Uniformly recurrent subgroups and simple C^*-algebras*

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Abstract

We study uniformly recurrent subgroups (URS) introduced by Glasner and Weiss [18]. Answering their query we show that any URS Z of a finitely generated group is the stability system of a minimal Z-proper action. We also show that for any sofic URS Z there is a Z-proper action admitting an invariant measure. We prove that for a URS Z all Z-proper actions admits an invariant measure if and only if Z is coamenable. In the second part of the paper we study the separable C^*-algebras associated to URS's. We prove that if a URS is generic then its C^*-algebra is simple. We give various examples of generic URS's with exact and nuclear C^*-algebras and an example of a URS Z for which the associated simple C^*-algebra is not exact and not even locally reflexive, in particular, it admits both a uniformly amenable trace and a nonuniformly amenable trace.

Keywords. uniformly recurrent subgroups, simple C^*-algebras, amenable traces, graph limits

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Contents

1 Introduction

Let Γ be a countable group and $\mathrm{Sub}(\Gamma)$ be the compact space of all subgroups of Γ. The group Γ acts on $\mathrm{Sub}(\Gamma)$ by conjugation. **Uniformly recurrent subgroups** (URS) were defined by Glasner and Weiss [18] as closed, invariant subsets $Z \subset \mathrm{Sub}(\Gamma)$ such that the action of Γ on Z is minimal (every orbit is dense). Now let (X, Γ, α) be a Γ-system (that is, X is a compact metric space and $\alpha : \Gamma \to \mathrm{Homeo}(X)$ is a homomorphism). For each point $x \in X$ one can define the topological stabilizer subgroup $\mathrm{Stab}^0_\alpha(x)$ by

$$\mathrm{Stab}^0_\alpha(x) = \{\gamma \in \Gamma \mid \gamma \text{ fixes some neighborhood of } x\}.$$

Let us consider the Γ-invariant subset $X^0 \subseteq X$ such that $x \in X^0$ if and only if $\mathrm{Stab}_\alpha(x) = \mathrm{Stab}^0_\alpha(x)$. The closure of the invariant subset $\mathrm{Stab}_\alpha(X^0) \subset \mathrm{Sub}(\Gamma)$ is called the **stability system** of (X, Γ, α) (see also [21],[23]). If the action is minimal, then the stability system of (X, Γ, α) is a URS. Glasner and Weiss proved (Proposition 6.1,[18]) that for every URS $Z \subset \mathrm{Sub}(\Gamma)$ there exists a topologically transitive (that is there is a dense orbit) system (X, Γ, α) with Z as its stability system. They asked (Problem 6.2., [18]), whether for any URS Z there exists a minimal system (X, Γ, α) with Z as its stability system. Recently, Kawabe [21] gave an affirmative answer for this question in the case of amenable groups.

Definition 1.1. Let Γ be a countable group and $Z \subset \mathrm{Sub}(\Gamma)$ be a URS. A system (X, Γ, α) is Z-proper if for any $x \in X$ $\mathrm{Stab}_\alpha(x) = \mathrm{Stab}^0_\alpha(x)$ and $\mathrm{Stab}_\alpha(X) \in Z$.

Before stating our first result we prove a lemma for the sake of completeness.

Lemma 1.1. *If (X, Γ, α) is a Z-proper system, then the map $Stab_\alpha : X \to Sub(\Gamma)$ is continuous.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X converging to an element $x \in X$. We need to show that $\gamma \in \mathrm{Stab}_\alpha(x)$ if and only if there exists some constant $N_\gamma > 0$ such that if $n \geq N_\gamma$ then $\gamma \in \mathrm{Stab}_\alpha(x_n)$. Clearly, if $\gamma \in \mathrm{Stab}_\alpha(x_n)$, then by the continuity of α, $\gamma \in \mathrm{Stab}_\alpha(x)$ for large enough n. In other words, for any Γ-system (X, Γ, α) the map $\mathrm{Stab}_\alpha : X \to \mathrm{Sub}(\Gamma)$ is upper-semicontinuous. It is important to note that for Γ-systems in general the map Stab_α is not necessarily continuous at all points $x \in X$. Let $(X, \alpha, \mathbb{Z})$ be the standard Bernoulli shift. That is, $X = \{0, 1\}^{\mathbb{Z}}$ and α is the left translation by $\mathbb{Z}$. Let $x_n \in X$ be defined the following way. For $n \geq 1$, let $x_n(k) = 1$ if $|k| \leq n$ and let $x_n(k) = 0$, otherwise. Also, let $x(k) = 1$ for any $k \in \mathbb{Z}$. Then, $x_n \to x$. On the other hand, $\mathrm{Stab}_\alpha(x_n) = \{0\}$ for all $n \geq 1$ and $\mathrm{Stab}_\alpha(x) = \mathbb{Z}$. Now, if (X, α, Γ) is Z-proper for some URS Z and $\gamma \in \mathrm{Stab}_\alpha(x)$, then $\gamma \in \mathrm{Stab}_\alpha(y)$ for some neighborhood $x \in U \subset X$. Hence, there exists some constant $N_\gamma > 0$ such that $\gamma \in Stab_\alpha(x_n)$ provided that $n \geq N_\gamma$. Therefore our lemma follows. $\square$

Theorem 1. *If Γ is a finitely generated group and $Z \subset Sub(\Gamma)$ is a URS, then there exists a minimal Z-proper system (X, Γ, α) (that is, Z is the stability system of (X, Γ, α)).*

In fact, we will show that X can be chosen as a Z-proper minimal Bernoulli subshift (see Definition 2.1). In the proof we will use the Lovász Local Lemma technique of Alon, Grytczuk, Haluszczak and Riordan [3] to construct a minimal action on the space of rooted colored Γ-Schreier graphs. This approach has already been used to construct free Γ-Bernoulli subshifts by Aubrun, Barbieri and Thomassé [2] . Very recently, Matte Bon and Tsankov [24] completely answered the query of Glasner and Weiss for uniformly recurrent subgroups of discrete and locally compact groups. The next result of the paper is about the existence of invariant measures on Z-proper Bernoulli subshifts. For a long time all finitely generated groups that had been known to have free Bernoulli subshifts were residually-finite. Then Dranishnikov and Schroeder [15] constructed a free Bernoulli subshift for any torsion-free hyperbolic group. Somewhat later Gao, Jackson and Seward proved that any countable group has free Bernoulli subshifts [16], [17]. On the other hand, Hjorth and Molberg [20] proved that for any countable group Γ there exists a free continuous action of Γ on a Cantor set admitting an invariant probability measure. We will prove the following result.

Theorem 2. *Let Γ be a finitely generated group and $Z \subset Sub(\Gamma)$ be a sofic URS (see Definition 4.1) then there exists a Z-proper Bernoulli shift with an invariant probability measure. In particular, for every finitely generated sofic group Γ there exists a free Bernoulli subshift with an invariant probability measure.*

Immediately after the first version of our paper appeared, using a measurable version of the Local Lemma, Bernhsteyn [5] proved that free Bernoulli subshift admitting an invariant probability measure exists for any countable group. He also noted that this result follows from a deep theorem of Seward and Tucker-Drob [26]. We can actually characterize those uniformly recurrent subgroups Z for which all the Z-proper actions admit invariant probability measures (Theorem 6).

The second part of the paper is about C^*-algebras. For any finitely generated group Γ and uniformly recurrent subgroup $Z \subset \mathrm{Sub}(G)$, we associate a separable C^*-algebra $C^*_r(Z)$. For any group Γ, if $Z = \{1\}$, the associated C^*-algebra $C^*_r(Z)$ is just the reduced C^*-algebra of the group. It is known that the reduced C^*-algebra of a group Γ is simple if and only if the group admits no non-trivial amenable uniformly recurrent subgroups [22]. We prove (Theorem 7) that if the URS Z is generic (see Subsection 2.3) then the C^*-algebra $C^*_r(Z)$ is always simple. Using the coloring scheme developed in the first part of the paper, we will show how to construct generic URS's from a single infinite graph of bounded vertex degrees. By this construction we obtain examples of generic URS's with nuclear (Theorem 8) and exact(but not nuclear) C^*-algebras (Proposition 7.1). Finally, we will construct a generic URS Z for which the simple C^*-algebra $C^*_r(Z)$ is not locally-reflexive (hence not exact). In fact, this algebra $C^*_r(Z)$ admits both a uniformly amenable and a non-uniformly amenable trace. We will see that the URS above is not Borel equivalent to a free minimal action of any countable group.

2 Schreier graphs

2.1 The space of rooted Schreier graphs

Let Γ be a finitely generated group with a generating system $Q = \{\gamma_i\}_{i=1}^n$. Let $H \in \mathrm{Sub}(\Gamma)$. Then the Schreier graph associated to H is constructed as follows.

- The vertex set of the Schreier graph of H is the coset space Γ/H (that is the group Γ acts on the vertex set of the Schreier graphs on the left).
- The vertices corresponding to the cosets aH and bH are connected by a directed edge labeled by the generator γ_i if $\gamma_i aH = bH$, or by γ_i^{-1} if $\gamma_i bH = aH$ (note that we allow loops and multiply labeled directed edges).

The coset class of H is called the *root* of the Schreier graph associated to H. The set of all rooted Schreier graphs will be denoted by Sch_Γ^Q. So, we have a map $S_\Gamma^Q : \mathrm{Sub}(\Gamma) \to \mathrm{Sch}_\Gamma^Q$ such that $S_\Gamma^Q(H)$ is the rooted Schreier graph associated to the subgroup H. We will consider the usual shortest path distance on the graph $S_\Gamma^Q(H)$ and denote the ball of radius r around the root H by $B_r(S_\Gamma^Q(H), H)$. Note that $B_r(S_\Gamma^Q(H), H)$ is a rooted edge-labeled graph. The space of all Schreier graphs Sch_Γ^Q is a compact metric space, where

$$d_{\mathrm{Sch}_\Gamma^Q}(S_\Gamma^Q(H_1), S_\Gamma^Q(H_2)) = 2^{-r},$$

if r is the largest integer for which the r-balls $B_r(S_\Gamma^Q(H_1), H_1)$ and $B_r(S_\Gamma^Q(H_2), H_2)$ are rooted-labeled isomorphic. We can define the action of the group Γ on the compact metric space Sch_Γ^Q in the following way. If $\gamma \in \Gamma$ and $H \in \mathrm{Sub}(\Gamma)$, then

$$\gamma(S_\Gamma^Q(H)) = S_\Gamma^Q(\gamma H \gamma^{-1}).$$

The graph $S_\Gamma^Q(\gamma H \gamma^{-1})$ can be regarded as the same graph as $S_\Gamma^Q(H)$ with the new root γH. We will use the root-change picture of the Γ-action on Sch_Γ^Q later in the paper. If $S = S_\Gamma^Q(H)$ is a Schreier graph and $x = \gamma H$ is another vertex of S, then $(S_\Gamma^Q(H), x)$ will denote the Schreier graph with underlying labeled graph S and root x. In this case (S_Γ^Q, x) is isomorphic to $S_\Gamma^Q(\gamma H \gamma^{-1})$ as rooted Schreier graphs. Clearly, $S_\Gamma^Q : \mathrm{Sub}(\Gamma) \to \mathrm{Sch}_\Gamma^Q$, is a homeomorphism commuting with the Γ-actions defined above. Let $Z \subset \mathrm{Sub}(\Gamma)$, then $S_\Gamma^Q(Z) \subset \mathrm{Sch}_\Gamma^Q$ is a closed Γ-invariant subspace of rooted Schreier graphs.

2.2 Schreier graphs and uniformly recurrent subgroups

Proposition 2.1. *Let Γ and Z be as above, $H \in Z$ and $S_\Gamma^Q(H)$ be the corresponding rooted Schreier graph. Then for any $x \in V(S_\Gamma^Q(H))$ and $R > 0$ there exists $S_{x,R} > 0$ such that for any $y \in V(S_\Gamma^Q(H))$, there is a $z \in V(S_\Gamma^Q(H))$ so that*

- $d_{S_\Gamma^Q(H)}(y,z) \leq S_{x,R}$
- *The rooted labeled balls $B_R(S_\Gamma^Q(H),x)$ and $B_R(S_\Gamma^Q(H),z)$ are isomorphic.*

Conversely, if $H \in Sub(\Gamma)$ has the repetition property as above, then its orbit closure in $Sub(\Gamma)$ is a uniformly recurrent subgroup.

