} \#\mathcal{W}_d. \tag{9}$$

First note that

$$\#\left(\mathcal{W}_d + \mathcal{W}_d\right) \le \#\left(\mathcal{Y}_d + \mathcal{Y}_d\right) \le 2D \qquad (10)$$

from the second condition in (8).

Furthermore, the product set of $\mathcal{W}_d$ is contained in

$$\{w \in \mathbb{Z}_p^* : \operatorname{ind}(d^2 w) \equiv 0 \pmod{T_d}\},$$

and so

$$\#\left(\mathcal{W}_d \cdot \mathcal{W}_d\right) \le \frac{p-1}{T_d} = dt. \qquad (11)$$

Hence, applying Lemma 1 and using the bounds (10) and (11) we see that

$$\min\left\{p\#\mathcal{W}_d, \frac{\left(\#\mathcal{W}_d\right)^4}{p}\right\} \ll pt.$$

Hence

$$\#\mathcal{W}_d \ll \max\{t, p^{1/2}t^{1/4}\}. \qquad (12)$$

Recalling the bound on the divisor function $\tau(k)$

$$\tau(k) = \sum_{d|k} 1 = k^{o(1)}, \qquad (13)$$

see [5, Theorem 315], and using (12) in (9), we conclude the proof. □

Corollary 3 *Uniformly over $t \mid p-1$ and all integers a with* $\gcd(a,p) = 1$ *of multiplicative order* $\operatorname{ord} a = t$*, we have, as $p \to \infty$,*

$$N(p;a) \le \max\{t, p^{1/2}t^{1/4}\}p^{o(1)}.$$

Next we show that if t is very small then the bound of Theorem 2 can be improved. For example, this applies to the most interesting special case of the congruence (1), namely the case $a = 1$.

Theorem 4 *Uniformly over $t \mid p-1$, we have, as $p \to \infty$,*

$$\sum_{\substack{a \in \mathbb{Z}_p^* \\ \operatorname{ord} a | t}} N(p;a) \le p^{1/3+o(1)}t^{2/3}.$$

Proof. We follow the proof of Theorem 2 up to (11), but finish the argument in a different way to derive a new bound for $\#\mathcal{Y}_d$. Let us define

$$s(b) = \#\{(y_1, y_2) \ : \ y_1, y_2 \in \mathcal{Y}_d, \ y_1 y_2 \equiv b \pmod p\}.$$

First note that $s(b) > 0$ only when $b \in \mathcal{W}_d \cdot \mathcal{W}_d$, and so

$$(\#\mathcal{Y}_d)^2 = \sum_{b \in \mathbb{Z}_p} s(b) \le \# \left(\mathcal{W}_d \cdot \mathcal{W}_d\right) \max_{b \in \mathbb{Z}_p} s(b). \tag{14}$$

If (y_1, y_2) is counted in $s(b)$ then on the one hand $y_1 y_2 \equiv b \pmod p$, on the other hand $1 \le y_1 y_2 \le D^2$ (where as before $D = (p-1)/d$), therefore $y_1 y_2 = b + kp$, where $0 \le k < \frac{p}{d^2}$. Thus the product $y_1 y_2$ can take at most $p/d^2 + 1$ possible values $y_1 y_2 = z$ and once z is fixed, there are $\tau(z) = z^{o(1)} = p^{o(1)}$ possibilities for the pair (y_1, y_2), see (13). Thus

$$s(b) \le (p/d^2 + 1)p^{o(1)},$$

which after inserting in (14) and recalling (11) yields

$$\#\mathcal{Y}_d \le \left((pt/d)^{1/2} + (td)^{1/2}\right) p^{o(1)}. \tag{15}$$

For $d \le p^{1/3} t^{-1/3}$ we use $\#\mathcal{Y}_d \le dt$ from the first condition of (8) and for $d \ge p^{2/3} t^{-1/3}$ we use $\#\mathcal{Y}_d \le D$ from the second condition of (8). Therefore we obtain

$$\#\mathcal{Y}_d \ll p^{1/3} t^{2/3} \qquad \text{and} \qquad \#\mathcal{Y}_d \ll p^{1/3} t^{1/3},$$

respectively.

