# An improvement on the maximum number of $k$-Dominating Independent Sets 

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#### Abstract

Erdős and Moser raised the question of determining the maximum number of maximal cliques or equivalently, the maximum number of maximal independent sets in a graph on $n$ vertices. Since then there has been a lot of research along these lines.

A $k$-dominating independent set is an independent set $D$ such that every vertex not contained in $D$ has at least $k$ neighbours in $D$. Let $m i_{k}(n)$ denote the maximum number of $k$-dominating independent sets in a graph on $n$ vertices, and let $\zeta_{k}:=$ $\lim _{n \rightarrow \infty} \sqrt[n]{m i_{k}(n)}$. Nagy initiated the study of $m i_{k}(n)$.

In this article we disprove a conjecture of Nagy and prove that for any even $k$ we have $$
1.489 \approx \sqrt[9]{36} \leq \zeta_{k}^{k} .
$$


We also prove that for any $k \geq 3$ we have

$$
\zeta_{k}^{k} \leq 2.053^{\frac{1}{1.053+1 / k}}<1.98
$$

improving the upper bound of Nagy.

Keywords: independent sets, $k$-dominating sets, almost twin vertices
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## 1 Introduction

Let $G=G(V, E)$ be a simple graph. For any vertex $v \in V(G)$ let us denote by $d(v)$ the degree of $v, N(v)$ denotes the set of neighbors of $v$, also called the open neighborhood of $v$ and $N[v]$ denotes the closed neighborhood, i.e. $N[v]:=N(v) \cup\{v\}$.

A subset $I \subset V(G)$ is called independent if it does not induce any edges. A maximal independent set is an independent set which is not a proper subset of another independent set (that is, it cannot be extended to a bigger independent set). A subset $D \subset V(G)$ is a dominating set in G if each vertex in $V(G) \backslash D$ is adjacent to at least one vertex of D , that is,

$$
\forall v \in V(G) \backslash D:|N(v) \cap D| \geq 1
$$

Erdős and Moser raised the question to determine the maximum number of maximal cliques that an $n$-vertex graph might contain. By taking complements, one sees that it is the same as the maximum number of maximal independent sets an $n$-vertex graph can have. A dominating and independent set $W$ of vertices is often called a kernel of the graph (due to Morgenstern and von Neumann [6]) and clearly, a subset $W$ is a kernel if and only if it is a maximal independent set.

The problem of finding the maximum possible number of kernels has been resolved in many graph families. To state (some of) these results, let $m i_{1}(n)$ denote the maximum number of maximal independent sets in graphs of order $n$, and let $m i_{1}(n, \mathcal{F})$ denote the maximum number of maximal independent sets in the $n$-vertex members of the graph family $\mathcal{F}$. Answering the question of Erdős and Moser, Moon and Moser proved the following well known theorem.

Theorem 1. (Moser, Moon, [5]) We have

$$
m i_{1}(n)= \begin{cases}3^{n / 3} & \text { if } n \equiv 0(\bmod 3) \\ \frac{4}{3} \cdot 3^{\lfloor n / 3\rfloor} & \text { if } n \equiv 1(\bmod 3) \\ 2 \cdot 3^{\lfloor n / 3\rfloor} & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

Moreover, they obtained the extremal graphs. If addition and multiplication by a positive integer denotes taking vertex disjoint union, then Moser and Moon proved that the equality is attained if and only if the graph $G$ is isomorphic to the graph $n / 3 K_{3}($ if $n \equiv 0(\bmod 3))$; to one of the graphs $(\lfloor n / 3\rfloor-1) K_{3}+K_{4}$ or $(\lfloor n / 3\rfloor-1) K_{3}+2 K_{2}($ if $n \equiv 1(\bmod 3))$; $\lfloor n / 3\rfloor K_{3}+K_{2}($ if $n \equiv 2(\bmod 3))$.

For the family of connected graphs the analogous question was raised by Wilf [11] and answered by the following result.