Proof. We proceed by contradiction. Suppose that there is some $x \in V(S_\Gamma^Q(H))$ such that for all $n \geq 1$ there exists $y_n \in V(S_\Gamma^Q(H))$ such that if $d_{S_\Gamma^Q(H)}(y,z) \leq n$, then $B_R(S_\Gamma^Q(H),x)$ and $B_R(S_\Gamma^Q(H),z)$ are not isomorphic. Let $S \in \mathrm{Sch}_\Gamma^Q$ be a rooted Schreier graph that is a limitpoint of the sequence of rooted Schreier graphs $\{S_\Gamma^Q(H),y_n\}_{n=1}^\infty$. Then, if $q \in V(S)$, the rooted balls $B_R(S_\Gamma^Q(H),x)$ and $B_R(S,q)$ are not isomorphic. Hence, the orbit closure of S in the Γ-space Sch_Γ^Q does not contain the Schreier graph $S_\Gamma^Q(H)$ in contradiction with the minimality of Z.
Now we prove the converse. Let $H \in \mathrm{Sub}(\Gamma)$ be a subgroup satisfying the condition of our lemma. Let $K, L \in \mathrm{Sub}(\Gamma)$ be elements of the orbit closure of H. It is enough to show that the orbit closure of K contains L. Let $R > 0$ be an integer. We need to show that there exists $x \in V(S_\Gamma^Q(K))$ such that $B_R(S_\Gamma^Q(K),x)$ is rooted-labeled isomorphic to $B_R(S_\Gamma^Q(L),L))$. Since L is in the orbit closure of H, we have $y \in V(S_\Gamma^Q(H))$ such that $B_R(S_\Gamma^Q(H),y)$ is rooted-labeled isomorphic to $B_R(S_\Gamma^Q(L),L))$. By our condition, if K is in the orbit closure of H, there exists $x \in V(S_\Gamma^Q(K))$ so that $B_R(S_\Gamma^Q(K),x)$ is rooted-labeled isomorphic to $B_R(S_\Gamma^Q(H),y)$. This finishes the proof of our proposition. $\square$

2.3 Genericity

Let Γ be as above and $Z \subset \mathrm{Sub}(\Gamma)$ be a URS. We say that Z is **generic** if for every $H \in Z$, the coset space Γ/H and the orbit of H in $\mathrm{Sub}(\Gamma)$ are Γ-isomorphic sets under the map $\phi : \Gamma/H \to \mathrm{Orb}(H)$, $\phi(gH) = gHg^{-1}$. That is, all the elements of Z are self-normalizing subgroups. We will give several examples of generic URS's in Section 5.

Proposition 2.2. *Let Z be a generic URS of Γ. Then for each $H \in Z$, $Stab_\alpha^0(H) = Stab_\alpha(H) = H$. That is, (Z,Γ,α) is a Z-proper system, where α is the conjugation action of Γ on Z. Hence, the stability system of a generic URS is itself.*

Proof. Let $H \in Z$. Then by genericity, $\mathrm{Stab}_\alpha(H)$ is the stabilizer of the root in $S_\Gamma^Q(H)$, that is, $\mathrm{Stab}_\alpha(H) = H$. Also, if $h \in H$, then h fixes the root of every element of Sch_Γ^Q that is close enough to $S_\Gamma^Q(H)$, hence $\mathrm{Stab}_\alpha^0(H) = \mathrm{Stab}_\alpha(H)$. $\square$

Proposition 2.3. *The uniformly recurrent subgroup Z is generic if and only if the following statement holds. For any $R > 0$ there exists $S > 0$ such that* *if $H \in Z$, $x, y \in V(S^Q_\Gamma(H))$, $0 < d_{S^Q_\Gamma(H)}(x,y) \leq R$, then the rooted balls $B_S(S^Q_\Gamma(H), x)$ and $B_S(S^Q_\Gamma(H), y)$ are not rooted-labeled isomorphic.*

Proof. Suppose that for any $n \geq 1$, there exists $H_n \in Z$ and $x_n, y_n \in \Gamma/H_n$ such that

- $0 < d_{\Gamma/H_n}(x_n, y_n) \leq R$.
- the n-balls around x_n and y_n are rooted-labeled isomorphic.

Let $(S^Q_\Gamma(H), H)$ be a limitpoint of the sequence $\{(S^Q_\Gamma(H_n), x_n)\}_{n=1}^\infty$ in S^Q_Γ. Then, there exists $\gamma \in \Gamma, \gamma \notin H$, so that $(S^Q_\Gamma(H), H)$ and $(S^Q_\Gamma(H), gH)$ are rooted-labeled isomorphic. Hence $\phi : \Gamma/H \to \mathrm{Orb}(H)$ is not a bijective map. On the other hand, it is clear that if the condition of our proposition is satisfied for any $H \in Z$, then $\phi : \Gamma/H \to \mathrm{Orb}(x)$ is always bijective, hence Z is generic. $\square$

2.4 The Bernoulli shift space of uniformly recurrent subgroups

Let Γ, Q be as in the previous subsection, $H \in \mathrm{Sub}(\Gamma)$ and let K be a finite alphabet. A rooted K-colored Schreier graph of H is the rooted Schreier graph $S^Q_\Gamma(H)$ equipped with a vertex-coloring $c : \Gamma/H \to K$. Let $\mathrm{Sch}^{K,Q}_\Gamma$ be the set of all rooted K-colored Schreier-graphs. Again, we have a compact, metric topology on $\mathrm{Sch}^{K,Q}_\Gamma$:

$$d_{\mathrm{Sch}^{K,Q}_\Gamma}(S,T) = 2^{-r},$$

if r is the largest integer such that the r-balls around the roots of the graphs S and T are rooted-colored-labeled isomorphic. We define $d_{\mathrm{Sch}^{K,Q}_\Gamma}(S,T) = 2$ if the 1-balls around the roots are nonisomorphic and even the colors of the roots are different. Again, Γ acts on the compact space $\mathrm{Sch}^{K,Q}_\Gamma$ by the root-changing map. Hence, we have a natural color-forgetting map $\mathrm{F} : \mathrm{Sch}^{K,Q}_\Gamma \to \mathrm{Sch}^{Q}_\Gamma$ that commutes with the Γ-actions. Notice that if a sequence $\{S_n\}_{n=1}^\infty \subset \mathrm{Sch}^{K,Q}_\Gamma$ converges to $S \in \mathrm{Sch}^{K,Q}_\Gamma$, then for any $r \geq 1$ there exists some integer $N_r \geq 1$ such that if $n \geq N_r$ then the r-balls around the roots of the graph S_n and the graph S are rooted-colored-labeled isomorphic. Let $H \in \mathrm{Sub}(\Gamma)$ and $c : \Gamma/H \to K$ be a vertex coloring that defines the element $S_{H,c} \in \mathrm{Sch}^{K,Q}_\Gamma$. Then of course, $\gamma(S_{H,c}) = S_{H,c}$ if $\gamma \in H$. On the other hand, if $\gamma(S_{H,c}) = S_{H,c}$ and $\gamma \notin H$ then we have the following lemma that is immediately follows from the definitions of the Γ-actions.

Lemma 2.1. *Let $\gamma \notin H$ and $\gamma(S_{H,c}) = S_{H,c}$. Then there exists a colored-labeled graph-automorphism of the K-colored labeled graph $S_{H,c}$ moving the vertex representing H to the vertex representing $\gamma(H) \neq H$.*

Note that we have a continuous Γ-equivariant map $\pi : \mathrm{Sch}_\Gamma^{K,Q} \to \mathrm{Sub}(\Gamma)$, where $\pi(t) = (S_\Gamma^Q)^{-1} \circ \mathrm{F}(t)$. Let Z be a URS of Γ. We say that the element $t \in \mathrm{Sch}_\Gamma^{K,Q}$ is Z**-regular** if $\pi(t) = H \in Z$ and $\mathrm{Stab}_\alpha(t) = H$, where α is the left action of Γ on $\mathrm{Sch}_\Gamma^{K,Q}$. Note that if $H \in Z$ and t is a K-coloring the Schreier graph $S_\Gamma^Q(H)$, then by Lemma 2.1, t is Z-regular if and only if there is no non-trivial colored-labeled automorphism of t.

Proposition 2.4. *Let $Y \subset Sch_\Gamma^{K,Q}$ be a closed Γ-invariant subset consisting of Z-regular elements. Let $(M, \Gamma, \alpha) \subset (Y, \Gamma, \alpha)$ be a minimal Γ-subsystem. Then M is Z-proper, that is, for any $m \in M$, $Stab_\alpha^0(m) = Stab_\alpha(m) \in Z$. Also, $\pi(M) = Z$.*

Proof. Let $h \in \mathrm{Stab}_\alpha(m)$. Then by Z-regularity h fixes the root of m. Therefore, h fixes the root of m' provided that $d_{\mathrm{Sch}_\Gamma^{K,Q}}(m, m')$ is small enough. Thus, $h \in \mathrm{Stab}_\alpha^0(m)$. Since π is a Γ-equivariant continuous map and M is a closed Γ-invariant subset, $\pi(M) = Z$. $\square$

Definition 2.1. Let Z be as above and K be a finite alphabet. Let $B^K(Z)$ be the Γ-invariant subset of all elements S of $\mathrm{Sch}_\Gamma^{K,Q}$ such that the underlying Schreier graph is in $\mathrm{Sch}_\Gamma^Q(Z)$. We call $B^K(Z)$ the K-Bernoulli shift space of Z. A closed Γ-invariant subset of $B^K(Z)$ is called a Bernoulli subshift of Z.

Note that if $Z = \{1\}$, then Z-properness is just the classical notion of Γ-freeness, and the Z-subshifts are the Bernoulli subshifts of Γ.

3 Lovász's Local Lemma and the proof of Theorem 1

Let Z be a URS of Γ. By Proposition 2.4, it is enough to construct a closed Γ-invariant subset $Y \subset \mathrm{Sch}_\Gamma^{K,Q}$ for some alphabet K such that all the elements of Y are Z-proper. This will give us a bit more than just a continuous action having stability system Z, Y will be a minimal Z-proper Bernoulli subshift. It is quite clear that the stability system of a minimal Z-proper Bernoulli subshift is always Z itself. Let $H \in Z$ and consider the Schreier graph $S = S_\Gamma^Q(H)$. Following [2] and [3] we call a coloring $c : \Gamma/H \to K$ nonrepetitive if for any path $(x_1, x_2, \dots, x_{2n})$ in S there exists some $1 \le i \le n$ such that $c(x_i) \neq c(x_{n+i})$. We call all the other colorings repetitive.

Theorem 3. *[Theorem 1 [3]] For any $d \ge 1$ there exists a constant $C(d) > 0$ such that any graph G (finite or infinite) with vertex degree bound d has a nonrepetitive coloring with an alphabet K, provided that $|K| \ge C(d)$.*

Proof. Since the proof in [3] is about edge-colorings and the proof in [2] is in slightly different setting, for completeness we give a proof using Lovász's Local Lemma, that closely follows the proof in [3]. Now, let us state the Local Lemma.

Theorem 4 (The Local Lemma)**.** *Let X be a finite set and* $\Pr$ *be a probability distribution on the subsets of X. For $1 \le i \le r$ let $\mathcal{A}_i$ be a set of events, where an "event" is just a subset of X. Suppose that for all $A \in \mathcal{A}_i$, $\Pr(A_i) = p_i$. Let $\mathcal{A} = \cup_{i=1}^r \mathcal{A}_i$. Suppose that there are real numbers $0 \le a_1, a_2, \dots, a_r < 1$ and $\Delta_{ij} \ge 0$, $i, j = 1, 2, \dots, r$ such that the following conditions hold:*

- *for any event $A \in \mathcal{A}_i$ there exists a set $D_A \subset \mathcal{A}$ with $|D_A \cap \mathcal{A}_j| \le \Delta_{ij}$ for all $1 \le j \le r$ such that A is independent of $\mathcal{A} \backslash (D_A \cup \{A\})$,*
- *$p_i \le a_i \prod_{j=1}^r (1 - a_j)^{\Delta_{ij}}$ for all $1 \le i \le r$.*

Then $\Pr(\cap_{A \in \mathcal{A}} \overline{A}) > 0$.

Let G be a finite graph with maximum degree d. It is enough to prove our theorem for finite graphs. Indeed, if G' is a connected infinite graph with vertex degree bound d, then for each ball around a given vertex p we have a nonrepetitive coloring. Picking a pointwise convergent subsequence of the colorings we obtain a nonrepetitive coloring of our infinite graph G'.
Let C be a large enough number, its exact value will be given later. Let X be the set of all random $\{1, 2, \dots, C\}$-colorings of G. Let $r = \operatorname{diam}(G)$ and for $1 \le i \le r$ and for any path P of length $2i - 1$ let $A(P)$ be the event that P is repetitive. Set

$$\mathcal{A}_i = \{A(P) : P \text{ is a path of length } 2i - 1 \text{ in } G\}.$$

Then $p_i = C^{-i}$. The number of paths of length $2j - 1$ that intersects a given path of length $2i - 1$ is less or equal than $4ijd^{2j}$. So, we can set $\Delta_{ij} = 4ijd^{2j}$. Let $a_i = \frac{1}{2^i d^{2i}}$. Since $a_i \le \frac{1}{2}$, we have that $(1 - a_i) \ge \exp(-2a_i)$. In order to be able to apply the Local Lemma, we need that for any $1 \le i \le r$

$$p_i \le a_i \prod_{j=1}^r \exp(-2a_j \Delta_{ij}).$$

That is

$$C^{-i} \le a_i \prod_{j=1}^r \exp(-8ija_j d^{2j}),$$

or equivalently

$$C \ge 2d^2 \exp\left(8 \sum_{j=1}^r \frac{j}{2^j}\right).$$

Since the infinite series $\sum_{j=1}^\infty \frac{j}{2^j}$ converges to 2, we obtain that for large enough C, the conditions of the Local Lemma are satisfied independently on the size of our finite graph G. This ends the proof of Theorem 3. $\square$

Let $|K| = C(|Q|)$ and let $c : \Gamma/H \to K$ be a nonrepetitive K-coloring that gives rise to an element $y \in \operatorname{Sch}_\Gamma^{K,Q}$. The following proposition finishes the proof of Theorem 1.