Finally, for $p^{1/3} t^{-1/3} \le d \le p^{2/3} t^{-1/3}$ we use (15) to derive

$$\#\mathcal{Y}_d \le \left(p^{1/3} t^{2/3} + p^{1/3} t^{1/3}\right) p^{o(1)} = p^{1/3 + o(1)} t^{2/3}.$$

Using these bounds with (13) in (9) we conclude the proof. $\square$

Corollary 5 *Uniformly over $t \mid p-1$ and all integers a with $\gcd(a,p) = 1$ of multiplicative order $\operatorname{ord} a = t$, we have, as $p \to \infty$,*

$$N(p; a) \le p^{1/3 + o(1)} t^{2/3}.$$

3 Elements of Large Order

Here we use a different argument, which is similar to the one used in [1], and a bound of [2], on the number of solutions of an exponential congruence, plays the crucial role. However, this approach is effective only for values of a of sufficiently large order.

We recall the following estimate, given in [2, Lemma 7], on the number of zeros of sparse polynomials over a finite field $\mathbb{F}_q$ of q elements.

Lemma 6 *For $n \geq 2$ given elements $a_1, \ldots, a_n \in \mathbb{F}_q^*$ and integers $k_1, \ldots, k_n$ in $\mathbb{Z}$ let us denote by Q the number of solutions of the equation*

$$\sum_{i=1}^{n} a_i X^{k_i} = 0, \qquad X \in \mathbb{F}_q^*.$$

Then

$$Q \leq 2q^{1-1/(n-1)} \Delta^{1/(n-1)} + O\left(q^{1-2/(n-1)} \Delta^{2/(n-1)}\right),$$

where

$$\Delta = \min_{1 \leq i \leq n} \max_{j \neq i} \gcd(k_j - k_i, q-1).$$

We are now ready to prove the main result of this section.

Theorem 7 *Uniformly over $t \mid p-1$ and all integers a with $\gcd(a,p) = 1$ of multiplicative order $\operatorname{ord} a = t$, we have, as $p \to \infty$,*

$$N(p;a) \leq p^{1+o(1)} t^{-1/12}.$$

Proof. Let a be a non-zero residue class modulo p of multiplicative order $t \mid p-1$. As before, we put

$$T = \frac{p-1}{t}$$

Clearly, there is a primitive root g modulo p with $a \equiv g^T \pmod{p}$. Using the discrete logarithm to base g, the congruence (1) is equivalent to

$$x \operatorname{ind} x \equiv T \pmod{p-1}.$$

Note the condition $\gcd(x, p-1) \mid T$. After fixing $d \mid T$ and considering only the solutions to (1) with $\gcd(x, p-1) = d$, they can be written as $x = dy$ and satisfy

$$y \operatorname{ind}(dy) \equiv T_d \pmod{D}, \quad 1 \leq y \leq D, \quad \gcd(y, D) = 1,$$

where, as before,

$$T_d = \frac{T}{d} \qquad \text{and} \qquad D = \frac{p-1}{d}.$$

Note that $t \mid D$. The congruence $yz \equiv 1 \pmod D$ defines a one–to–one correspondence between the integers $\{1 \le y \le D \ : \ \gcd(y, D) = 1\}$ and $z \in \mathbb{Z}_D^*$.

Furthermore, the relation $yz \equiv 1 \pmod D$ defines a one–to–M_d correspondence between the set $\{1 \le y \le D \ : \ \gcd(y, D) = 1\}$ and $z \in \mathbb{Z}_{p-1}^*$, where M_d is the number of residue classes in $\mathbb{Z}_{p-1}^*$ in the form $z + kD$. These residue classes are automatically coprime to D, but we have to ensure that they are coprime to d as well (and thus belong to $\mathbb{Z}_{p-1}^*$). Thus using $\mu(k)$ to denote the Möbius function, by [5, Theorem 263] (which is essentially the inclusion-exclusion principle) we obtain