Theorem 2. (Füredi [2], Griggs, Grinstead, Guichard [3]) Let $\mathcal{F}_{\text {con }}$ be the family of connected graphs. Then

$$
m i_{1}\left(n, \mathcal{F}_{\text {con }}\right)= \begin{cases}\frac{2}{3} \cdot 3^{n / 3}+\frac{1}{2} \cdot 2^{n / 3} & \text { if } n \equiv 0(\bmod 3) \\ 3^{\lfloor n / 3\rfloor}+\frac{1}{2} \cdot 3^{\lfloor n / 3\rfloor} & \text { if } n \equiv 1(\bmod 3) \\ \frac{4}{3} \cdot 3^{\lfloor n / 3\rfloor}+\frac{4}{3} \cdot 3^{\lfloor n / 3\rfloor} & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

The extremal graphs are determined as well. In these graphs, there is a vertex of maximum degree, and its removal yields a member of the extremal graphs list of the previous theorem.

Wilf [11] and Sagan [10] investigated the case of trees and proved the following theorem.
Theorem 3. Let $\mathcal{T}$ be the family of trees. Then we have

$$
m i_{1}(n, \mathcal{T})= \begin{cases}\frac{1}{2} \cdot 2^{n / 2}+1 & \text { if } n \equiv 0(\bmod 2) \\ 2^{\lfloor n / 2\rfloor} & \text { if } n \equiv 1(\bmod 2)\end{cases}
$$

Hujter and Tuza determined the maximal number of kernels in triangle free graphs by proving the following result.

Theorem 4. ([4]) Let $\mathcal{T}_{\Delta}$ be the family of triangle-free graphs. Then for any integer $n \geq 4$ we have

$$
m i_{1}\left(n, \mathcal{T}_{\Delta}\right)= \begin{cases}2^{n / 2} & \text { if } n \equiv 0(\bmod 2) \\ 5 \cdot 2^{(n-5) / 2} & \text { if } n \equiv 1(\bmod 2)\end{cases}
$$

Other related results can be found in the survey of Chang and Jou [1].
There are lots of variants of domination studied in the literature. A quite natural and often considered one is $k$-domination. A set $D$ is called $k$-dominating if each vertex in $V(G) \backslash D$ is adjacent to at least $k$ vertices of $D$. In other words,

$$
\forall v \in V(G) \backslash D:|N(v) \cap D| \geq k
$$

A $k$-dominating independent set is called a $k$-DIS for short. Note that 1-DISes are exactly maximal independent sets. This notion was introduced by Włoch [12]. Nagy [7, 8] addressed the problem of determining the maximum number of $k$-dominating independent sets (for a given $k \geq 2$ ) in an $n$-vertex graph. Generalizing $m i_{1}(n)$ and $m i_{1}(\mathcal{F})$ we introduce the following notation.

Notation 5. For $n, k \geq 1$ let $m i_{k}(n)$ denote the maximum number of $k$-DISes in graphs of order $n$, and let $m i_{k}(n, \mathcal{F})$ denote the maximum number of $k$-DISes in an n-vertex graph from the family $\mathcal{F}$. If $\mathcal{F}$ consists of a single graph $G$, we denote by $m i_{k}(G)$ the number of $k$-DISes in $G$.

In [8] Nagy proved that for all $k \geq 1$

$$
\zeta_{k}:=\lim _{n \rightarrow \infty} \sqrt[n]{m i_{k}(n)}
$$

exists. Theorem 1 implies $\zeta_{1}=\sqrt[3]{3}$ and, by definition, for $k \geq 2$ we have $\zeta_{k} \in[1, \sqrt[3]{3}]$. The following upper and lower bounds were established on the values of $\zeta_{k}$.

Theorem 6. (Theorem 1.7 [8]) For all $k \geq 3$ we have:

$$
\sqrt{2} \leq \zeta_{k}^{k} \leq 2^{\frac{k}{k+1}}
$$

Theorem 7. (Theorem 1.6 [8]) We have

$$
1.489 \approx \sqrt[9]{36} \leq \zeta_{2}^{2} \leq \sqrt[5]{9} \approx 1.551
$$

Nagy conjectured in [8] (Conjecture 2, p19) that the lower bound of Theorem 6] will be the value of $\zeta_{k}^{k}$. Our following theorem disproves this conjecture.

Theorem 8. For any even $k$ we have

$$
\sqrt[9]{36} \leq \zeta_{k}^{k}
$$

Furthermore, $\lim _{\infty} \zeta_{k}^{k}$ exists and is at least $\sqrt[9]{36}$.
In this paper, our aim is to show that there is a constant $\eta>0$ such that $\zeta_{k}^{k}<2-\eta$ for all $k \geq 3$, thus improving Theorem 6.