Proposition 3.1. *All elements of the orbit closure Y of y in $Sch_\Gamma^{K,Q}$ are Z-regular.*

Proof. Let $x \in Y$ with underlying Schreier graph $S_\Gamma^Q(H')$ and coloring $c' : \Gamma/H' \to K$. Since Z is a URS, $H' \in Z$. Indeed, $\pi^{-1}(Z)$ is a closed Γ-invariant set and $y \in \pi^{-1}(Z)$. Clearly, $\alpha(\gamma)(x) = x$ if $\gamma \in H'$. Now suppose that $\alpha(\gamma)(x) = x$ and $\gamma \notin H'$ (that is x is not Z-proper). By Lemma 2.1, there exists a colored-labeled automorphism θ of the graph x moving $\mathrm{root}(x)$ to $\alpha(\gamma)(\mathrm{root}(x)) \neq \mathrm{root}(x)$. Note that if a is a vertex of x, then $\theta(a) \neq a$. Indeed, if a labeled automorphism of a Schreier graph fixes one vertex, it must fix all the other vertices as well. Now we proceed similarly as in the proof of Lemma 2 [3] or in the proof of Theorem 2.6 [2]. Let $a \in V(x)$ be a vertex such that there is no $b \in x$ such that $\mathrm{dist}_x(b, \theta(b)) < \mathrm{dist}_x(a, \theta(a))$. Let $(a = a_1, a_2, \dots, a_{n+1} = \theta(a))$ be a shortest path between a and $\theta(a)$. For $1 \leq i \leq n$, let $\alpha(\gamma_{k_i})(a_i) = a_{i+1}$. Then let $a_{n+2} = \alpha(\gamma_{k_1})(a_{n+1}), a_{n+3} = \alpha(\gamma_{k_2})(a_{n+2}), \dots, a_{2n} = \alpha(\gamma_{k_n})(a_{2n-1})$. Since θ is a colored-labeled automorphism, for any $1 \leq i \leq n$

$$c(a_i) = c(a_{i+n}). \tag{1}$$

Lemma 3.1. *The walk $(a_1, a_2, \dots, a_{2n})$ is a path.*

Proof. Suppose that the walk above crosses itself, that is for some i, j, $a_j = a_{n+i}$. If $(n+1) - j \geq (n+i) - (n+1) = i - 1$, then $\mathrm{dist}_x(a_2, \theta(a_2)) = \mathrm{dist}_x(a_2, a_{n+2}) < \mathrm{dist}_x(a, \theta(a))$. On the other hand, if $(n+1) - j \leq (n+i) - (n+1) = i - 1$, then $\mathrm{dist}_x(a_n, \theta(a_n)) = \mathrm{dist}_x(a_n, a_{2n-1}) < \mathrm{dist}_x(a, \theta(a))$. Therefore, $(a_1, a_2, \dots, a_{2n})$ is a path. $\square$

By (1) and the previous lemma, the K-colored Schreier-graph x contains a repetitive path. Since x is in the orbit closure of y, this implies that y contains a repetitive path as well, in contradiction with our assumption. $\square$

4 Sofic groups, sofic URS's and invariant measures

4.1 Sofic groups

First, let us recall the notion of a finitely generated sofic group. Let Γ be a finitely generated infinite group with a symmetric generating system $Q = \{\gamma_i\}_{i=1}^n$ and a surjective homomorphism $\kappa : \mathbb{F}_n \to \Gamma$ from the free group $\mathbb{F}_n$ with generating system $\overline{Q} = \{r_i\}_{i=1}^n$ mapping r_i to γ_i. Let Cay_Γ^Q be the Cayley graph of Γ with respect to the generating system Q, that is the Schreier graph corresponding to the subgroup $H = \{1_\Gamma\}$. Let $\{G_k\}_{k=1}^\infty$ be a sequence of finite $\mathbb{F}_n$-Schreier graphs. We call a vertex $p \in V(G_k)$ a (Γ, r)-vertex if there exists a rooted isomorphism

$$\Psi : B_r(G_k, p) \to B_r(\mathrm{Cay}_\Gamma^Q, 1_\Gamma)$$

such that if e is a directed edge in the ball $B_r(G_k, p)$ labeled by r_i, then the edge $\Psi(e)$ is labeled by γ_i. We say that $\{G_k\}_{k=1}^{\infty}$ is a sofic approximation of $\mathrm{Cay}_{\Gamma}^{Q}$, if for any $r \geq 1$ and a real number $\varepsilon > 0$ there exists $N_{r,\varepsilon} \geq 1$ such that if $k \geq N_{r,\varepsilon}$ then there exists a subset $V_k \subset V(G_k)$ consisting of (Γ, r)-vertices such that $|V_k| \geq (1-\varepsilon)|V(G_k)|$. A finitely generated group Γ is called sofic if the Cayley-graphs of Γ admit sofic approximations. Sofic groups were introduced by Gromov in [19] under the name of initially subamenable groups, the word "sofic" was coined by Weiss in [30]. It is important to note that all the amenable, residually-finite and residually amenable groups are sofic, but there exist finitely generated sofic groups that are not residually amenable (see the book of Capraro and Lupini [14] on sofic groups). It is still an open question whether all groups are sofic.

4.2 Sofic URS's and the proof of Theorem 2

We can extend the notion of soficifty from groups to URS's in the following way. Let Γ,Q, $\kappa : \mathbb{F}_n \to \Gamma$ be as above and let $Z \subset \mathrm{Sub}(\Gamma)$ be a uniformly recurrent subgroup. Again, let $\{G_k\}_{k=1}^{\infty}$ be a sequence of finite $\mathbb{F}_n$-Schreier graphs. We call a vertex $p \in V(G_k)$ be a (Z, r)-vertex if there exists a rooted isomorphism $\Psi : B_r(G_k, p) \to B_r(S_{\Gamma}^{Q}(H), H)$, where $H \in Z$ such that if e is a directed edge in the ball $B_r(G_k, p)$ labeled by r_i, then the edge $\Psi(e)$ is labeled by γ_i. Similarly to the case of groups, we say that $\{G_k\}_{k=1}^{\infty}$ is a sofic approximation of the uniformly recurrent subgroup Z if or any $r \geq 1$ and a real number $\varepsilon > 0$ there exists $N_{r,\varepsilon} \geq 1$ such that if $k \geq N_{r,\varepsilon}$ then there exists a subset $V_k \subset V(G_k)$ consisting of (Z, r)-vertices such that $|V_k| \geq (1-\varepsilon)|V(G_k)|$.

Definition 4.1. A uniformly recurrent subgroup is sofic if it admits a sofic approximation system (note that soficity does not depend upon the choice of the generating system)

In Section 5 we will construct a large variety of generic and non-generic URS's. The rest of this subsection is devoted to the proof of Theorem 2. Let Z be a sofic URS and $\{G_k\}_{k=1}^{\infty}$ be a sofic approximation of Z. Using Theorem 3, for each $k \geq 1$ let us choose a nonrepetitive coloring $c_k : V(G_k) \to K$, where $|K| \geq C(|Q|)$. We can associate a probability measure μ_k on the space of K-colored $\mathbb{F}_n$-Schreier graphs $\mathrm{Sch}_{\mathbb{F}_n}^{\overline{Q},K}$, where $\overline{Q} = \{r_i\}_{i=1}^{n}$ is the generating system of the free group $\mathbb{F}_n$. Note that the origin of this construction can be traced back to the paper of Benjamini and Schramm [7]. For a vertex $p \in V(G_k)$ we consider the rooted K-colored Schreier graph $(G_k^{c_k}, p)$. The measure μ_k is defined as

$$\mu_k = \frac{1}{|V(G_k)|} \sum_{p \in V(G_k)} \delta(G_k^{c_k}, p)\,,$$

where $\delta(G_k^{c_k}, p)$ is the Dirac-measure on $\mathrm{Sch}_{\mathbb{F}_n}^{\overline{Q},K}$ concentrated on the rooted K-colored Schreier graph $(G_k^{c_k}, p)$. Clearly, μ_k is invariant under the action of $\mathbb{F}_n$. Since the space of $\mathbb{F}_n$-invariant probability measures on the compact space

$\mathrm{Sch}_{\mathbb{F}_n}^{\overline{Q},K}$ is compact with respect to the weak-topology, we have a convergent subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ converging weakly to some probability measure μ. We consider the K-Bernoulli shift space $B^K(Z)$ as an $\mathbb{F}_n$-space, where for $h \in \mathbb{F}_n$ and $f \in B^K(Z)$

$$\overline{\beta}(h)(f) = \beta(\kappa(h))(f)\,,$$

where β is the left Γ-action on $B^K(Z)$. Hence, we have an injective Γ-equivariant map $\Phi_\kappa : B^K(Z) \to \mathrm{Sch}_{\mathbb{F}_n}^{\overline{Q},K}$.

Lemma 4.1. *The probability measure μ is concentrated on the $\mathbb{F}_n$-invariant closed set Ω of nonrepetitive K-colorings in $\Phi_\kappa(B^K(Z))$.*

Proof. Let $U_r \subset \mathrm{Sch}_{\mathbb{F}_n}^{\overline{Q},K}$ be the clopen set of K-colored Schreier graphs G such that the ball $B_r(G, \mathrm{root}(G))$ is not rooted-labeled isomorphic to $B_r(S_\Gamma^Q(H), H)$ for some $H \in Z$. By our assumptions on the sofic approximations, $\lim_{k\to\infty} \mu_k(U_r) = 0$, hence $\mu(U_r) = 0$. Now let $V_r \subset \mathrm{Sch}_{\mathbb{F}_n}^{\overline{Q},K}$ be the clopen set of K-colored Schreier graphs G such that the ball $B_r(G, \mathrm{root}(G))$ contains a repetitive path. By our assumptions on the colorings c_k, $\mu_k(V_r) = 0$ for any $k \geq 1$. Hence $\mu(V_r) = 0$. Therefore μ is concentrated on Ω. □

Now we can finish the proof of Theorem 2. We can identify Ω with a Γ-invariant closed subset $\overline{\Omega}$ of $B^K(Z)$ on which the Γ-action is Z-proper by Proposition 3.1. That is, our construction gave rise to a Z-proper Bernoulli subshift with an invariant measure. □

Note that we have a Γ-equivariant continuous map from the Z-proper space above to Z itself mapping x into $\mathrm{Stab}(x)$. Recall that a Γ-invariant measure on $\mathrm{Sub}(\Gamma)$ is called an invariant random subgroup.

Proposition 4.1. *Any sofic URS admits an invariant measure.*

Recall that a Γ-invariant measure on $\mathrm{Sub}(\Gamma)$ is called an invariant random subgroup [1]. Example 3.3 in [18] shows that there exists a uniformly recurrent subgroups $Z \subset \mathrm{Sub}(\mathbb{F}_2)$ that does not admit invariant random subgroups, hence Z is not sofic. In Section 5, we provide further examples of uniformly recurrent subgroups that does not carry invariant measures.

5 Coamenable uniformly recurrent subgroups

5.1 Colored graphs

Let Γ_k be the k-fold free product of cyclic groups of rank 2, with free generators $A = \{a_i\}_{i=1}^k$. Let G be an arbitrary infinite, simple connected graph of bounded vertex degrees and a proper edge-coloring by k-colors $\{c_1, c_2, \ldots, c_k\}$. Observe that the edge-coloring of G (and picking an arbitrary root) gives rise to a Γ_k-Schreier graph (S, x). The action of Γ_k on $V(G)$ is defined the following way. If $x \in V(G)$ and $1 \leq i \leq k$, then

- If there is no c_i-colored edge adjacent to x, then $a_i(x) = x$.

- If there there exists an edge (x, y) colored by c_i, then $a_i(x) = y$.

Let X be the orbit closure of the rooted Schreier graph (S, x) above. Then it contains a minimal system (M, Γ_k, β). Then $(S^Q_{\Gamma_k})^{-1}(M)$ is a uniformly recurrent subgroup, where $S^Q_{\Gamma_k} : \mathrm{Sub}(\Gamma_k) \to \mathrm{Sch}^Q_{\Gamma_k}$ is the map defined in Subsection 2.1.

