

# On the maximum size of connected hypergraphs without a path of given length

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## Abstract

In this note we asymptotically determine the maximum number of hyperedges possible in an  $r$ -uniform, connected  $n$ -vertex hypergraph without a Berge path of length  $k$ , as  $n$  and  $k$  tend to infinity. We show that, unlike in the graph case, the multiplicative constant is smaller with the assumption of connectivity.

## 1 Introduction

Let  $P_k$  denote a path consisting of  $k$  edges in a graph  $G$ . There are several notions of paths in hypergraphs the most basic of which is due to Berge. A Berge path of length  $k$  is a set of  $k + 1$  distinct vertices  $v_1, v_2, \dots, v_{k+1}$  and  $k$  distinct hyperedges  $h_1, h_2, \dots, h_k$  such that for  $1 \leq i \leq k$ ,  $v_i, v_{i+1} \in h_i$ . A Berge path is also denoted simply as  $P_k$ , and the vertices  $v_i$  are called basic vertices. If  $v_1 = v$  and  $v_{k+1} = w$ , then we call the Berge path a Berge  $v$ - $w$ -path. A hypergraph  $\mathcal{H}$  is called connected if for any  $v \in V(\mathcal{H})$  and  $w \in V(\mathcal{H})$  there is a Berge  $v$ - $w$ -path. Let  $N_s(G)$  denote the number of  $s$ -vertex cliques in the graph  $G$ .

A classical result of Erdős and Gallai [3] asserts that

**Theorem 1** (Erdős-Gallai). *Let  $G$  be a graph on  $n$  vertices not containing  $P_k$  as a subgraph, then*

$$|E(G)| \leq \frac{(k-1)n}{2}.$$

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In fact, Erdős and Gallai deduced this result as a corollary of the following stronger result about cycles,

**Theorem 2** (Erdős-Gallai). *Let  $G$  be a graph on  $n$  vertices with no cycle of length at least  $k$ , then*

$$|E(G)| \leq \frac{(k-1)(n-1)}{2}.$$

Kopylov [5] and later Balister, Győri, Lehel and Schelp [1] determined the maximum number of edges possible in a connected  $P_k$ -free graph.

**Theorem 3.** *Let  $G$  be a connected  $n$ -vertex graph with no  $P_k$ ,  $n > k \geq 3$ . Then  $|E(G)|$  is bounded above by*

$$\max\left\{\binom{k-1}{2} + n - k + 1, \left(\left\lceil \frac{k+1}{2} \right\rceil\right) + \left\lfloor \frac{k-1}{2} \right\rfloor \left(n - \left\lceil \frac{k+1}{2} \right\rceil\right)\right\}.$$

Observe that, although the upper bound is lower in the connected case, it is nonetheless the same asymptotically. Balister, Győri, Lehel and Schelp also determined the extremal cases.

**Definition 1.** *The graph  $H_{n,k,a}$  consists of 3 disjoint vertex sets  $A, B, C$  with  $|A| = a$ ,  $|B| = n - k + a$  and  $|C| = k - 2a$ .  $H_{n,k,a}$  contains all edges in  $A \cup C$  and all edges between  $A$  and  $B$ .  $B$  is taken to be an independent set. The number of  $s$ -cliques in this graph is*

$$f_s(n, k, a) = \binom{k-a}{s} + (n-k+a) \binom{a}{s-1}.$$

The upper bound of Theorem 3 is attained for the graph  $H_{n,k,1}$  or  $H_{n,k, \lfloor \frac{k-1}{2} \rfloor}$ .

We now mention some recent results of Luo [6] which will be essential in our proof.