Theorem 9. For $k \geq 3$ we have

$$
\zeta_{k}^{k} \leq 2.053^{\frac{1}{1.053+1 / k}}<1.98
$$

Remark 10. It is easy to see that $1.98<2^{k /(k+1)}$ for $k \geq 588503$. In fact, the following calculation shows that Theorem 9 improves Theorem for all $k \geq 3$. We want to show that

$$
2^{k /(k+1)}>(2+\varepsilon)^{1 /(1+\varepsilon+1 / k)}
$$

for $\varepsilon=0.053$ and any $k \geq 3$. After rearranging we get

$$
2^{\varepsilon}>(1+\varepsilon / 2)^{1+1 / k}
$$

which is true for $\varepsilon=0.053$ and $k=3$. Therefore, it is true for any larger $k$.

The remainder of the paper is organized as follows. In Section 2 we prove Theorem 8, in Section 3 we prove Theorem 9 and we finish the article with some remarks and open questions in Section 4.

## 2 Constructions - Proof of Theorem 8

In this section we gather some observations that are related to lower bound constructions. To be more formal, we introduce the following function: let $m(k, t)$ denote the smallest integer $n$ such that there exists a graph on $n$ vertices that contains at least $t k$-DISes. For our constructions we will need two types of graph products: the lexicographic product $G \cdot H$
of two graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \cdot H$ if and only if either $u$ is adjacent with $x$ in $G$ or $u=x$ and $v$ is adjacent with $y$ in $H$.

The cartesian product $G \times H$ of two graphs $G$ and $H$ also has vertex set $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \cdot H$ if and only if both $u$ is adjacent with $x$ in $G$ and $v$ is adjacent with $y$ in $H$.

All our lower bounds follow from the following remark.
Proposition 11. For any positive integers $k, l, t$, we have
(i) $m i_{k}(n) \geq t^{\left\lfloor\frac{n}{m(k, t)}\right\rfloor}$, and
(ii) $m(k l, t) \leq l m(k, t)$.

Proof. To prove (i) observe that if $G$ is a graph on $m(k, t)$ vertices containing at least $t$ $k$-DISes, then the graph $G^{\prime}$ consisting of $\left\lfloor\frac{n}{m(k, t)}\right\rfloor$ disjoint copies of $G$ and possibly some isolated vertices, contains at least $t^{\left\lfloor\frac{n}{m(k, t)}\right\rfloor}$ many $k$-DISes. Indeed, all isolated vertices must be contained in every $k$-DIS of $G^{\prime}$, and to form a $k$-DIS of $G^{\prime}$, one has to pick a $k$-DIS in every copy of $G$.

To prove (ii) let $G$ be a graph on $m(k, t)$ vertices containing at least $t k$-DISes. Then, if we denote by $E_{l}$ the empty graph on $l$ vertices, the graph $G^{\prime}=G \cdot E_{l}$ has $l m(k, t)$ vertices and if $I$ is a $k$-DIS in $G$, then $I^{\prime}=\{(u, v): u \in I\}$ is a $(k l)$-DIS in $G^{\prime}$.

Proof of Theorem 8. First note (as observed by Nagy already) that $K_{3} \times K_{3}$ contains 6 2-DISes on 9 vertices. Therefore, by (ii) of Proposition 11, for every even $k$ we have

$$
m(k, 6) \leq \frac{k}{2} m(2,6) \leq \frac{9 k}{2}
$$

Part (i) of Proposition 11 yields the statement for even $k$.

Proposition 12. $m(k, 2)=2 k, m(k, 3)=3 k$.
Proof. The upper bounds are given by $K_{k, k}$ and $K_{k, k, k}$. For the lower bounds, note that if $A$ and $B$ are two different $k$-DISes, then we have $|A \backslash B| \geq k$ and $|B \backslash A| \geq k$. Indeed, e.g., if $v \in A \backslash B$ then $N(v)$ must contain at least $k$ vertices in $B$, while none of these are in $A$. This observation immediately shows we need at least $2 k$ vertices for $2 k$-DISes. One can easily see by analyzing possible intersection sizes that it also shows we need at least $3 k$ vertices for $3 k$-DISes.

Note that $K_{k, k, \ldots, k}$ gives $m(k, t) \leq t k$. Nagy [8] showed $m(2,4)=8$ and $m(2,6)=9$.

