SMALL UNIONS OF AFFINE SUBSPACES AND SKELETONS VIA BAIRE CATEGORY

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ABSTRACT. Our aim is to find the minimal Hausdorff dimension of the union of scaled and/or rotated copies of the k-skeleton of a fixed polytope centered at the points of a given set. For many of these problems, we show that a typical arrangement in the sense of Baire category gives minimal Hausdorff dimension. In particular, this proves a conjecture of R. Thornton.

Our results also show that Nikodym sets are typical among all sets which contain, for every $x \in \mathbb{R}^n$, a punctured hyperplane $H \setminus \{x\}$ through x. With similar methods we also construct a Borel subset of \mathbb{R}^n of Lebesgue measure zero containing a hyperplane at every positive distance from every point.

1. INTRODUCTION

E. Stein [19] proved in 1976 that for any $n \geq 3$, if a set $A \subset \mathbb{R}^n$ contains a sphere centered at each point of a set $C \subset \mathbb{R}^n$ of positive Lebesgue measure, then A also has positive Lebesgue measure. It was shown by Mitsis [18] that the same holds if we only assume that C is a Borel subset of \mathbb{R}^n of Hausdorff dimension greater than 1. The analogous results are also true in the case n = 2; this was proved independently by Bourgain [2] and Marstrand [15] for circles centered at the points of an arbitrary set $C \subset \mathbb{R}^2$ of positive Lebesgue measure, and by Wolff [21] for $C \subset \mathbb{R}^2$ of Hausdorff dimension greater than 1. In fact, Bourgain proved a stronger result, which extends to other curves with non-zero curvature.

Inspired by these results, the authors in [12] studied what happens if the circles are replaced by axis-parallel squares. They constructed a closed set A of Hausdorff dimension 1 that contains the boundary of an axis-parallel square centered at each point in \mathbb{R}^2 (see [12, Theorem 1.1]). Thornton studied in [20] the higher dimensional versions: the problem when $0 \leq k < n$ and $A \subset \mathbb{R}^n$ contains the k-skeleton of an n-dimensional axis-parallel cube centered at every point of a compact set of given dimension d for some fixed $d \in [0, n]$. (Recall that the k-skeleton of a polytope is the union of its k-dimensional faces.) He found the smallest possible dimension of such a compact A in the cases when we consider box dimension and packing dimension. He conjectured that the smallest possible Hausdorff dimension of A is $\max(d - 1, k)$, which would be the generalization of [12, Theorem 1.4], which addresses the case n = 2, k = 0.

In this paper we prove Thornton's conjecture not only for cubes but for general polytopes of \mathbb{R}^n . It turns out that it plays an important role whether 0 is contained in one of the k-dimensional affine subspaces defined by the k-skeleton of the polytope (see Theorem 2.1). This is even more true if instead of just scaling, we also allow rotations. In this case, we ask what the minimal Hausdorff dimension of a set is that contains a scaled and rotated copy of the k-skeleton of a given polytope centered

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at each point of C. Obviously, it must have dimension at least k if C is nonempty. It turns out that this is sharp: we show that there is a Borel set of dimension k that contains a scaled and rotated copy of the k-skeleton of a polytope centered at each point of \mathbb{R}^n , provided that 0 is not in any of the k-dimensional affine subspaces defined by the k-skeleton. On the other hand, if 0 belongs to one of these affine subspaces, then the problem becomes much harder (see Remark 3.3).

As mentioned above at the end of the second paragraph, a (very) special case of Theorem 2.1, namely, when n = 2 and S consists of the 4 vertices of a square centered at the origin, was already proved in [12]. Our proof of Theorem 2.1 is much simpler than the proof in [12]. In fact, in all our results mentioned above, we will show that, in the sense of Baire category, the minimal dimension is attained by residually many sets. As it often happens, it is much easier to show that some properties hold for residually many sets than to try to construct a set for which they hold. In our case, after proving residuality for k-dimensional affine subspaces, we automatically obtain residuality for countable unions of k-dimensional subsets of k-dimensional affine subspaces, hence k-skeletons.

If we allow rotations but do not allow scaling, the question becomes: what is the minimal Hausdorff dimension of a set that contains a rotated copy of the k-skeleton of a given polytope centered at each point of C? We do not know the answer to this question for a general compact set C. However, as the following simple example shows, it is no longer true that a typical construction has minimal dimension.

Let $C \subset \mathbb{R}^2$ denote the unit circle centered at 0, and let the "polytope" be a single point of C. Then $\{0\}$ is a set of dimension 0 that contains, centered at each point of C, a rotated copy of our "polytope". (That is, it contains a point at distance 1 from each point of C.) On the other hand, it is easy to show that, if A contains a *nonzero* point at distance 1 from each point of C, then A has dimension at least 1. In particular, a "typical" A has dimension 1 and not 0. The same example also shows that the minimal dimension can be different depending on whether the "polytope" consists of one point or two points.