**Theorem 4** (Luo). *Let  $n-1 \geq k \geq 4$ . Let  $G$  be a connected  $n$ -vertex graph with no  $P_k$ , then the number of  $s$ -cliques in  $G$  is at most*

$$\max\{f_s(n, k, \lfloor (k-1)/2 \rfloor), f_s(n, k, 1)\}.$$

As a corollary, she also showed

**Corollary 1** (Luo). *Let  $n \geq k \geq 3$ . Assume that  $G$  is an  $n$ -vertex graph with no cycle of length  $k$  or more, then*

$$N_s(G) \leq \frac{n-1}{k-2} \binom{k-1}{s}.$$

Győri, Katona and Lemons [4] initiated the study of Berge  $P_k$ -free hypergraphs. They proved

**Theorem 5** (Győri-Katona-Lemons). *Let  $\mathcal{H}$  be an  $r$ -uniform hypergraph with no Berge path of length  $k$ . If  $k > r + 1 > 3$ , we have*

$$|E(\mathcal{H})| \leq \frac{n}{k} \binom{k}{r}.$$

If  $r \geq k > 2$ , we have

$$|E(\mathcal{H})| \leq \frac{n(k-1)}{r+1}.$$

The case when  $k = r + 1$  was settled later [2]:

**Theorem 6** (Davoodi-Győri-Methuku-Tompkins). *Let  $\mathcal{H}$  be an  $n$ -vertex  $r$ -uniform hypergraph. If  $|E(\mathcal{H})| > n$ , then  $\mathcal{H}$  contains a Berge path of length at least  $r + 1$ .*

Our main result is the asymptotic upper bound for the connected version of Theorem 5, as  $n$  and  $k$  tend to infinity.

**Theorem 7.** *Let  $\mathcal{H}_{n,k}$  be a largest  $r$ -uniform connected  $n$ -vertex hypergraph with no Berge path of length  $k$ , then*

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{|E(\mathcal{H}_{n,k})|}{k^{r-1}n} = \frac{1}{2^{r-1}(r-1)!}.$$

A construction yielding the bound in Theorem 7 is given by partitioning an  $n$ -vertex set into two classes  $A$ , of size  $\lfloor \frac{k-1}{2} \rfloor$ , and  $B$ , of size  $n - \lfloor \frac{k-1}{2} \rfloor$  and taking  $X \cup \{y\}$  as a hyperedge for every  $(r-1)$ -element subset  $X$  of  $A$  and every element  $y \in B$ . This hypergraph has no Berge  $P_k$  as we could have at most  $\lfloor \frac{k-1}{2} \rfloor$  basic vertices in  $A$  and  $\lfloor \frac{k-1}{2} \rfloor + 1$  basic vertices in  $B$ , thus yielding less than the required  $k + 1$  basic vertices.

Observe that in Theorem 5 the corresponding limiting value of the constant factor is  $\frac{1}{r!}$  which is  $\frac{2^{r-1}}{r}$  times larger than in the connected case. Note that the ideas of the proof of Theorem 7 can be used to prove that the limiting value of the constant factor in Theorem 5 is  $\frac{1}{r!}$ .

## 2 Proof of Theorem 7

We will use the following simple corollary of Theorem 4.

**Corollary 2.** *Let  $G$  be a connected graph on  $n$  vertices with no  $P_k$ , then  $G$  has at most*

$$\frac{k^{r-1}n}{2^{r-1}(r-1)!}$$

$r$ -cliques if  $n \geq c_{k,r}$  for some constant  $c_{k,r}$  depending only on  $k$  and  $r$ .

*Proof.* From Theorem 4, it follows that for large enough  $n$ , the number of  $r$ -cliques is at most

$$\binom{n - \lfloor \frac{k-1}{2} \rfloor}{r-1} + \binom{\lfloor \frac{k-1}{2} \rfloor}{r-1} + \binom{\lfloor \frac{k-1}{2} \rfloor}{r-2} < n \binom{\frac{k}{2}}{r-1}. \quad \square$$