## 3 Proof of Theorem 9

First of all we fix $k \geq 3$. Let $\varepsilon=0.053$ and choose $c$ such that

$$
c^{k}=(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / k}} .
$$

We need to show that $m i_{k}(n) \leq A c^{n}$ for some absolute constant $A$. We will proceed by induction on $n$ and the base case is covered by a large enough choice of $A$. Let $G$ be a graph on $n$ vertices containing maximum possible number of $k$-DISes. We assume that every vertex belongs to at least one $k$-DIS, as otherwise we can delete the vertex without decreasing the number of $k$-DISes. Let $v$ be a vertex of minimum degree in $G$ that we denote by $\delta$. Note that we may assume $\delta \geq k$. Indeed, if a vertex $v$ has degree less than $k$, then it is easy to see that it must be contained in every $k$-DIS of $G$. Then it follows that the number of $k$-DISes in $G$ is at most $m i_{k}(n-|N(v)|-1)$ (where $N(v)$ denotes the set of vertices adjacent to $v$ ) and we are done by induction.

Consider the following two cases:
Case 1: $\delta \geq(1+\varepsilon) k$.
In this case we use Proposition 5.1 from [8]. Following an inductive argument of Füredi [2], Nagy proved that we have

$$
m i_{k}(n)=m i_{k}(G) \leq c_{0} \max _{\delta \in \mathbb{Z}^{+}}\left\{\left(\frac{k+\delta}{k}\right)^{\frac{n}{\delta+1}}\right\}
$$

for some universal constant $c_{0}$. Let $\delta=\left(1+\varepsilon^{\prime}\right) k$. Then we have

$$
m i_{k}(n) \leq c_{0}\left(2+\varepsilon^{\prime}\right)^{\frac{1}{1+\varepsilon^{\prime}+1 / k} \frac{n}{k}}
$$

By Proposition 14 (see Appendix), the right hand side of the above inequality is monotone decreasing in $\varepsilon^{\prime}$. Since $\delta \geq(1+\varepsilon) k$, we have $\varepsilon^{\prime} \geq \varepsilon$. So for fixed $k \geq 3$ we conclude that

$$
m i_{k}(n) \leq c_{0}(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / k} \frac{n}{k}}=O\left(c^{n}\right)
$$

Case 2: $\delta \leq(1+\varepsilon) k$.
In this case we combine the inductive argument with a new idea. Let $v$ be a vertex of degree $\delta$. The number of $k$-DISes containing $v$ is at most $m i_{k}(n-\delta-1)$ and to bound the number of $k$-DISes not containing $v$, we introduce the following auxiliary graph. We say that two non-adjacent vertices $x, y$ of $G$ are almost twins if

$$
|N(x) \backslash N(y)|,|N(y) \backslash N(x)|<k
$$

hold. We define $T_{G}$ to be the graph with vertex set $N(v)$ and $x, y$ form an edge in $T_{G}$ if they are almost twins in $G$.

Proposition 13. If $x, y$ belong to the same connected component in $T_{G}$, then they belong to the same $k$-DISes of $G$. In particular, they are not connected.

Proof. It is enough to prove the statement for vertices adjacent in $T_{G}$. If $x$ belongs to a $k$-DIS $I$ with $y \notin I$, then there should be at least $k$ neighbors of $y$ in $I$ and as $x \in I$, we
must have $N(x) \cap I=\emptyset$. This implies $|N(y) \backslash N(x)| \geq k$ which contradicts the fact that $x$ and $y$ are almost twins.

If a pair of vertices $x, y \in N(v)$ belong to different components of $T_{G}$ then the $k$-DISes $I$ containing both of $x$ and $y$ are disjoint from $N(x) \cup N(y)$, and $I \backslash\{x, y\}$ should form a $k$-DIS in $G \backslash(N(x) \cup N(y) \cup\{x, y\})$. As $x$ and $y$ are not almost twins, $|N(x) \cup N(y)| \geq \delta+k$ as wlog. $|N(y) \backslash N(x)| \geq k$ and $|N(x)| \geq k$. Thus, the number of $k$-DISes containing both of $x$ and $y$ is at most $m i_{k}(n-\delta-k)$.