Proposition 5.1. *For any infinite simple, connected graph G of bounded vertex degree, there exists $k > 0$ and a edge-coloring of G with k colors such that all the uniformly recurrent subgroups that can be obtained as above are necessarily generic.*

Proof. First, consider an arbitrary proper edge-coloring $c : E(G) \to L$ and a proper nonrepetitive vertex-coloring $\rho : V(G) \to D$ by some finite sets L and D (the product of a nonrepetitive and a proper vertex-coloring is always a proper nonrepetitive vertex-coloring). Now we construct a new proper edge-coloring ζ of G by the set $D_2 \times L$, where D_2 is the set of 2-elements subset of D. Let $\zeta(e) = \{\rho(x), \rho(y)\} \times c(e)$, where x, y are the endpoints of e. Since ρ is proper, $\rho(x) \neq \rho(y)$. Hence we obtain a Schreier graph $T \in \mathrm{Sch}^A_{\Gamma_k}$, where $k = |D_2 \times L|$ and $A = \{a_1, a_2, \dots, a_k\}$. Let (M, Γ_k, β) be a minimal subsystem in the orbit closure of T. By Proposition 2.3, it is enough to show that if $T' \in M$ and $x \neq y \in V(T')$, then (T', x) and (T', y) are not rooted-labeled isomorphic. We construct a nonrepetitive vertex-coloring $\rho' : V(T') \to D$ in the following way. If $\deg(z) > 1$, let $\rho'(z) = d$, where d is the unique element in the intersection of the D_2-components of the edges adjacent to z. If $\deg(z) = 1$, then let $\rho'(z) = d$, where the D_2-component of z is $\{d, d'\}$ and $\rho'(z') = d'$, for the only neighbour of z. Since T' is in the orbit closure of T, the coloring ρ' is nonrepetitive, hence by Proposition 3.1, $T'(x)$ and $T'(y)$ are not rooted-labeled isomorphic. $\square$.

5.2 Coamenability

Let Γ be a finitely generated group and $H \in \mathrm{Sub}(\Gamma)$. Recall that H is **coamenable** if the action of Γ on Γ/H is amenable. That is, there exists a sequence of finite subsets $\{F_k\}_{k=1}^\infty \subset \Gamma/H$ such that for any $g \in \Gamma$,

$$\lim_{n\to\infty} \frac{|gF_k \cup F_k|}{|F_k|} = 1\,.$$

We call a URS $Z \subset \mathrm{Sub}(\Gamma)$ coamenable if for all $H \in Z$, H is coamenable.

Proposition 5.2. *Let $Z \subset Sub(\Gamma)$ be a URS such that there exists $H \in Z$ so that H is coamenable. Then Z is coamenable.*

Proof. Fix a generating system $Q = \{\gamma_i\}_{i=1}^n$ for Γ. Let $\{F_k\}_{k=1}^\infty$ be finite subsets in Γ/H such that for any $g \in \Gamma$, $\lim_{n\to\infty} \frac{|gF_k \cup F_k|}{|F_k|} = 1$. Let $x_k \in F_k$ and $l(k) > 0$ such that if $y \in F_k$ then $B_{l(k)}(\Gamma/H, x_k)$ contains $B_k(\Gamma/H, y)$. Let $K \in Z$. Since Z is a URS, for any $k \geq 1$, there exists $x'_k \in \Gamma/K$ such that $B_{l(k)}(\Gamma/H, x_k)$ is rooted-labeled isomorphic to $B_{l(k)}(\Gamma/K, x'_k)$. For $k \geq 1$, let

$F'_k \subset \Gamma/K$ be the image of F_k by the isomorphism above. If $g \in \Gamma$ then $gF'_k \subset B_{l(k)}(\Gamma/K, x'_k)$ provided that k is large enough (depending on g). Hence, $\lim_{k\to\infty} \frac{|gF'_k \cup F'_k|}{|F'_k|} = 1$. ☐

Let G be an arbitrary graph that is amenable in the sense that there exists a sequence of subsets $\{F_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} \frac{|B_1(F_k)|}{|F_k|} = 1$, where $x \in B_1(F_k)$ if either $x \in F_k$ or there exists $y \in F_k$ adjacent to x. Repeating the proof of the previous proposition one can immeadiately see that all the Γ_k-URS's constructed from G as in Subsection 5.1 are coamenable. Barlow [Proposition 4.[4]] has shown that for any $\alpha \geq 1$ there exists a bounded degree infinite graph G_α and positive constants C^1_α and C^2_α such that

$$C^1_\alpha r^\alpha \leq B_r(G_\alpha, x) \leq C^2_\alpha r^\alpha \tag{2}$$

holds for all $x \in V(G_\alpha)$. Such graphs are clearly amenable. Hence we can see that as opposed to finitely generated group case, for any $\alpha \geq 1$ there exist generic coamenable uniformly recurrent subgroups $Z \subset \mathrm{Sub}(\Gamma_k)$ so that the volume growth rate of the individual Schreier graphs $S^A_{\Gamma_k}(H), H \in Z$ are always α.

5.3 Coamenable uniformly recurrent subgroups are sofic

The following theorem (or rather the construction in the proof) will be crucial in Section 9.

Theorem 5. *Let Γ be a finitely generated group and $Z \subset Sub(\Gamma)$ be a coamenable URS. Then Z is sofic.*

Proof. Fix a generating system $Q = \{\gamma_i\}_{i=1}^n$. Again, let $\mathbb{F}_n$ be the free group with free generating system $\overline{Q} = \{r_i\}_{i=1}^n$ and $\kappa : \mathbb{F}_n \to \Gamma$ be the corresponding quotient map. Every continuous action α of Γ can be regarded as a $\mathbb{F}_n$-action $\alpha \circ \kappa$. In particular, we have a $\mathbb{F}_n$-invariant embedding $\lambda : S^Q_\Gamma(Z) \to \mathrm{Sch}^{\overline{Q}}_{\mathbb{F}_n}$. Now let $H \in Z$ and consider the Schreier graph $S^Q_\Gamma(H)$. Since Z is coamenable, the isoperimetric constant of $S^Q_\Gamma(H)$ is zero, that is, we have a sequence of finite induced subgraphs $\{H_k\}_{k=1}^\infty \subset S^Q_\Gamma(H)$ so that

$$\lim_{k\to\infty} \frac{|\partial H_k|}{|V(H_k)|} = 0\,,$$

where ∂H_k is the set of vertices in H_k for which there exists
$y \in V(S^Q_\Gamma(H)) \backslash V(H_k)$ with $\gamma_i x = y$ or $\gamma_i y = x$ for some $1 \leq i \leq n$. Now we construct a sequence of $\mathbb{F}_n$-Schreier graphs $\{G_k\}_{k=1}^\infty$ that form a sofic approximation of Z:

- $V(G_k) = V(H_k)$.
- If $\gamma_i x = y$ for some $x, y \in V(H_k)$, then $r_i x = y$.

- Then the action of r_i is extended to the set $V(G_k)$ arbitrarily.

Let $W_r(G_k) \subset V(G_k) = V(H_k)$ be the set of vertices p for which $d_{G_k}(p, \partial H_k) > r$. Clearly, if $p \in W_r(G_k)$ then $B_r(W(G_k), p)$ is rooted-labeled isomorphic to $B_r(S^Q_\Gamma(H), p)$. That is, all the vertices of $W_r(G_K)$ are (Z, r)-vertices. Since, $\frac{|\partial H_k|}{|V(G_k)|} \to 0$, we have that $\frac{|W_r(G_k)|}{|V(G_k)|} = 1$. Hence the Schreier graphs $\{G_k\}_{k=1}^\infty$ form a sofic approximation of the URS Z. $\square$

As in Subsection 4.2, for each $k \geq 1$ we have an $\mathbb{F}_n$-invariant probability measure on $\mathrm{Sch}^{\overline{Q}}_{\mathbb{F}_n}$

$$\mu_k = \frac{1}{|V(G_k)|} \sum_{p \in V(G_k)} \delta(G_k, p)\,.$$

Let $\{\mu_{n_k}\}_{l=1}^\infty$ be a weakly convergent sequence converging to an $\mathbb{F}_n$-invariant measure μ on $\mathrm{Sch}^{\overline{Q}}_{\mathbb{F}_n}$. By our previous lemma, the measure μ is concentrated on $\lambda(S^Q_\Gamma(Z))$. Note that the probability measure μ depends only on the sequence of subgraphs $\{H_{k_l}\}_{l=1}^\infty$. We say that the sequence of subgraphs $\{H_l\}_{l=1}^\infty$ is **convergent in the sense of Benjamini and Schramm** if the associated probability measures $\{\mu_l\}_{l=1}^\infty$ converge to some invariant measure μ on $\lambda(S^Q_\Gamma(Z))$. In this case the measure preserving action $(Z, \Gamma, \lambda^{-1}(\mu))$ is called the **limit** of the sequence $\{H_l\}_{l=1}^\infty$. Also note, that if Z is a generic URS, then $(Z, \Gamma, \lambda^{-1}(\mu))$ is a totally nonfree action in the sense of Vershik [29].

5.4 A characterization of coamenability

As in the previous sections let Γ be a finitely generated group with generating system $Q = \{\gamma_i\}_{i=1}^r$ and $Z \subset \mathrm{Sub}(\Gamma)$ be a URS. The goal of this subsection is to prove the following characterization of coamenability.

Theorem 6. *The URS Z is coamenable if and only if every Z-proper continuous action of Γ admits an invariant measure.*

Proof. First, let Z be coamenable and $\alpha : \Gamma \curvearrowright X$ be a continuous Z-proper action. Let $x \in X$ and $Stab_\alpha(x) = H \in Z$. Then the orbit graph of x is isomorphic to $S^Q_\Gamma(H)$. Let $\{F_k\}_{k=1}^\infty \subset \Gamma/H$ be a sequence of finite sets such that for any $g \in \Gamma$,

$$\lim_{k\to\infty} \frac{|gF_k \cup F_k|}{|F_k|} = 1\,. \tag{3}$$

Now we proceed in exactly the same way as in the proof of the classical Krylov-Bogoliubov Theorem. Fix an ultrafilter ω on the natural numbers and let $\lim_\omega$ be the corresponding ultralimit. We define a bounded linear functional $T : \mathbb{C}[X] \to \mathbb{C}$ in the following way. For a continuous function $f : X \to \mathbb{C}$ set

$$T(f) = \lim_\omega \frac{\sum_{\overline{\gamma} \in F_k} f(\overline{\gamma} x)}{|F_k|}\,.$$

Then, by (3), $T(\gamma(f)) = T(f)$ for all $\gamma \in \Gamma$, $T(1) = 1$, therefore $T : \mathbb{C}[X] \to \mathbb{C}$ is a Γ-invariant bounded functional associated to a Γ-invariant Borel probability measure μ. Now, let Z be a URS that is not coamenable and let $H \in Z$. Then the graph $S^Q_\Gamma(H)$ has positive isoperimetric constant so by Theorem 3.1 [9], we have maps $\phi_1 : \Gamma/H \to \Gamma/H$, $\phi_2 : \Gamma/H \to \Gamma/H$ so that $\phi_1(\Gamma/H) \cap \phi_2(\Gamma/H) = \emptyset$ and there exists a positive constant $C > 0$ so that for all $p \in \Gamma/H$

$$d_{S^Q_\Gamma(H)}(\phi_1(p), p)) < C \quad \text{and} \quad d_{S^Q_\Gamma(H)}(\phi_2(p), p)) < C\,.$$

Now we build a vertex-coloring for the graph $S^Q_\Gamma(H)$ to encode ϕ_1 and ϕ_2. First, we pick a nonrepetitive coloring $c_1 : \Gamma/H \to D$, where D is some finite set. Then we choose a coloring $c_2 : \Gamma/H \to E$ for some finite set E so that $c_2(p) \neq c_2(q)$, whenever

$$0 < d_{S^Q_\Gamma(H)}(p, q) \leq 3C\,.$$

We need two more colorings of the vertices of $S^Q_\Gamma(H)$:

$$c_3 : \Gamma/H \to \{E \times \{1\}\} \cup \{*\}$$

and

$$c_4 : \Gamma/H \to \{E \times \{2\}\} \cup \{*\}$$

satisfying the following properties. If there exists $p \in \Gamma/H$ so that $\phi_1(p) = q$ and $c_2(p) = e$, then $c_3(q) = e \times \{1\}$. If such p does not exist, set $c_3(q) = *$. If there exists $p \in \Gamma/H$ so that $\phi_2(p) = q$ and $c_2(p) = e$, then $c_4(q) = e \times \{2\}$. If such p does not exist, set $c_4(q) = *$. Let $M = D \times E \times \{\{E \times \{1\}\} \cup \{*\}\} \times \{\{E \times \{2\}\} \cup \{*\}\}$. Our final coloring $c : \Gamma/H \to M$ is defined by

$$c(p) = c_1(p) \times c_2(p) \times c_3(p) \times c_4(p)\,.$$

Let X be the orbit closure of the M-colored graph $S^{Q,c}_\Gamma(H)$ in the space $\mathrm{Sch}^{M,Q}_\Gamma$. Observe that c is nonrepetitive since even its first component is nonrepetitive, that is the action β on X is Z-proper. Now we need to show that β admits no Γ-invariant measure. We define continuous injective maps $\Phi_1 : X \to X$ and $\Phi_2 : X \to X$ such that

- $\Phi_1(X) \cap \Phi_2(X) = \emptyset$
- For each $x \in X$, $\Phi_1(x), \Phi_2(x) \in \mathrm{Orb}(x)$.