$$\begin{aligned} M_d &= \sum_{k=1}^{d} \sum_{f \mid \gcd(z+kD,d)} \mu(f) = \sum_{f \mid d} \mu(f) \sum_{\substack{k=1 \\ z+kD \equiv 0 \pmod f}}^{d} 1 \\ &= \sum_{\substack{f \mid d \\ \gcd(f,D)=1}} \mu(f) \frac{d}{f} = d\, \frac{\varphi(m)}{m}, \end{aligned}$$

where $\varphi(k)$ is the Euler function and m is the product of primes q with $q \mid d$ and $q \nmid D$, see [5, Equation (16.3.1)]. In particular $m \le d \le p$ and recalling the well-known estimate on the Euler function, see [5, Theorem 328] we obtain

$$M_d = dp^{o(1)}.$$

From now on the integer $1 \le y \le D$ and the residue class $z \in \mathbb{Z}_{p-1}^*$ with or without subscripts are always connected by $yz \equiv 1 \pmod D$, even if this is not explicitly stated.

Let us define

$$\mathcal{Z}_d = \{z \in \mathbb{Z}_{p-1}^* \ : \ \text{ind}\,(dy) \equiv Dz/t \pmod D, \ 1 \le y \le D\}.$$

(we recall our convention that we always have $yz \equiv 1 \pmod D$). We have

$$N(p, a) = \sum_{d \mid T} \frac{1}{M_d} \# \mathcal{Z}_d \le p^{o(1)} \sum_{d \mid T} \frac{1}{d} \# \mathcal{Z}_d. \qquad (16)$$

The congruence $\operatorname{ind}(dy) \equiv Dz/t \pmod D$ is equivalent to

$$dy \equiv \rho g^{Dz/t} \pmod p,$$

for some $\rho \in \mathbb{Z}_p^*$ with $\rho^d \equiv 1 \pmod p$. Thus we split $\mathcal{Z}_d$ into subsets $\mathcal{Z}_{d,\rho}$ getting

$$\#\mathcal{Z}_d = \sum_{\rho^d \equiv 1 \pmod p} \#\mathcal{Z}_{d,\rho}, \tag{17}$$

where

$$\mathcal{Z}_{d,\rho} = \{z \in \mathbb{Z}_{p-1}^* \ : \ dy \equiv \rho g^{Dz/t} \pmod p, \ 1 \le y \le D\}$$

(and again we recall our convention that $yz \equiv 1 \pmod D$).

Clearly,

$$(\#\mathcal{Z}_{d,\rho})^2 = \#\{z_1, z_2 \in \mathbb{Z}_{p-1}^* \ : \ dy_j \equiv \rho g^{Dz_j/t} \pmod p, \ j = 1, 2\}.$$

We have by adding the two congruences that

$$\begin{aligned}(\#\mathcal{Z}_{d,\rho})^2 & \\ &\le \#\{z_1, z_2 \in \mathbb{Z}_{p-1}^* \ : \ d(y_1+y_2) \equiv \rho\left(g^{Dz_1/t} + g^{Dz_2/t}\right) \pmod p\} \\ &= \sum_{v \in \mathbb{Z}} \#\{z_1, z_2 \in \mathbb{Z}_{p-1}^* \ : \ d(y_1+y_2) = v, \\ &\qquad\qquad \rho\left(g^{Dz_1/t} + g^{Dz_2/t}\right) \equiv v \pmod p\}.\end{aligned}$$

The sum over $v \in \mathbb{Z}$ is empty unless $v = dw$, where $2 \le w \le 2D$ and we get by the Cauchy–Schwarz inequality that

$$\begin{aligned}(\#\mathcal{Z}_{d,\rho})^4 \le 2D\#\{z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^* \ : \ & d(y_1+y_2) = d(y_3+y_4) \\ & \equiv \rho\left(g^{Dz_1/t} + g^{Dz_2/t}\right) \equiv \rho\left(g^{Dz_3/t} + g^{Dz_4/t}\right) \pmod p\}.\end{aligned}$$