On the other hand, if $x$ and $y$ are in the same component $C$ of $T_{G}$, then by Proposition 13 any $k$-DIS $I$ containing both of $x$ and $y$ contains all vertices of $C$, is disjoint from $N(C)$ and $I \backslash C$ is a $k$-DIS in $G \backslash(N(C) \cup C)$ and by the second part of Proposition $13 N(C)$ and $C$ are disjoint. As $|N(C)| \geq \delta$, the number of $k$-DISes containing both of $x$ and $y$ is at most $m i_{k}(n-\delta-|C|)$.

Writing $s_{1}, s_{2}, \ldots, s_{j}$ for the sizes of the components of $T_{G}$, we obtain

$$
\begin{equation*}
m i_{k}(n) \leq m i_{k}(n-\delta-1)+\frac{\sum_{i=1}^{j}\binom{s_{i}}{2} m i_{k}\left(n-\delta-s_{i}\right)+\left(\binom{\delta}{2}-\sum_{i=1}^{j}\binom{s_{i}}{2}\right) m i_{k}(n-\delta-k)}{\binom{k}{2}} \tag{1}
\end{equation*}
$$

as every $k$-DIS $I$ with $v \notin I$ was counted at least $\binom{k}{2}$ times since $I$ must $k$-dominate $v$.
Let us choose $B=\beta k$ with $\beta=0.8$. This implies $2 \leq B \leq k$ as $k \geq 3$. Suppose that in $T_{G}$ the union of components of size at most $B$ is $s$. Then the number of pairs of vertices within these components is $\sum_{s_{i} \leq B}\binom{s_{i}}{2} \leq \frac{s(B-1)}{2}$. Also, the number of pairs within components of size larger than $B$ is $\sum_{s_{i}>B}\binom{s_{i}}{2} \leq\binom{\delta-s}{2}$. Observe that either $s=\delta$ or $s<\delta-B$.

Observe that $m i_{k}(n-\delta-2) \geq m i_{k}(n-\delta-B) \geq m i_{k}(n-\delta-k)$. Thus majoring all $\binom{\delta}{2}$ summands in the following sum we get:

$$
\begin{gathered}
\sum_{i=1}^{j}\binom{s_{i}}{2} m i_{k}\left(n-\delta-s_{i}\right)+\left(\binom{\delta}{2}-\sum_{i=1}^{j}\binom{s_{i}}{2}\right) m i_{k}(n-\delta-k) \leq \\
\leq \sum_{s_{i} \leq B}\binom{s_{i}}{2} m i_{k}(n-\delta-2)+\sum_{s_{i}>B}\binom{s_{i}}{2} m i_{k}(n-\delta-B)+\left(\binom{\delta}{2}-\sum_{i=1}^{j}\binom{s_{i}}{2} m i_{k}(n-\delta-k) \leq\right. \\
\leq \frac{s(B-1)}{2} m i_{k}(n-\delta-2)+\binom{\delta-s}{2} m i_{k}(n-\delta-B)+\left(\binom{\delta}{2}-\frac{s(B-1)}{2}-\binom{\delta-s}{2}\right) m i_{k}(n-\delta-k)
\end{gathered}
$$

As $\binom{\delta}{2}=\frac{s(B-1)}{2}+\left[s(\delta-s)+\frac{s(s-B)}{2}\right]+\binom{\delta-s}{2}$, this implies that the right hand side of (1) is at most

$$
\begin{equation*}
m i_{k}(n-\delta-1)+\frac{s(B-1)}{2\binom{k}{2}} m i_{k}(n-\delta-2)+\frac{\left(s(\delta-s)+\frac{s(s-B)}{2}\right)}{\binom{k}{2}} m i_{k}(n-\delta-k)+\frac{\binom{\delta-s}{2}}{\binom{k}{2}} m i_{k}(n-\delta-B) . \tag{2}
\end{equation*}
$$

Recall that we want to prove that $m i_{k}(n) \leq A c^{n}$ for some constant $A$. Using (2), by induction after simplifying it would be enough to show

$$
E:=c^{n}-\left[c^{n-\delta-1}+\frac{s(B-1)}{2\binom{k}{2}} c^{n-\delta-2}+\frac{\left(s(\delta-s)+\frac{s(s-B)}{2}\right)}{\binom{k}{2}} c^{n-\delta-k}+\frac{\binom{\delta-s}{2}}{\binom{k}{2}} c^{n-\delta-B}\right] \geq 0 .
$$

Using that $k \leq \delta$ and simplifying we obtain

$$
\begin{equation*}
\frac{E}{c^{n-\delta-k}} \geq c^{2 k}-\left[c^{k-1}+\frac{s(B-1)}{k(k-1)} c^{k-2}+\frac{s(2 \delta-s-B)}{k(k-1)}+\frac{(\delta-s)(\delta-s-1)}{k(k-1)} c^{k-B}\right] . \tag{3}
\end{equation*}
$$

We consider two cases, depending on whether $s$ is equal to $\delta$ or not. In the latter case, $s<\delta-B$, as noted already.