Given an  $r$ -uniform hypergraph  $\mathcal{H}$  we define the shadow graph of  $\mathcal{H}$ , denoted  $\partial\mathcal{H}$  to be the graph on the same vertex set with edge set:

$$E(\partial\mathcal{H}) := \{\{x, y\} : \{x, y\} \subset e \in E(\mathcal{H})\}.$$

**Definition 2.** If  $r = 3$ , then we call an edge  $e \in E(\partial\mathcal{H})$  fat if there are at least 2 distinct hyperedges  $h_1, h_2$  with  $e \subset h_1, h_2$ . If  $r > 3$ , then we call an edge  $e \in E(\partial\mathcal{H})$  fat if there are at least  $k$  distinct hyperedges  $h_1, h_2, \dots, h_k$  in  $\mathcal{H}$  with  $e \subset h_i$  for  $1 \leq i \leq k$ .

We call an edge  $e \in E(\partial\mathcal{H})$  thin if it is not fat.

Thus, the set  $E(\partial\mathcal{H})$  decomposes into the set of fat edges and the set of thin edges. We will refer to the graph whose edges consist of all fat edges in  $\partial\mathcal{H}$  as the *fat graph* and denote it by  $F$ .

**Lemma 1.** *There is no  $P_k$  in the fat graph  $F$  of the hypergraph  $\mathcal{H}$ .*

*Proof.* Suppose we have such a  $P_k$  with edges  $e_1, e_2, \dots, e_k$ . For  $r = 3$ , if a hyperedge contains two edges from the path, then it must contain consecutive edges  $e_i, e_{i+1}$ . Select hyperedges  $h_1, h_2, \dots, h_k$  where  $e_i \subset h_i$  in such a way that  $h_{i+1}$  is different from  $h_i$  for all  $1 \leq i \leq k-1$ , and these edges yield the required Berge path.

Suppose now that  $r > 3$ , we will find a Berge path of length  $k$  in  $\mathcal{H}$ , greedily. For  $e_1$ , select an arbitrary hyperedge  $h_1$  containing it. Suppose we have found a distinct hyperedge  $h_i$  containing the fat edge  $e_i$  for all  $1 \leq i < i^*$ . Since the edge  $e_{i^*}$  is fat, there are at least  $k$  different hyperedges  $h_{i^*}^1, h_{i^*}^2, \dots, h_{i^*}^k$  containing it. Select one of them, say  $h_{i^*}^j$ , which is not equal to any of  $h_1, h_2, \dots, h_{i^*-1}$ . Thus, we may find distinct hyperedges  $h_1, h_2, \dots, h_k$  where  $e_i \subset h_i$  for  $1 \leq i \leq k$ , and thus, we have a Berge path of length  $k$ .  $\square$

We call a hyperedge  $h \in E(\mathcal{H})$  fat if  $h$  contains no thin edge. Let  $\mathcal{F}$  denote the hypergraph on the same set of vertices as  $\mathcal{H}$  consisting of the fat hyperedges, then

**Lemma 2.** *If  $r = 3$ , then*

$$|E(\mathcal{H} \setminus \mathcal{F})| \leq \frac{(k-1)n}{2}.$$

*If  $r > 3$ , then*

$$|E(\mathcal{H} \setminus \mathcal{F})| \leq \frac{(k-1)^2 n}{2}.$$

*Proof.* Arbitrarily select a thin edge from each  $h \in \mathcal{H} \setminus \mathcal{F}$ . Let  $G$  be the graph consisting of the selected thin edges. We know that each edge in  $G$  was selected at most once if  $r = 3$  and at most  $k-1$  times in the  $r > 3$ . Thus, we have that  $|\mathcal{H} \setminus \mathcal{F}| \leq |E(G)|$  for  $r = 3$  and  $|\mathcal{H} \setminus \mathcal{F}| \leq (k-1)|E(G)|$  for  $r > 3$ . Moreover,  $G$  is  $P_k$ -free since a  $P_k$  in  $G$  would imply a Berge  $P_k$  in  $\mathcal{H}$  by considering any hyperedge from which each edge was selected. It follows by Theorem 1 that  $|E(G)| \leq \frac{(k-1)n}{2}$ , so  $|\mathcal{H} \setminus \mathcal{F}| \leq \frac{(k-1)n}{2}$  if  $r = 3$ , and  $|\mathcal{H} \setminus \mathcal{F}| \leq \frac{(k-1)^2 n}{2}$  if  $r > 3$ .  $\square$

Any hyperedge of  $\mathcal{F}$  contains only fat edges, so it corresponds to a unique  $r$ -clique in  $F$ . This implies the following.