Clearly, when $z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^*$ are fixed, then the condition

$$\begin{aligned}d(y_1+y_2) &= d(y_3+y_4) \\ &\equiv \rho\left(g^{Dz_1/t} + g^{Dz_2/t}\right) \equiv \rho\left(g^{Dz_3/t} + g^{Dz_4/t}\right) \pmod p\end{aligned}$$

defines ρ uniquely. Hence

$$\begin{aligned}\sum_{\rho^d \equiv 1 \pmod p} & (\#\mathcal{Z}_{d,\rho})^4 \\ &\le 2D\#\{z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^* \ : \ y_1+y_2 = y_3+y_4, \\ &\qquad g^{Dz_1/t} + g^{Dz_2/t} \equiv g^{Dz_3/t} + g^{Dz_4/t} \pmod p\}.\end{aligned}$$

Relaxing the condition $y_1+y_2=y_3+y_4$ to $y_1+y_2\equiv y_3+y_4 \pmod D$ only increases the number of solution (but allows us to think about y_j as a residue class modulo D defined by $y_jz_j\equiv 1 \pmod D$, $j=1,2,3,4$). Thus

$$\sum_{\rho^d\equiv 1 \pmod p} (\#\mathcal{Z}_{d,\rho})^4 \le 2D\#\{z_1,z_2,z_3,z_4\in\mathbb{Z}_{p-1}^* \;:\; y_1+y_2\equiv y_3+y_4 \pmod D, \quad g^{Dz_1/t}+g^{Dz_2/t}\equiv g^{Dz_3/t}+g^{Dz_4/t} \pmod p\}.$$

Finally, after the substitution $z_j\to wz_j$ for $w\in\mathbb{Z}_{p-1}^*$ (and thus $y_j\to w^{-1}y_j$), $j=1,2,3,4$, where w^{-1} is defined modulo D, we obtain that any solution is computed with $\varphi(p-1)$ multiplicity, that is

$$\sum_{\rho^d\equiv 1 \pmod p} (\#\mathcal{Z}_{d,\rho})^4 \le \frac{2D}{\varphi(p-1)}\#\{z_1,z_2,z_3,z_4,w\in\mathbb{Z}_{p-1}^* \;: \; y_1+y_2\equiv y_3+y_4 \pmod D, \quad (g^w)^{Dz_1/t}+(g^w)^{Dz_2/t}\equiv (g^w)^{Dz_3/t}+(g^w)^{Dz_4/t} \pmod p\}. \tag{18}$$

Writing $X\equiv g^w \pmod p$ and $k_j=Dz_j/t=(p-1)z_j/dt=T_dz_j$, after fixing z_1,z_2,z_3,z_4, the number of $w\in\mathbb{Z}_{p-1}^*$ satisfying the congruence in (18) is bounded by the number of solutions to the congruence $X^{k_1}+X^{k_2}\equiv X^{k_3}+X^{k_4} \pmod p$, and this is bounded in Lemma 6, applied with $n=4$, by $O\left(p^{2/3}\Delta^{1/3}\right)$, where

$$\Delta = \min_{1\le i<j\le 4}\gcd\left(T_d(z_i-z_j),p-1\right)=T_d\min_{1\le i<j\le 4}\gcd\left(z_i-z_j,dt\right).$$

For every fixed i,j, $1\le i<j\le 4$ and $\delta\mid dt$ there are $(p-1)^2/\delta$ choices for (z_i,z_j) with

$$\gcd(z_i-z_j,dt)=\delta.$$

When z_i and z_j are fixed the congruence $y_1+y_2\equiv y_3+y_4 \pmod D$ implies that there are $dp^{1+o(1)}$ choices for the remaining two variables. (Recall that each y determines $M_d=dp^{o(1)}$ different choices of z.) Thus, putting everything together in (18) and recalling (13), we obtain

$$\sum_{\rho^d\equiv 1 \pmod p} (\#\mathcal{Z}_{d,\rho})^4 \le \frac{2D}{\varphi(p-1)}\sum_{\delta\mid dt}p^{2/3}(T_d\delta)^{1/3}\frac{(p-1)^2}{\delta}dp^{1+o(1)}$$
$$= dDp^{8/3+o(1)}T_d^{1/3}\sum_{\delta\mid dt}\delta^{-2/3}=p^{11/3+o(1)}T_d^{1/3}=\frac{p^{4+o(1)}}{(dt)^{1/3}}.$$