Thus the equivalence relation defined by the action is compressible, so it cannot admit an invariant measure [10]. The construction of Φ_1 and Φ_2 goes as follows. If $x \in X$, then $c(x) = (c_1(x), c_2(x), c_3(x), c_4(x))$ is well-defined and there exist a unique $y \in X$ and a unique $z \in X$ such that

- $c_2(x) \times 1 = c_3(y)$, $c_2(x) \times 2 = c_4(z)$
- $d_{\mathrm{Orb}(x)}(x, y) \leq C$, $d_{\mathrm{Orb}(x)}(x, z) \leq C$.

We set $\Phi_1(x) = y$, $\Phi_2(x) = z$, Clearly, Φ_1 and Φ_2 are continuous and $\Phi_1(X) \cap \Phi_2(X) = \emptyset$. $\square$

Let Γ be as above and $Z \subset \mathrm{Sub}(\Gamma)$ be a URS that is not coamenable and X is a Z-proper action without invariant measure as above. Let $Y \subset X$ be a minimal Z-proper M-Bernoulli subshift. Let Γ_M be the M-fold free product of the finite group of two elements with free generator system $\{a_m\}_{m \in M}$. Then we can associate to Y a nonsofic generic URS $Z \subset \mathrm{Sub}(\Gamma * \Gamma_M)$ in the following way. Let $S = S^{M,Q}_\Gamma(H)$ be an element of Y. Let $V = V(S) \times \{0,1\}$. We define an action of the group $\Gamma * \Gamma_M$ as follows. The group Γ acts on $V(S) \star \{0\}$ as Γ acts on $V(S)$. Also, the group Γ acts on $V(S) \star \{1\}$ trivially. If $x \in V(S)$, $c(x) = m$, then $a_m(x \times \{0\}) = x \times \{1\}$ and $a_m(x \times \{1\}) = x \times \{0\}$. Otherwise, let $a_m(x \times \{0\}) = x \times \{0\}$ and $a_m(x \times \{1\}) = x \times \{1\}$. It is not hard to see that the resulting $(\Gamma * \Gamma_M)$-Schreier graph satisfies the conditions of Proposition 2.1 and 2.3, hence the associated URS is generic and does not admit invariant measures.

6 The C^*-algebras of uniformly recurrent subgroups

6.1 The algebra of local kernels

Let Γ be a finitely generated group with generating system $Q = \{\gamma_i\}_{i=1}^n$ and $Z \subset \mathrm{Sub}(\Gamma)$ be a URS of Γ. Let $H \in Z$ and $S = S^Q_\Gamma(H)$ be the Schreier graph of H. A **local kernel** is a function $K : \Gamma/H \times \Gamma/H \to \mathbb{C}$ satisfying the following properties.

- There exists an integer $R > 0$ (depending on K) such that $K(x,y) = 0$ if $d_S(x,y) > R$.
- If $B_R(S,y)$ is rooted-labeled isomorphic to $B_R(S,z)$ then $K(y,\gamma y) = K(z,\gamma z)$ provided that $d_S(y,\gamma y) = d_S(z,\gamma z) \leq R$.

We will call the smallest R satisfying the two conditions above the **width** of K. It is easy to see that the local kernels form a unital $*$-algebra $\mathbb{C}Z$ with respect to the following operations:

- $(K+L)(x,y) = K(x,y) + L(x,y)$
- $KL(x,y) = \sum_{z \in \Gamma/H} K(x,z)L(z,y)$
- $K^*(x,y) = \overline{K(y,x)}$.

By minimality, the algebra $\mathbb{C}Z$ does not depend on the choice of H or the generating system Q only on the URS Z itself. We will call the concrete realization of the algebra of local kernels az above the representation of $\mathbb{C}Z$ on $\mathbb{C}^{\Gamma/H}$. One can observe that if Z consists only of the unit element, then $\mathbb{C}Z$ is the complex group algebra of Γ.

6.2 The construction of $C^*_r(Z)$

Let Γ be a finitely generated group (with a fixed generating system $Q = \{\gamma_i\}_{i=1}^n$) and $Z \subset \mathrm{Sub}(\Gamma)$ be a URS. Let $H \in Z$ and consider the algebra $\mathbb{C}Z$ as above represented on the vector space $\mathbb{C}^{\Gamma/H}$. The we have a bounder linear representation of $\mathbb{C}Z$ on $l^2(\Gamma/H)$ by

$$K(f)(x) = \sum K(x,y)f(y)\,,$$

where $f \in l^2(\Gamma/H)$.

Definition 6.1. The C^*-algebra of Z, $C^*_r(Z)$ is defined as the norm closure of $\mathbb{C}Z$ in $B(l^2(\Gamma/H))$.

Note that we used a specific subgroup H in order to equip the algebra $\mathbb{C}Z$ with a norm. However, we have the following proposition.

Proposition 6.1. *The norm on $\mathbb{C}Z$ and hence the definition of $C^*_r(Z)$ does not depend on the choice of the subgroup H.*

Proof. Let $K \in \mathbb{C}Z$ be a local kernel of width R and let $H, L \in Z$. Let K_H respectively K_L be the representation of K on $l^2(\Gamma/H)$ respectively on $l^2(\Gamma/L)$. We need to show that $\|K_H\| = \|K_L\|$. Let $\varepsilon > 0$ and $f \in l^2(\Gamma/H), \|f\| = 1$ such that f is supported on a ball $B_T(S^Q_\Gamma(H), x)$ and $\|K_H(f)\| \geq \|K_H\| - \varepsilon$. Observe that $K_H(f)$ is supported on the ball $B_{T+R}(S^Q_\Gamma(H), x)$ and $\|K_H(f)\| \geq \|K_H\| - \varepsilon$. By Proposition 2.1, there exists $y \in \Gamma/L$ such that the balls $B_{T+R}(S^Q_\Gamma(H), x)$ and $B_{T+R}(S^Q_\Gamma(L), y)$ are rooted-labeled isomorphic. Hence, there exists $f' \in l^2(\Gamma/L)$ supported on $B_T(S^Q_\Gamma(L), y)$, $\|f'\| = 1$ such that $\|K_H(f)\| = \|K_L(f')\|$. Therefore, $\|K_H\| \leq \|K_L\|$. Similarly, $\|K_L\| \leq \|K_H\|$, that is, $\|K_H\| = \|K_L\|$. □

6.3 The C^*-algebras of generic URS's are simple

The goal of this section is to prove the following theorem.

Theorem 7. *Let Γ be as above and $Z \subset Sub(\Gamma)$ be a generic URS. Then the C^*-algebra $C^*_r(Z)$ is simple.*

Proof. Let $H \in Z$. For each $r \geq 1$ we define an equivalence relation on Γ/H in the following way. If $p, q \in \Gamma/H$, then $p \equiv_r q$ if the balls $B_r(S^Q_\Gamma(H), p)$ and $B_r(S^Q_\Gamma(H), q)$ are rooted-labeled isomorphic. The following lemma is a straightforward consequence of Proposition 2.1 and Proposition 2.3.

Lemma 6.1. *Let $\equiv_r$ be the equivalence relation as above. Then:*

1. *For any $n \geq 1$ there exists r_n such that if $p \neq q$ and $p \equiv_{r_n} q$, then $d_{S^Q_\Gamma(H)}(p, q) \geq n$.*

2. *For every $r \geq 1$ there exists t_r such that for any $p \in \Gamma/H$ the ball $B_{t_r}(S^Q_\Gamma(H), p)$ intersects all the equivalence classes of E_r (in particular, the number of equivalence classes is finite).*

3. *If $r \leq s$, then $p \equiv_s q$ implies $p \equiv_r q$.*

4. *Let E_r denote the classes of $\equiv_r$. Then we have an inverse system of surjective maps*
$$E_1 \leftarrow E_2 \leftarrow \dots$$
and a natural homeomorphism $\iota_H : \lim_{\leftarrow} E_r \to Z$, between the compact space $\lim_{\leftarrow} E_r$ and the uniformly recurrent subgroup Z.

Note that if $\alpha \in E_r$, then $\iota_H(\alpha)$ is the clopen set of Schreier graphs $S^Q_\Gamma(L)$, $L \in Z$, such that the ball $B_r(S^Q_\Gamma(L), L)$ is rooted-labeled isomorphic to the ball $B_r(S^Q_\Gamma(H), x)$, where $x \in \alpha$.
Now let us consider the commutative C^*-algebra $l^\infty(\Gamma/H)$. For any $r \geq 1$ and $\alpha \in E_r$ we have a projection $e_\alpha \in l^\infty(\Gamma/H)$, where $e_\alpha(x) = 1$ if $x \in \alpha$ and zero otherwise. The projections $\{e_\alpha\}_{r\geq 1, \alpha \in E_r}$ generates a *-subalgebra $\mathcal{A}$ in $l^\infty(\Gamma/H)$ and by the previous lemma the closure of $\mathcal{A}$ in $l^\infty(\Gamma/H)$ is isomorphic to $\mathbb{C}[Z]$ (the C^*-algebra of continuous complex-valued functions on the compact metrizable space Z). Indeed, under this isomorphism $\lambda_H : \overline{\mathcal{A}} \to \mathbb{C}[Z]$, $\lambda_H(e_\alpha)$ is the characteristic function of the clopen set $\iota_H(\alpha)$. It is easy to see that the isomorphism $\lambda_H : \overline{\mathcal{A}} \to \mathbb{C}[Z]$ commutes with the respective Γ-actions. Now let us consider the representation of $C^*_r(Z)$ on $l^2(\Gamma/H)$. For $K \in C^*_r(Z)$ let $K(x,y) = \langle K(\delta_y), \delta_x \rangle$, be the kernel of K. We have a bounded linear map $Q_r : C^*_r(Z) \to C^*_r(Z)$ given by
$$Q_r(K) = \sum_{\alpha \in E_r} e_\alpha K e_\alpha \,.$$

Lemma 6.2. *For any $r \geq 1$, $\|Q_r\| \leq 1$.*

Proof. Let $h \in l^2(\Gamma/H)$, $\|h\| = 1$. For any $K \in C^*_r(Z)$ we have that
$$\|(Q_r(K))(h)\|^2 = \|\sum_{\alpha \in E_r} e_\alpha K e_\alpha(h)\|^2 = \sum_{\alpha \in E_r} \|e_\alpha K e_\alpha(h)\|^2 \leq$$
$$\leq \|K\|^2 \sum_{\alpha \in E_r} \|e_\alpha(h)\|^2 = \|K\|^2 \,.$$
Therefore $\|Q_r K\| \leq \|K\|$. ∎

Observe that we have a natural injective homomorphism $\rho : \mathcal{A} \to \mathbb{C}Z$ defined in the following way.

- $\rho(a)(x,x) = a(x)$.
- $\rho(a)(x,y) = 0$ if $x \neq y$.

Clearly, ρ is preserving the norm, so we can extend it to a unital embedding $\overline{\rho} : \mathcal{A} \to C_r^*(Z)$. Also, we have a map $\kappa : \Gamma \to C_r^*(Z)$ such that $\kappa(g)(x,y) = 1$, whenever $g^{-1}x = y$ and $\kappa(g)(x,y) = 0$ otherwise.

Lemma 6.3. *For any $g, h \in \Gamma$, $\kappa(g)\kappa(h) = \kappa(gh)$.*

First, we have that

$$\kappa(g)\kappa(h)(x,y) = \sum_{z\in\Gamma/H} \kappa(g)(x,z)\kappa(h)(z,y)\,.$$

Hence, $\kappa(g)\kappa(h)(x,y) = 1$ if $y = h^{-1}g^{-1}x$ and $\kappa(g)\kappa(h)(x,y) = 0$ otherwise. Therefore, $\kappa(g)\kappa(h) = \kappa(gh)$. □

Lemma 6.4. *For any $g \in \Gamma$ and $a \in \overline{\mathcal{A}}$*

$$\rho(g(a)) = \kappa(g)\rho(a)\kappa(g^{-1})\,.$$

Proof. On one hand, $\rho(g(a))(x,y) = a(g^{-1}(x))$ if $x \neq y$,
otherwise $\rho(g(a))(x,y) = 0$. On the other hand,

$$\kappa(g)\rho(a)\kappa^{-1}(g)(x,x) = \sum_{y\in\Gamma/H} \kappa(g)(x,y)\rho(a)(y,y)\kappa^{-1}(g)(y,x) = a(g^{-1}(x))\,.$$

Also, $\kappa(g)\rho(a)\kappa^{-1}(g)(x,y) = 0$ if $x \neq y$. □

Let us consider the linear operator $D : C_r^*(Z) \to C_r^*(Z)$ such that for $x \in \Gamma/H$ $D(K)(x,x) = K(x,x)$, $D(K)(x,y) = 0$ if $x \neq y$. The operator D is bounded with norm 1 since

$$\|D(K)\| = \sup_{x\in\Gamma/H} |K(x,x)| = \sup_{x\in\Gamma/H} |\langle K(\delta_x), \delta_x\rangle|\,.$$

Lemma 6.5. *Let $K \in \mathbb{C}Z$. Then $Q_r(K) = D(K)$ provided that r is large enough.*