**Observation 1.** *The number of hyperedges in  $E(\mathcal{F})$  is at most the number of  $r$ -cliques in the fat graph  $F$ .*

To this end we will upper bound the number of  $r$ -cliques in  $F$ , by making use of the following important lemma.

**Lemma 3.** *There are no two disjoint cycles of length at least  $k/2 + 1$  in the fat graph  $F$ .*

*Proof.* Let  $C$  and  $D$  be two such cycles. By connectivity, there are vertices  $v \in V(C)$  and  $w \in V(D)$  and a Berge path from  $v$  to  $w$  in  $\mathcal{H}$  containing no additional vertices of  $C$  or  $D$  as defining vertices. This path can be extended using the hyperedges containing the edges of  $C$  and  $D$  to produce a Berge path of length  $k$  in  $\mathcal{H}$  (note that here we used that the edges of  $C$  and  $D$  are fat), a contradiction.  $\square$

Assume that  $F$  has connected components  $C_1, C_2, \dots, C_t$ . Trivially,

$$N_r(F) = \sum_{i=1}^t N_r(C_i). \quad (1)$$

If  $|V(C_i)| \leq k/2$ , then trivially

$$N_r(C_i) \leq \binom{|V(C_i)|}{r} \leq \frac{|V(C_i)|^r}{r!} \leq \frac{k^{r-1} |V(C_i)|}{2^{r-1}(r-1)!}.$$

So we can assume  $|V(C_i)| \geq k/2$ . By Lemma 3, we have that for all but at most one  $i$ ,  $C_i$  does not contain a cycle of length at least  $k/2 + 1$ . So by Corollary 1, for all but at most one  $i$ , say  $i_0$ , we have

$$N_r(C_i) \leq \frac{|V(C_i)| - 1}{k/2 - 2} \binom{k/2 - 1}{r} \leq \frac{k^{r-1} |V(C_i)|}{2^{r-1}(r-1)!} + O(k^{r-2}).$$

If  $|V(C_{i_0})| \geq c_{k,r}$ , then by Lemma 1 and by Corollary 2 we have

$$N_r(C_{i_0}) \leq \frac{k^{r-1} |V(C_{i_0})|}{2^{r-1}(r-1)!}.$$

Otherwise,  $N_r(C_{i_0}) \leq \binom{|V(C_{i_0})|}{r} = o(n)$ . Therefore, by (1), we have

$$\begin{aligned} N_r(F) &= \sum_{i=1}^t N_r(C_i) \leq \\ &\leq \sum_{i=1}^t \left( \frac{k^{r-1} |V(C_i)|}{2^{r-1}(r-1)!} + O(k^{r-2}) \right) + o(n) \leq \frac{k^{r-1} n}{2^{r-1}(r-1)!} + O(k^{r-2})n + o(n). \end{aligned}$$

Therefore, by Observation 1,

$$|E(\mathcal{F})| \leq N_r(F) \leq \frac{k^{r-1}n}{2^{r-1}(r-1)!} + O(k^{r-2})n + o(n). \quad (2)$$

Since  $|E(\mathcal{H})| = |E(\mathcal{H} \setminus \mathcal{F})| + |E(\mathcal{F})|$ , adding up the upper bounds in (2) and Lemma 2, we obtain the desired upper bound on  $|E(\mathcal{H})|$ . □

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