Putting this to (17), we get by the Hölder inequality

$$\#\mathcal{Z}_d \le d^{3/4} \left(\sum_{\rho^d \equiv 1 \pmod p} (\#\mathcal{Z}_{d,\rho})^4 \right)^{1/4} \le \frac{p^{1+o(1)}}{t^{1/12}}\, d^{2/3}.$$

Finally (16) and (13) gives

$$N(p,a) \le \sum_{d|(p-1)/t} \frac{p^{1+o(1)}}{t^{1/12} d^{1/3}} \le \frac{p^{1+o(1)}}{t^{1/12}},$$

and we conclude the proof. □

4 Symmetric Congruence

We now improve the bound (6) on the number of solutions to the symmetric congruence (3).

Theorem 8 *We have, as $p \to \infty$.*

$$M(p) \le p^{48/25+o(1)}.$$

Proof. From (4) we obtain

$$M(p) \le \sum_{t|p-1} \sum_{\substack{a\in\mathbb{Z}_p^* \\ \operatorname{ord} a = t}} N(p;a)^2.$$

We fix some parameter ϑ and for $t \le \vartheta$ we use Theorem 2 to estimate

$$\begin{aligned} \sum_{\substack{a\in\mathbb{Z}_p^* \\ \operatorname{ord} a = t}} N(p;a)^2 &\le \left(\sum_{\substack{a\in\mathbb{Z}_p^* \\ \operatorname{ord} a = t}} N(p;a) \right)^2 \\ &\le \max\{t^2 p^{o(1)}, p^{1+o(1)} t^{1/2}\} \le \max\{\vartheta^2 p^{o(1)}, p^{1+o(1)} \vartheta^{1/2}\}. \end{aligned}$$

For $t \ge \vartheta$ we use Theorem 7 together with (5) to estimate

$$\sum_{\substack{a\in\mathbb{Z}_p^* \\ \operatorname{ord} a = t}} N(p;a)^2 \le p^{1+o(1)} t^{-1/12} \sum_{\substack{a\in\mathbb{Z}_p^* \\ \operatorname{ord} a = t}} N(p;a) \le p^{2+o(1)} \vartheta^{-1/12}.$$

Taking

$$\vartheta = p^{24/25}$$

to balance the above estimates, we obtain the bound

$$\sum_{\substack{a\in\mathbb{Z}_p^* \\ \mathrm{ord}\, a=t}} N(p;a)^2 \le p^{48/25+o(1)}$$

and using (13), we conclude the proof. □

5 Concluding Remarks

Clearly Theorem 2 is nontrivial provided that $t \le p^{1-\varepsilon}$ for some $\varepsilon > 0$, while Theorem 7 is nontrivial provided $t \ge p^{\varepsilon}$, for an arbitrary $\varepsilon > 0$ and a sufficiently large p. In particular, using Corollary 3 for $t \le p^{12/13}$ and Theorem 7 for $t > p^{12/13}$, we derive (2).

It is also easy to see that all but $o(p)$ elements $a \in \mathbb{Z}_p^*$ are of multiplicative order $t = p^{1+o(1)}$. Thus for almost all $a \in \mathbb{Z}_p^*$ we have $N(p;a) \le p^{11/12+o(1)}$ by Theorem 7.

Similar results can also be established for several other congruences. For example, the same arguments as those used in the proof of Theorem 4 imply that the congruence

$$x^{x-1} \equiv 1 \pmod p, \qquad 1 \le x \le p-1,$$

has $O\left(p^{1/3+o(1)}\right)$ solutions. This means that the function $x \mapsto x^x \pmod p$ has $O(p^{1/3+o(1)})$ fixed points in the interval $1 \le x \le p-1$.

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