Proof. Let $s > 0$ be the width of K and let $r > 0$ be so large that if $p \equiv_r q$ and $p \neq q$, then $d_{S^Q_\Gamma(H)}(p,q) > s$. Then, if $\alpha \in E_r$ we have that $(e_\alpha K e_\alpha)(x,y) = 0$ if $x \neq y$ or $x \notin \alpha$, otherwise $(e_\alpha K e_\alpha)(x,x) = K(x,x)$. Therefore, $Q_r(K) = D(K)$. □

Lemma 6.6. *Let $K \in C_r^*(Z)$. Then $\lim_{r\to\infty} Q_r(K) = D(K)$.*

Proof. Let $K_n \to K$ such that $K_n \in \mathbb{C}Z$. Then, by the previous lemma $\|Q_r(K) - D(K_n)\| \leq \|K - K_n\|$, provided that r is large enough. Since $D(K_n) \to D(K)$, we have that $\lim_{r\to\infty} Q_r(K) = D(K)$. □

Lemma 6.7. *Let $I \lhd C_r^*(K)$ be a closed ideal. Suppose that $I \cap D(C_r^*(Z)) \neq \{0\}$. Then $I = C_r^*(K)$.*

Proof. Recall that $D(C_r^*(Z)) = \rho(\overline{\mathcal{A}})$, so by Lemma 6.4 we have a nonzero, Γ-invariant closed ideal in $\mathcal{A} \cong \mathbb{C}[Z]$. However, any Γ-invariant closed ideal in $\mathcal{A} \cong \mathbb{C}[Z]$ is in the form of $I(Y)$, where Y is a Γ-invariant closed set in Z and $I(Y)$ is the set of continuous functions vanishing at Y. By minimality, Y must be empty, hence I contains the unit, that is, $I = C_r^*(K)$. $\square$

Now, we finish the proof of our theorem. Let I be a closed ideal of $C_r^*(Z)$ and $0 \neq K \in I$. Then $K^*K \in I$ and $D(K^*K) \neq 0$. Since $D(K^*K) = \lim_{r\to\infty} Q_r(K^*K)$ and $Q_r(K^*K) \in I$ for any $r \geq 1$, we have that $1 \in I$. $\square$

Remark Let $Z \subset \mathrm{Sub}(\Gamma)$ be a not necessarily generic URS, where Γ is a finitely generated group as above. Let $Y \subset S_\Gamma^{K,Q}(Z)$ be a minimal Z-proper Bernoulli subshift. Then the local kernels on Y can be defined using the rooted-labeled-colored neighborhoods and the resulting C^*-algebra is always simple.

7 Exactness and nuclearity

7.1 Property A vs. Local Property A

First let us recall the notion of **Property** A from [25]. Let G be an infinite graph of bounded vertex degrees. We say the G has Property A if there exists a sequence of maps $\{\varsigma^n : V(G) \to l^2(V(G)\}_{n=1}^\infty$ such that

- Each ς_x^n has length 1.
- If $d_G(x,y) \leq n$, then $\|\varsigma_x^n - \varsigma_y^n\| \leq \frac{1}{n}$.
- For any $n \geq 1$ we have $R_n > 0$ such that the vector ς_x^n is supported in the ball $B_{R_n}(G,x)$.

We also need the notion of the **uniform Roe algebra** of the graph G. First, we consider the $*$-algebra of **bounded kernels** $K : V(G) \times V(G) \to \mathbb{C}$, that is

- there exists some positive integer R depending on K such that $K(x,y) = 0$ if $d_G(x,y) > R$,
- there exists some positive integer M depending on K such that $|K(x,y)| < M$.

The uniform Roe algebra $C_u^*(G)$ is the norm closure of the bounded kernels in $B(l^2(V(G)))$. Observe that if $Z \subset \mathrm{Sub}(\Gamma)$ is a unformly recurrent subgroup and $H \in Z$, $S = S_\Gamma^Q(H)$, then $C_r^*(Z) \subset C_u^*(S)$. According to Proposition 11.41 [25], if G has Property A then the algebra $C_u^*(G)$ is nuclear. All C^*-subalgebras of a nuclear C^*-algebra are exact, hence we have the following proposition.

Proposition 7.1. *Let $Z \subset Sub(\Gamma)$ and $H \in Z$ as above, such that S_Γ^Q has Property A. Then $C_r^*(Z)$ is exact.*

Example: Let G be the underlying graph of the Cayley graph of an exact group (say, a hyperbolic group or an amenable group) and let S be a colored graph associated to a generic URS Z as in Proposition 5.1. Then by the previous proposition, $C_r(Z)$ is a simple exact C^*-algebra.

Now we introduce the notion of Schreier graphs with **Local Property** A.

Definition 7.1. Let $S = S^Q_\Gamma(H)$ be a Schreier graph. We say that S has Local Property A, if the sequence $\{\varsigma^n\}_{n=1}^\infty$ can be chosen locally, that is for any $n \geq 1$, there exists $S_n > R_n$ so that for $x, y \in V(G)$ the balls $B_{S_n}(G,x)$ and $B_{S_n}(G,y)$ are rooted-labeled isomorphic under the map $\theta : B_{S_n}(G,x) \to B_{S_n}(G,y)$, then $\varsigma^n_y = \theta(\varsigma^n_x)$.

The main result of this section is the following theorem.

Theorem 8. *Let Γ be a finitely generated group, $Z \subset Sub(\Gamma)$ a uniformly recurrent subgroup and $H \in Z$ so that $S^Q_\Gamma(H)$ has local Property A. Then $C^*_r(Z)$ is nuclear.*

Proof. We closely follow the proof of Proposition 11.41 [25]. The nuclearity of the uniform Roe algebra for a graph S having Property A has been proved the following way (we will denote by X the vertex set of S). First, a sequence of unital completely positive maps $\Phi_n : C^*_u(S) \to l^\infty(X) \otimes M_{N_n}(\mathbb{C})$ were constructed, where $M_{N_n}(\mathbb{C})$ is the algebra of $N_n \times N_n$-matrices. Then, a sequence of unital completely positive maps $\Psi_n : l^\infty(X) \otimes M_{N_n} \to C^*_u(S)$ were given in such a way that $\{\Psi_n \circ \Phi_n\}_{n=1}^\infty$ tends to the identity in the point-norm topology. Hence, the nuclearity of the uniform Roe algebra $C^*_u(S)$ follows. It is enough to see that Φ_n maps the subalgebra $C^*_r(Z) \subset C^*_u(S)$ into $\mathbb{C}[Z] \otimes M_{N_n} \subset l^\infty(X) \otimes M_{N_n}$ and Ψ_n maps $\mathbb{C}[Z] \otimes M_{N_n}$ into $C^*_r(Z)$. Then the nuclearity of $C^*_r(Z)$ automatically follows. So, let us examine the maps Φ_n, Ψ_n. For each $n \geq 1$, we choose $N_n > 0$ such that $|B_{R_n}(S,x)| \leq N_n$ for all $x \in V(S) = X$. Then, for each $x \in X$ we choose a subset $H^n_x \supset B_{R_n}(S,x)$ of size N_n "locally". That is, if $B_{S_n}(S,x)$ and $B_{S_n}(S,y)$ are rooted-labeled isomorphic under the map θ, then $\theta(H^n_x) = H^n_y$. Now for each $x \in X$ let $P_n(x) : l^2(X) \to l^2(H^n_x)$ be the orthogonal projection. We set

$$\Phi_n : C^*_u(S) \to l^\infty(X) \otimes M_{N_n}(\mathbb{C})$$

by mapping T to $\{P_n(x) T P_n(x)\}_{x \in X}$ in the same way as in [25]. The only difference between the approach of us and the one of [25] is the local choice of the projections P_n. Clearly, if $T \in \mathbb{C}Z$ is a local kernel, then $\Phi_n(T) \in \mathcal{A} \otimes M_{N_n}(\mathbb{C})$, where $\mathcal{A}$ is the algebra defined in Subsection 6.3. Hence, Φ_n maps the algebra $C^*_r(Z)$ into $\mathbb{C}[Z] \otimes M_{N_n}(\mathbb{C})$.
The maps $\Psi_n : l^\infty(X) \times M_{N_n}(\mathbb{C}) \to C^*_u(X)$ are defined by mapping $\{T_x\}_{x \in X}$, $T_x \in B(l^2(H^n_x)) \cong M_{N_n}(\mathbb{C})$ to $\sum_x M_n(x)^* T_x M_n(x)$, where $M_n(x)$ denotes the operator of pointwise multiplication by the function $y \to \varsigma^n_y(x)$. By the definition of Local Property A, the vectors ς^n_y are a priori locally defined, hence Ψ_n maps $\mathcal{A} \otimes M_{N_n}(\mathbb{C})$ into $\mathbb{C}Z$. That is, Ψ_n maps $\mathbb{C}[Z] \otimes M_{N_n}(\mathbb{C})$ into $C^*_r(Z)$. Now our theorem follows. $\square$

7.2 Two examples for Local Property A

A tracial example. Let Z be the generic URS constructed at the end of Subsection 5.2. That is, if $H \in Z$, then $S = S^Q_{\Gamma_k}(H)$ is a colored graph satisfying

$$C^1_\alpha r^\alpha \leq B_r(S, x) \leq C^2_\alpha r^\alpha \tag{4}$$

uniformly for some positive constants C^1_α and C^2_α.

Proposition 7.2. *The graph S has Local Property A.*

Proof. For a fixed vertex $w \in V(S)$ the unit vector ς^k_w is defined the following way.

- $\varsigma^k_w(z) = \frac{1}{\sqrt{|B_k(S,w)|}}$ if $z \in B_k(S, w)$.
- $\varsigma^k_w(z) = 0$ otherwise.

Lemma 7.1. *Let $x, y \in V(S)$ be arbitrary adjacent vertices. Then $\|\varsigma^k_y - \varsigma^k_x\|^2 \leq 2d\rho + \left(\frac{1}{\sqrt{2\rho d+1}} - 1\right)^2$.*

Proof. Let d be a bound for the vertex degrees of S and let

$$\rho = \frac{|\partial B_k(S, x)|}{|B_k(S, x)|}\,.$$

We can suppose that $|B_k(S, x)| \leq |B_k(S, y)|$. Then we have that

$$\|\varsigma^k_y - \varsigma^k_x\|^2 \leq 2d\frac{|\partial B_k(S, x)|}{|B_k(S, x)|} + |B_k(S, x)| \left(\frac{1}{\sqrt{|B_k(S, y)}} - \frac{1}{\sqrt{|B_k(S, x)}}\right)^2 .$$

Now, $|B_k(S, x)| \leq |B_k(S, y)| \leq (2\rho d + 1)|B_k(S, x)|$. Therefore,

$$\left(\frac{1}{\sqrt{|B_k(S, y)}} - \frac{1}{\sqrt{|B_k(S, x)}}\right)^2 \leq \frac{1}{|B_k(S, x)|} \left(\frac{1}{\sqrt{2\rho d + 1}} - 1\right)^2 ,$$

hence our lemma follows. ∎

Thus, in order to prove our proposition, it is enough to show that for every $\varepsilon > 0$, there exists $K > 0$ such that for each $x \in V(S)$

$$\frac{|\partial B_k(S, x)|}{|B_k(S, x)|} \leq \varepsilon\,. \tag{5}$$

Note that (4) implies that S has the doubling condition, hence (5) follows from Theorem 4 [28]. ∎

A non-tracial example. Let T be a 3-regular tree. It is well-known that T has Property A. The construction goes as follows. First, we pick an infinite ray

$R = (x_0, x_1, \dots)$ towards the infinity. Then for each $t \in T$, there is a unique adjacent vertex $\phi(t)$ towards R (if $t = x_i$, $\phi(t) = x_{i+1}$). Then for a vertex s, we choose the path

$$P_s^n = (s, \phi(s), \phi^2(s), \dots, \phi^{n^2-1}(s)) .$$

The unit vector ς_s^n is associated to the path P_s^n as above, that is $\varsigma_s^n(z) = \frac{1}{n}$ if $z \in P_s$ and $\varsigma_s^n(z) = 0$ otherwise.

Proposition 7.3. *One can properly color T by finitely many colors to obtain a Schreier graph of Local Property A that generates a generic URS.*

Proof. Our goal is to choose a coloring that encodes ϕ. First, pick any finite proper coloring $c : E(T) \to K$ for some finite set K such that $c(e) \neq c(f)$ if $e \neq f$ and the distance of e and f is less than 3. Now we recolor the edge $(a, \phi(a))$ by $c(a, \phi(a)) \times c(\phi(a), \phi^2(a))$. Hence, we obtained a proper coloring $c' : E(T) \to K \times K$ such that ϕ is encoded in the coloring so the paths P_s^n (and thus the unit vectors ς_s^n) can be chosen locally. Now let $m : E(T) \to A$ be the coloring given in Proposition 5.1. Then $m \times c' : E(T) \to A \times K \times K$ provides a proper coloring of T, such that the resulting Schreier graph has Local Property A and generates a generic URS. $\square$

8 The Feldman-Moore construction revisited

Let $\alpha : \Gamma \to (X, \mu)$ be a measure preserving action of a finitely generated group Γ on a standard probability measure space (X, μ). The following construction is due to Feldman and Moore [13]. We call a bounded measurable function $K : X \times X \to \mathbb{C}$ an FM-kernel if

- $K(x, y) \neq 0$ implies that x and y are on the same orbit.
- There exists a constant w_K such that if x and y are on the orbit graph S, then $d_S(x, y) > w_K$ implies that $K(x, y) = 0$.