Case 2.1: $s=\delta$
In this case, the right hand side of (3) simplifies to

$$
c^{2 k}-c^{k-1}-\frac{\delta(B-1)}{k(k-1)} c^{k-2}-\frac{\delta(\delta-B)}{k(k-1)} .
$$

Since $\delta$ is at most $(1+\varepsilon) k$ and replacing $B$ by $\beta k$, the right hand side of the above inequality is at least

$$
\begin{gathered}
c^{2 k}-c^{k-1}-(1+\varepsilon) \frac{(\beta k-1)}{(k-1)} c^{k-2}-(1+\varepsilon)(1+\varepsilon-\beta)\left(\frac{k}{k-1}\right)=: f_{0}(k, \varepsilon, \beta) \\
\quad \geq c^{2 k}-c^{k}-(1+\varepsilon) \beta c^{k}-(1+\varepsilon)(1+\varepsilon-\beta)\left(1+\frac{1}{1000}\right)=: f_{1}(k, \varepsilon, \beta)
\end{gathered}
$$

for $k>1000$.
Recall that $\varepsilon=0.053$ and $\beta=0.8$. Note that the function $a^{2}-a-(1+\varepsilon) \beta a-(1+$ $\varepsilon)(1+\varepsilon-\beta)\left(1+\frac{1}{1000}\right)$ is increasing in the range $a \geq 1$. At $a=(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / 1000}}$ the function is positive, thus also for all $k>1000$ at $a=(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / k}}=c^{k}$ the function is positive. This means $f_{1}(k, \varepsilon, \beta)>0$, which implies $f_{0}(k, \varepsilon, \beta)>0$ for $k>1000$. It is easy to check by a simple computer calculation that $f_{0}(k, \varepsilon, \beta)>0$ for $k \leq 1000$ as well.

Case 2.2: $s<\delta-B$.
Note that $\max _{s<\delta-B}\{s(2 \delta-s-B)\}<(\delta-B) \delta$. Using this, the right hand side of (3) is at least

$$
\begin{gathered}
c^{2 k}-c^{k-1}-\frac{(\delta-B)(B-1)}{k(k-1)} c^{k-2}-\frac{(\delta-B) \delta}{k(k-1)}-\frac{\delta(\delta-1)}{k(k-1)} c^{k-B} \geq \\
c^{2 k}-c^{k}-(1+\varepsilon-\beta) \frac{(\beta k-1)}{(k-1)} c^{k}-(1+\varepsilon-\beta)(1+\varepsilon) \frac{k}{k-1} \\
-\frac{(1+\varepsilon)(k(1+\varepsilon)-1)}{(k-1)} c^{k-\beta k}:=f_{2}(k, \varepsilon, \beta) \\
\geq c^{2 k}-c^{k}-(1+\varepsilon-\beta) \beta c^{k}-(1+\varepsilon-\beta)(1+\varepsilon+2 / 1000)-(1+\varepsilon)(1+\varepsilon(1+1 / 1000)) c^{(1-\beta) k}:=f_{3}(k, \varepsilon, \beta)
\end{gathered}
$$

for $k>1000$. In the last inequality for bounding the third term we used that $2 / 1000 \geq$ $(1+\varepsilon) /(k-1)$ for $k>1000$ as $\varepsilon=0.053$.

Recall that $\beta=0.8$ and so $1-\beta=\frac{1}{5}$. Observe that the function $a^{10}-a^{5}-(1+\varepsilon-$ $\beta) \beta a^{5}-(1+\varepsilon-\beta)(1+\varepsilon+2 / 1000)-(1+\varepsilon)(1+\varepsilon(1+1 / 1000)) a$ is increasing in $a$ if $a>1$. As for $a=(2+\varepsilon)^{\frac{0.2}{1+\varepsilon+1 / 1000000}}$ the function is positive, also for all $k>1000000$ for the value $a=(2+\varepsilon)^{\frac{0.2}{1+\varepsilon+1 / k}}=c^{k-\beta k}$ the function is positive. This means $f_{3}(k, \varepsilon, \beta)>0$, which implies $f_{2}(k, \varepsilon, \beta)>0$ for $k>1000000$. It is easy to check by a simple computer calculation that $f_{2}(k, \varepsilon, \beta)>0$ for $k \leq 1000000$.

Since $\varepsilon=0.053$ and $c^{k}=(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / k}} \leq(2+\varepsilon)^{\frac{1}{1+\varepsilon}}$ for any $k \geq 3$, we get $c^{k} \leq 1.98$ for any $k \geq 3$, completing the proof of Theorem 9 .

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## References

[1] G. J. Chang and M. J. Jou, Survey on counting maximal independent sets. In: Proceedings of the Second Asian Mathematical Conference, (1995) 265-275.
[2] Z. Füredi, The number of independent sets in connected graphs. J. Graph Theory 11 (1987) 463-470.
[3] J. R. Griggs, C. M. Grinstead, and D. Guichard, The number of maximal independent sets in a connected graph. Discrete Mathematics, 68(2-3) (1988) 211-220.
[4] M. Hujter and Zs. Tuza, The number of maximal independent sets in triangle-free graphs. SIAM J. Discrete Math., 6 (1993) 284-288.
[5] J. W. Moon and L. Moser, On cliques in graphs. Israel J. Math. 3 (1965) 23-28.
[6] O. Morgenstern and J. Von Neumann, Theory of games and economic behavior. Princeton university press, (1945).
[7] Z. L. Nagy, Generalizing Erdős, Moon and Moser's result - The number of $k$-dominating independent sets. Electronic Notes in Discrete Mathematics, proceedings of Eurocomb'17, 61 (2017) 909-915.
[8] Z. L. Nagy, On the Number of $k$-Dominating Independent Sets. Journal of Graph Theory, 84(4) (2017) 566-580.
[9] Problem Booklet of Workshop on Graph and Hypergraph Domination. https://renyi.hu/conferences/graphdom/dominationworkshopbooklet.pdf
[10] B. E. Sagan, A note on independent sets in trees. SIAM Journal on discrete mathematics, 1(1) (1988) 105-108.
[11] H. S. Wilf, The number of maximal independent sets in a tree. SIAM Journal on Algebraic Discrete Methods, 7(1) (1986) 125-130.
[12] A. Włoch, On 2-dominating kernels in graphs. Australas. J. Combin., 53 (2012) 273284.
[13] https://www.wolframcloud.com/

## Appendix

Proposition 14. Suppose $k \geq 3$ is fixed. Then the function

$$
f(\varepsilon)=(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / k}}
$$

is monotone decreasing in $\varepsilon$ for $\varepsilon \in[0, \infty)$.

Proof. As $f$ is differentiable, it is enough to prove that the derivative of $f$ is not positive.

$$
f^{\prime}(\varepsilon)=\left(e^{\ln (2+\varepsilon) \frac{1}{1+\varepsilon+\frac{T}{k}}}\right)^{\prime}=(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / k}}\left(\frac{1}{(2+\varepsilon)\left(1+\varepsilon+\frac{1}{k}\right)}-\frac{\ln (2+\varepsilon)}{\left(1+\varepsilon+\frac{1}{k}\right)^{2}}\right),
$$

so as $(2+\varepsilon)^{\frac{1}{1+\varepsilon+1 / k}} \geq 0$, it is enough to prove that

$$
\frac{1}{(2+\varepsilon)\left(1+\varepsilon+\frac{1}{k}\right)}-\frac{\ln (2+\varepsilon)}{\left(1+\varepsilon+\frac{1}{k}\right)^{2}} \leq 0
$$

Simplifying (and using that $1+\varepsilon+\frac{1}{k} \geq 0$ and $2+\varepsilon \geq 0$ ), we get

$$
1+\varepsilon+\frac{1}{k} \leq(2+\varepsilon) \ln (2+\varepsilon)
$$

it is easy to check that for $\varepsilon=0$ the above inequality holds as $k \geq 3$. Now note that the derivative of the right hand side with respect to $\varepsilon$, namely $1+\ln (2+\varepsilon)$, is larger than the derivative of the left hand side, namely 1 . Therefore the above inequality holds for all $\varepsilon \geq 0$, and we are done.