The FM-kernels for the unital $*$-algebra $FM(\alpha)$, where

- $(K + L)(x, y) = K(x, y) + L(x, y)$.
- $KL(x, y) = \sum_{z \in X} K(x, z) L(z, y)$.
- $K^*(x, y) = \overline{K(y, x)}$.

The trace function Tr_α is defined on $FM(\alpha)$ by

$$\mathrm{Tr}_\alpha(K) = \int_X K(x, x) d\mu(x) .$$

Then, by the GNS-construction we can obtain a tracial von Neumann-algebra $FM(\alpha) \subset M(\alpha)$ in such a way that the trace on $M(\alpha)$ is the extension of Tr_α. Let us very briefly recall the construction. We define a pre-Hilbert space structure on $FM(\alpha)$ by $\langle A, B\rangle = \mathrm{Tr}_\alpha(B^*A)$. Then $L_A(B) = AB$ defines a map of $LM(\alpha)$ into $B(FM(\alpha))$. Then $M(\alpha)$ is the weak closure of the image. In particular, $\{K_n\}_{n=1}^\infty \subset M(\alpha)$ converges to $K \in M(\alpha)$ weakly if and only if for any $A, B \in LM(\alpha)$, $\lim_{n\to\infty} \mathrm{Tr}(AK_nB) = \mathrm{Tr}(AKB)$. Now, let $Z \subset \mathrm{Sub}(\Gamma)$ be a URS and μ be a Γ-invariant Borel probability measure on Z. Again, β denotes the Γ-action on Z. By definition, we have a natural homomorphism: $\phi_\beta : \mathbb{C}Z \to M(\beta)$.

Proposition 8.1. *The map ϕ_β is injective and $\phi_\beta(\mathbb{C}Z)$ is weakly dense in the von Neumann algebra $M(\beta)$. Furthermore, the map ϕ_β extends to a continuous embedding $\overline{\phi}_\beta : C_r^*(Z) \to M(\beta)$.*

Proof. First note, that if $K : X \times X \to \mathbb{C}$ is an FM-kernel, then K can be written as $\sum_{i=1}^t M_{f_i} K_{g_i}$, where

- For any $1 \leq i \leq t$, f_i is a bounded μ-measurable function.
- $M_{f_i} \in FM(\beta)$ is supported on the diagonal and $M_{f_i}(x, x) = f_i(x)$.
- $g_i \in \Gamma$ and $K_{g_i}(x, y) = 1$ if $\beta(g_i)(y) = x$, otherwise $K_{g_i}(x, y) = 0$.

Let $0 \neq K \in \mathbb{C}Z$. In order to prove that ϕ_β is injective, it is enough to show that $\mathrm{Tr}_\beta(\phi_\beta(K^*K)) \neq 0$. Let

$$U = \{x \in X \mid K^*K(x, x) \neq 0\}.$$

Then U is a nonempty open set, so by minimality of the action β, $\mu(U) > 0$ since μ is Γ-invariant. Therefore, $\mathrm{Tr}_\beta(\phi_\beta(K^*K)) \neq 0$. Now we show that $\phi_\beta(\mathbb{C}Z)$ is weakly dense in the von Neumann algebra $M(\beta)$.

Lemma 8.1. *Let $K_n \in FM(\beta)$, $K \in FM(\beta)$ such that*

- $\sup_{x,y\in X} |K_n(x, y)| < \infty$,
- $\sup_{n\geq 1} w_{k_n} < \infty$,
- *For μ-almost every x, $K_n(x, y) \to K(x, y)$ for all $y \in Orb(x)$.*

then $Tr_\beta(AKB) = \lim_{n\to\infty} Tr_\beta(AK_nB)$ holds for any pair $A, B \in FM(\beta)$, hence by the GNS-construction $\{K_n\}_{n=1}^\infty$ weakly converges to K.

Proof. Recall that

$$\mathrm{Tr}_\beta(AKB) = \int_X \sum_{y,z\in \mathrm{Orb}(x)} A(x, y)K(y, z)B(z, x)\, d\mu(x).$$

By our condition, for almost every $x \in X$,

$$\lim_{n\to\infty} \sum_{y,z\in \mathrm{Orb}(x)} A(x, y)K_n(y, z)B(z, x) = \sum_{y,z\in \mathrm{Orb}(x)} A(x, y)K(y, z)B(z, x),$$

hence by Lebesgue's Theorem

$$\mathrm{Tr}_\beta(AKB) = \lim_{n\to\infty} \mathrm{Tr}_\beta(AK_nB)\,. \qquad \square$$

Now, let $K \in FM(\beta)$. We need to find a sequence $\{K_n\}_{n=1}^\infty \subset \phi_\beta(\mathbb{C}Z)$ that weakly converges to K. Let $K = \sum_{i=1}^t M_{f_i} K_{\gamma_i}$. By Lemma 6.1, for every $\alpha \in E_r$ we have a clopen set $W_\alpha \subset Z$ such that $\chi_{W_\alpha} \in \mathbb{C}Z$ and $\cup_{\alpha\in E_r} W_\alpha$ forms a partition of Z. Furthermore, if $U \in Z$ is an open set, then we have a sequence $\{Q_r^A \subset E_r\}$ so that

$$\cup_{\alpha\in Q_1^A} W_\alpha \subset \cup_{\alpha\in Q_2^A} W_\alpha \subset \dots$$

and

$$\bigcup_{r=1}^\infty (\cup_{\alpha\in Q_r^A W_\alpha}) = U\,. \tag{6}$$

Since Z is homeomorphic to the Cantor set and μ is a Borel measure, for any μ-measurable set $A \subset Z$ we have a sequence of open sets $\{U_n\}_{n=1}^\infty \subset Z$ such that

$$\{\chi_{U_n}\}_{n=1}^\infty \to \mu_A \tag{7}$$

μ-almost everywhere. Therefore, by (6) and (7), for any $1 \leq i \leq t$, we have a uniformly bounded sequence of functions $\{g_{ij}\}_{j=1}^\infty$ tending to f_i almost everywhere, such that for any $i, j \geq 1$, $g_{ij} \in \mathcal{A}$. For $r \geq 1$, let $K_r = \sum_{i=1}^t M_{g_{ir}} K_{\gamma_i} \in \phi_\beta(\mathbb{C}Z)$. Then for μ-almost every $x \in X$

$$\lim_{r\to\infty} K_r(x,y) = K(x,y)$$

provided that $y \in \mathrm{Orb}(x)$. Therefore by Lemma 8.1, $\{K_r\}_{r=1}^\infty$ weakly converges to K. Hence, $\phi_\beta(\mathbb{C}Z)$ is weakly dense in $LM(\beta)$ and thus $\phi_\beta(\mathbb{C}Z)$ is weakly dense in $M(\beta)$ as well. Now we prove that ϕ_β extends to $C_r^*(Z)$. First note that $\overline{Tr}_\beta(K) = \int K(x,x)\, d\mu(x)$ is a continuous trace on $C_r^*(Z)$ extending Tr_β. Indeed, $\overline{Tr}_\beta(K) \leq \sup_{x\in X} |K(x,x)| \leq \|K\|$. Let $\mathcal{N}$ be the von Neumann algebra obtained from $C_r^*(Z)$ by the GNS-construction using the continuous trace $\overline{Tr}_\beta$. The weak closure of $\mathbb{C}Z$ in $\mathcal{N}$ is isomorphic to M_β, hence it is enough to prove that $\phi_\beta(\mathbb{C}Z)$ is weakly dense in $C_r^*(Z)$. Let $A, B, K \in C_r^*(Z)$, $\{K_n\}_{n=1}^\infty \subset \mathbb{C}Z$, such that $K_n \to K$ in norm. Then by the continuity of the trace, $\lim_{n\to\infty} \overline{Tr}_\beta(AK_nB) = \overline{Tr}_\beta(AKB)$. Hence $\mathbb{C}Z$ is in fact weakly dense in $C_r^*(Z)$. $\square$

9 Coamenability and amenable traces

9.1 Amenable trace revisited

First, let us recall the notion of amenable traces from [6]. Let $\mathcal{A}$ be a C^*-algebra of bounded operators on the standard separable Hilbert space $\mathcal{H}$. Let $\{P_n\}_{n=1}^\infty$ be a sequence of finite dimensional projections in $\mathcal{H}$ such that

- For any $a \in \mathcal{A}$
$$\lim_{n\to\infty} \frac{\|AP_n - P_nA\|_{HS}}{\|P_n\|_{HS}} = 0\,.$$

- $$\tau(A) = \lim_{n\to\infty} \frac{\langle AP_n, P_n\rangle_{HS}}{\|P_n\|^2_{HS}}$$
defines a continuous trace on $\mathcal{A}$, where $\langle A, B\rangle_{HS} = \mathrm{Tr}(B^*A)$ for Hilbert-Schmidt operators.

Then τ is called an **amenable trace**. Now let Γ be a finitely generated group as above and $Z \subset \mathrm{Sub}(\Gamma)$ be a coamenable generic URS. Let $H \in Z$, and consider the usual representation of $C^*_r(Z)$ on $l^2(\Gamma/H)$ by kernels. Let $\{T_n\}_{n=1}^\infty$ be a sequence of induced subgraphs in $S = S^Q_\Gamma(H)$ such that $\lim_{n\to\infty} \frac{|\partial T_n|}{|V(T_n)|} = 0$. Also, let us suppose that the sequence $\{T_n\}_{n=1}^\infty$ is convergent in the sense of Benjamini and Schramm as defined in Subsection 5.3. Observe that convergence means that for any $r \geq 1$ and $\alpha \in E_r$
$$\lim_{n\to\infty} \frac{|V(T_n) \cap \alpha|}{|V(T_n)|} = t(\alpha)$$
exists and $t(\alpha) = \mu(\lambda_H(\alpha))$ (see Subsection 6.3), where the Γ-invariant probability measure μ on Z is the limit of the sequence $\{T_n\}_{n=1}^\infty$. We define the amenable trace τ similarly as in [11]. For $n \geq 1$, let $P_n : l^2(\Gamma/H) \to l^2(V(T_n)) \subset l^2(\Gamma/H)$ be the orthogonal projection.

Proposition 9.1. *For any $K \in C^*_r(Z)$*
$$\tau(K) = \lim_{n\to\infty} \frac{\langle AP_n, P_n\rangle_{HS}}{\|P_n\|^2_{HS}}$$
exists and $\tau(K) = \overline{Tr}_\mu(K)$ (as defined in Subsection 8). Also, for any $K \in C^*_r(Z)$,
$$\lim_{n\to\infty} \frac{\|KP_n - P_nK\|_{HS}}{\|P_n\|_{HS}} = 0\,,$$
hence τ is an amenable trace.

Proof. Let us start with a simple observation.

Lemma 9.1. *Let $\{H_n : \Gamma/H \times \Gamma/H \to \mathbb{C}\}_{n=1}^\infty$ be a sequence of maps such that*

- *There exists $K > 0$, $|H_n(x,y)| \leq K$, for any $n \geq 1$ and $x, y \in \Gamma/H$.*
- $\lim_{n\to\infty} \frac{|Q_n|}{|V(T_n)|} = 0$, *where*
$$Q_n = \{(x,y) \in \Gamma/H \times \Gamma/H \mid H_n(x,y) \neq 0\}\,.$$

Then $\lim_{n\to\infty} \frac{|Tr(H_n)|}{\|P_n\|^2_{HS}} = 0$.

Lemma 9.2. *Let* $K \in \mathbb{C}Z$. *Then*

$$\lim_{n\to\infty} \frac{\|KP_n - P_nK\|^2_{HS}}{\dim P_n} = 0\,.$$

Proof. First we have that

$$\|KP_n - P_nK\|^2_{HS} = \mathrm{Tr}((P_nK^* - K^*P_n)(KP_n - P_nK)) = \mathrm{Tr}(P_nK^*KP_n) -$$

$$-\mathrm{Tr}(K^*P_nKP_n) - \mathrm{Tr}(P_nK^*P_nK) + \mathrm{Tr}(K^*P_nK)\,.$$

For $n \geq 1$, let $K_n : \Gamma/H \to \Gamma/H \to \mathbb{C}$ be defined in the following way. $K_n(x,y) = K(x,y)$ if $x, y \in V(T_n)$, otherwise, $K_n(x,y) = 0$. That is, for any $n \geq 1$, K_n is a trace-class operator. Now, we have that

$$\mathrm{Tr}(P_nK^*KP_n) = \mathrm{Tr}(P_nK_n^*K_nP_n) + \mathrm{Tr}(P_n(K-K_n)^*K_nP_n) +$$

$$+\mathrm{Tr}(P_nK^*(K-K_n)P_n)\,.$$

Sublemma 9.1. *Both the sequences*

$$\{P_nK^*(K-K_n)P_n\}_{n=1}^\infty \quad \textit{and} \quad \{P_n(K-K_n)^*K_nP_n\}_{n=1}^\infty$$

satisfy the conditions of our Lemma 9.1.

Proof. Notice that

$$(P_nK^*(K-K_n)P_n)(x,y) = \sum_{z\in\Gamma/H} P_n(x,x)K^*(x,z)(K-K_n)(z,y)P_n(y,y)\,.$$

Observe that $(K-K_n)(z,y)P_n(y,y) \neq 0$ implies that $y \in V(T_n)$, $z \notin V(T_n)$, $d_{S^Q_\Gamma(H)}(z,y) \leq w_K$. Since $\lim_{n\to\infty} \frac{|\partial T_n|}{|V(T_n)|} = 0$, our sublemma immediately follows. ∎

Repeating the arguments of our sublemma it follows that

$$\lim_{n\to\infty} \frac{\|KP_n - P_nK\|^2_{HS}}{\dim P_n} =$$

$$= \frac{\mathrm{Tr}(P_nK_n^*K_nP_n) - \mathrm{Tr}(K_n^*P_nK_nP_n)}{\dim P_n} +$$

$$+ \frac{\mathrm{Tr}(K_n^*P_nP_nK_n) - \mathrm{Tr}(P_nK_n^*K_nP_n)}{\dim P_n} = 0$$

since K_n, P_n are trace-class operators. ∎

Now let $K \in C^*(Z)$ and $\varepsilon > 0$. Let $L \in \mathbb{C}Z$ such that $\|K - L\| < \varepsilon$. By the previous lemma, there exists $N > 0$ such that if $n \geq N$, then $\|LP_n - P_n L\|_{HS} < \varepsilon \|P_n\|_{HS}$. We have that

$$\|KP_n - LP_n\|_{HS} \leq \|K - L\| \|P_n\|_{HS}$$

and

$$\|P_n K - P_n L\|_{HS} \leq \|K - L\| \|P_n\|_{HS} \,.$$

Therefore, if $n \geq N$, $\|KP_n - P_n K\|_{HS} < 3\varepsilon \|P_n\|_{HS}$. Hence our proposition follows. □

9.2 Uniformly amenable traces

Recall from [6] that an amenable trace τ is **uniformly amenable** if the resulting von Neumann algebra $\pi_\tau(\mathcal{A})''$ is hyperfinite. Let $\alpha : \Gamma \curvearrowright (Z, \mu)$ be the Benjamini-Schramm limit of the graph sequence $\{V(T_n)\}_{n=1}^\infty$ as in the previous subsection. Since $C_r^*(Z)$ is dense in $M(\alpha)$, τ is a uniformly amenable trace if and only if the equivalence relation generated by the action α is hyperfinite. By Theorem 1. [12], the equivalence relation above is hyperfinite if and only if $\{T_n\}_{n=1}^\infty$ is a **hyperfinite graph sequence**. That is, for any $\varepsilon > 0$ there exists $K > 0$ such that for any $n \geq 1$ one can remove from the graph T_n $\varepsilon |V(T_n)|$ edges in such a way that all the components of the remaining graph has at most K elements. Therefore we have the following proposition.

Proposition 9.2. *Let $Z \subset Sub(\Gamma)$ be a generic URS and $H \in Z$. If $S_\Gamma^Q(H)$ admits a convergent hyperfinite sequence of finite subgraphs $\{T_n\}_{n=1}^\infty$ such that $\frac{|\partial T_n|}{|V(T_n)|} \to 0$, then $C_r^*(Z)$ has a uniformly amenable trace. If $S_\Gamma^Q(H)$ admits a convergent nonhyperfinite sequence of finite subgraphs $\{T_n\}_{n=1}^\infty$ such that $\frac{|\partial T_n|}{|V(T_n)|} \to 0$, then $C_r^*(Z)$ has a non-uniformly amenable trace, that is, by Theorem 4.3.3. $C_r^*(Z)$ is not locally reflexive, hence it is a nonexact C^*-algebra.*

10 A nonexact example

10.1 The construction

In Section 4.3.3 of [8], we constructed a Schreier graph $S = S_\Gamma^Q(H)$ of a group Γ such that that the orbit closure of S is a generic URS. Also, S contains a convergent, hyperfinite sequence of finite subgraphs $\{T_n\}_{n=1}^\infty$ with $\frac{|\partial T_n|}{|V(T_n)|} \to 0$ and a convergent nonhyperfinite sequence of finite subgraphs $\{W_n\}_{n=1}^\infty$ with $\frac{|\partial W_n|}{|V(W_n)|} \to 0$. Let Z be the orbit closure of H in $\mathrm{Sub}(\Gamma)$. Then by Proposition 9.2, $C_r^*(Z)$ is not a locally reflexive (hence nonexact) simple, unital separable C^*-algebra with both uniform amenable and non-uniform amenable traces. For completeness, we present a somewhat slicker construction, that is very similar to the one given in [8]

Step 0. For $n \geq 1$, let C_n be the cycle of length 2^{n+1}. Let Γ_3 be the free group of three cycle groups of rank 2. Let $\Gamma_3 \subset M_1 \supset M_2 \supset \dots, \cap_{n=1}^{\infty} M_n = \{1\}$ be a sequence of finite index normal subgroups. For $n \geq 1$ let V_n be the underlying graph of the Cayley-graph of Γ_3/M_n. That is, the sequence $\{V_n\}_{n=1}^{\infty}$ itself is a convergent nonhyperfinite sequence of finite graphs.

Step 1. Let $G_1 = C_1$ and $H_1 = C_2$.

Step n. Let us suppose that the graphs $G_2 \subset G_4 \subset \dots \subset G_{2n-2}$, $G_1 \subset G_3 \subset \dots \subset G_{2n-1}$ and $\{H_i\}_{i=1}^{2n-1}$ are already defined in such a way that

- If $1 \leq i \leq n-1$, then $H_{2i} = V_{k_i}$ for some $k_i > 0$, $H_{2i+1} = C_{l_i}$ for some $l_i > 0$.
- For any $1 \leq i \leq n-1$ we have disjoint subsets $\{R^j_{2i}\}_{j=1}^{i} \subset H_{2i}, \{R^j_{2i+1}\}_{j=1}^{i} \subset H_{2i+1}$ so that
$$|R^j_{2i}||V(G_j)| \leq \frac{1}{10^j}|V(H_{2i})| \ , \ |R^j_{2i+1}||V(G_j)| \leq \frac{1}{10^j}|V(H_{2i+1})| \, .$$
- We have positive integers $T_1 < T_2 < \dots T_{n-1}$ such that for any $1 \leq j \leq i \leq n-1$
$$R^j_{2i} \cap B_{T_j}(H_{2i}, x) \neq \emptyset, R^j_{2i+1} \cap B_{T_j}(H_{2i+1}, y) \neq \emptyset \, ,$$
for all $x \in V(H_{2i}), y \in V(H_{2i+1})$.
- The graph G_{2i} is constructed in such a way that for any $1 \leq j \leq i$ a copy of G_j is connected to all the vertices of $R^j_{2i} \subset V(H_{2i})$. The graph G_{2i+1} is constructed in such a way that for any $1 \leq j \leq i$ a copy of G_j is connected to all the vertices of $R^j_{2i+1} \subset V(H_{2i+1})$. Connecting a graph A to a graph B means that we add a disjoint copy of A to B plus an extra edge beween a vertex of A and a vertex of B.

Now we construct the graphs G_{2n} and G_{2n+1}. We pick a graph $H_{2n} = V_{k_n}$ and $H_{2n+1} = C_{l_n}$ in such a way that
$$|V(G_n)| \leq \frac{1}{10^n}|V(H_{2n})| \ , \ |V(G_n)| \leq \frac{1}{10^n}|V(H_{2n+1})| \, .$$
We define T_n as the maximum of the diameters of the graphs H_{2n} and H_{2n+1}. Now we use the fact that we have normal covering maps $\zeta_{2n} : H_{2n} \to H_{2n-2}$ and $\zeta_{2n+1} : H_{2n+1} \to H_{2n-1}$. For $1 \leq j \leq n-1$, set $R^j_{2n} = \zeta^{-1}_{2n}(R^j_{2n-2})$ and $R^j_{2n+1} = \zeta^{-1}_{2n+1}(R^j_{2n-1})$. That is,
$$|R^j_{2n}||V(G_j)| \leq \frac{1}{10^j}|V(H_{2n})| \ , \ |R^j_{2n+1}||V(G_j)| \leq \frac{1}{10^j}|V(H_{2n+1})| \, .$$
Now for any $1 \leq j \leq n-1$, we connect a copy of G_j to the vertices of R^j_{2n} and a copy of G_j to the vertices of R^j_{2n+1}. Finally, we connect a copy of G_n to a single

vertex a of $V(H_{2n})$ not covered by any R^j_{2n} and a copy of G_n to a single vertex b of $V(H_{2n+1})$ not covered by any R^j_{2n+1}. Then we set $a = R^n_{2n}$, $b = R^n_{2n+1}$. Then, for any $1 \leq j \leq n$, $R^j_{2n} \cap B_{T_j}(H_{2n}, x) \neq \emptyset$ and $R^j_{2n+1} \cap B_{T_j}(H_{2n+1}, y) \neq \emptyset$, for all $x \in V(H_{2n})$ and $y \in V(H_{2n+1})$. Also, we have that

$$|R^n_{2n}||V(G_n)| \leq \frac{1}{10^n}|V(H_{2n}| \ , \ |R^n_{2n+1}||V(G_n)| \leq \frac{1}{10^n}|V(H_{2n+1}| \, .$$

Now, we have graphs $G_1 \subset G_3 \subset G_5 \subset \dots$ and set $G = \cup_{i=1}^\infty G_{2i-1}$. By the self-similar nature of our construction, it is easy to see that for any $x \in V(G)$ and $r \geq 1$, there exists $N_{x,r} > 0$ such that if $y \in V(G)$, then there is a $z \in B_{N_{x,r}}(G, y)$ such that $B_r(G, x)$ and $B_r(G, z)$ are isomorphic as rooted graphs. Notice that $G_1 \subset G_3 \subset \dots$ are forming a hyperfinite sequence of subgraphs. so that $\frac{|\partial G_{k_i}|}{|V(G_{k_i})|} \to 0$. Also, we have subgraphs G_2, G_4,... connected by one single edge to H_{2n+1}, that are forming such a sequence of finite graphs that no subsequence of them is hyperfinite and $\frac{|\partial G_{2i}|}{|V(G_{2i})|} \to 0$. Indeed, in the construction $H_{2n} \subset G_{2n}$, $|V(H_{2n})| > |V(G_{2n})|/2$ and the sequence of finite graphs $\{H_{2n}\}_{n=1}^\infty$ is a large girth graph sequence so no subsequence of $\{G_{2n}\}_{n=1}^\infty$ can be hyperfinite (since that would mean that a subsequence of $\{H_{2n}\}_{n=1}^\infty$ is hyperfinite as well). Now we can apply the coloring construction of Proposition 5.1 to obtain the colored graph S (and hence a URS $Z \subset \mathrm{Sub}(\Gamma_k)$ we are sought of. In the graph S there is a convergent hyperfinite sequence of finite subgraphs $\{T_n\}_{n=1}^\infty$ with $\frac{|\partial T_n|}{|V(T_n)|} \to 0$ and there is a convergent nonhyperfinite sequence of finite subgraphs $\{T'_n\}_{n=1}^\infty$ with $\frac{|\partial T'_n|}{|V(T'_n)|} \to 0$

10.2 Two more interesting properties of the nonexact URS

Let $Z \subset \mathrm{Sub}(\Gamma)$ be the generic URS constructed above.

Proposition 10.1. *There is no free continuous action of any countable group that is Borel orbit equivalent to Z.*

Proof. Suppose that Z is Borel orbit equivalent to a free continuous action θ of the countable group Δ. If Δ is amenable, then any invariant measure of the action θ makes the equivalence relation hyperfinite. However, Z admits invariant measure that makes it a nonhyperfinite measurable equivalence relation. If Δ is nonamenable then all the invariant measures make the equivalence relation of the action θ nonhyperfinite. However, Z has an invariant measure that makes its equivalence relation hyperfinite (see also [20] for a minimal action of a group that is not Borel equivalent to a free continuous action). $\square$

If $K_1, K_2 \in Z$ then their Schreier graphs are locally indistinguishable. However, it is possible that their Schreier graphs look globally quite different.

Proposition 10.2. *There exists $K_1, K_2 \in Z$ such that $S^Q_\Gamma(K_1)$ is one-ended and $S^Q_\Gamma(K_2)$ is two-ended.*

Proof. Clearly, there must be an element K_1 in Z such that the underlying graph of $S^Q_\Gamma(K_1)$ is isomorphic to the graph G in our construction and G is clearly one-ended. Now we pick a sequence of points $x_n \in H_{2n+1}$. Consider a limitpoint $T = S^Q_\Gamma(K_2)$ of the rooted Schreier graphs $\{(S^Q_\Gamma(K_1), x_n)\}^\infty_{n=1}$ in the compact space Sch^Q_Γ. It is easy to see that T has multiple ends. $\square$

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