However, we will show that a typical construction does have minimal dimension, provided that C has full dimension, i.e., dim C = n for $C \subset \mathbb{R}^n$. In this case, the minimal (as well as typical) dimension of a set A that contains a rotated copy of the k-skeleton of a polytope centered at each point of C is k + 1. Somewhat surprisingly, we obtain that the smallest possible dimension (and also the typical dimension) is still k + 1 if we want the k-skeleton of a rotated copy of the polytope of every size centered at every point.

Let us state our results more precisely. Throughout this paper, by a scaled copy of a fixed set $S \subset \mathbb{R}^n$ we mean a set of the form $x + rS = \{x + rs : s \in S\}$, where $x \in \mathbb{R}^n$ and r > 0. We say that x + rS is a scaled copy of S centered at x. (That is, the center of S is assumed to be the origin.) Similarly, a rotated copy of S centered at $x \in \mathbb{R}^n$ is $x + T(S) = \{x + T(s) : s \in S\}$, where $T \in SO(n)$. Combining these two, we define a scaled and rotated copy of S centered at $x \in \mathbb{R}^n$ by $x + rT(S) = \{x + rT(s) : s \in S\}$, where r > 0 and $T \in SO(n)$.

In this paper we will consider only Hausdorff dimension, and we will denote by dim E the Hausdorff dimension of a set E. We list here the special cases of our results when the polytope is a cube and the set of centers is \mathbb{R}^n . (The first statement was already proved in [20].)

Corollary 1.1. For any integers $0 \le k < n$, the minimal dimension of a Borel set $A \subset \mathbb{R}^n$ that contains the k-skeleton of

- (1) a scaled copy of a cube centered at every point of \mathbb{R}^n is n-1;
- (2) a scaled and rotated copy of a cube centered at every point of \mathbb{R}^n is k;

- (3) a rotated copy of a cube centered at every point of \mathbb{R}^n is k+1;
- (4) a rotated cube of every size centered at every point of \mathbb{R}^n is k+1.

In fact, the same results hold if the k-skeleton of a cube is replaced by any $S \subset \mathbb{R}^n$ with dim S = k that can be covered by a countable union of k-dimensional affine subspaces that do not contain 0.

For k = n - 1 it is natural to ask if, in addition to dimension k + 1 = n, we can also guarantee positive Lebesgue measure in the settings (3) and (4). As we will see, we cannot guarantee positive measure. We show that there are residually many *Nikodym sets*, i.e., sets of measure zero which contain a punctured hyperplane through every point. The existence of Nikodym sets in \mathbb{R}^n for every $n \ge 2$ was proved by Falconer [6]. We also obtain residually many sets of measure zero which contain a hyperplane at every positive distance from every point. By combining our these two results, we get the following.

Corollary 1.2. Let $S \subset \mathbb{R}^n$ $(n \geq 2)$ be a set that can be covered by countably many hyperplanes and suppose that $0 \notin S$. Then there exists a set of Lebesgue measure zero that contains a scaled and rotated copy of S of every scale centered at every point of \mathbb{R}^n .

Note that here we need only the assumption $0 \notin S$ (which clearly cannot be dropped), while in Corollary 1.1 we needed the stronger assumption that the covering affine subspaces do not contain 0. Also, Corollary 1.2 is clearly false for n = 1.

One can ask what happens for those sets S to which neither the classical results nor our results can be applied. One of the simplest such case is when, say, n = 1 and S = C - 1/2, where C is the classical triadic Cantor set in the interval [0, 1]. We do not know how large a set A can be that contains a scaled copy of S centered at each $x \in \mathbb{R}$. Does it always have positive Lebesgue measure, or Hausdorff dimension at least 1? In [14] Laba and Pramanik construct random Cantor sets for which such a set must have positive Lebesgue measure, and by the result of Máthé [16], there exist Cantor sets for which such a set A can have zero measure. Hochman [10] and Bourgain [3] prove that for any porous Cantor set C with dim C > 0, such a set Amust have Hausdorff dimension strictly larger than dim C and at least 1/2.

Finally we remark that T. W. Körner [13] observed in 2003 that small Kakeyatype sets can be constructed using Baire category argument. He proved that if we consider the Hausdorff metric on the space of all compact sets that contain line segments in every possible direction between two fixed parallel line segments, then in this space, residually many sets have zero Lebesgue measure. As we will see, in our results we obtain residually many sets in a different type of metric space: we consider Hausdorff metric in a "code space".

2. Scaled copies

In this section we consider only scaled (not rotated) copies of S. We will prove the following theorem:

Theorem 2.1. Let S be the k-skeleton of an arbitrary polytope in \mathbb{R}^n for some $0 \le k < n$, and let $d \in [0, n]$ be arbitrary.

(i) Suppose that 0 is not contained in any of the k-dimensional affine subspaces defined by S. Then the smallest possible dimension of a compact set A that contains a scaled copy of S centered at each point of some d-dimensional compact set C is $\max(d-1,k)$.

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 (ii) Suppose that 0 is contained in at least one of the k-dimensional affine subspaces defined by S. Then the smallest possible dimension of a compact set A that contains a scaled copy of S centered at each point of some ddimensional compact set C is max(d, k).

Thornton's conjecture mentioned in the introduction is clearly a special case of part (i) of this theorem.

In fact, our main goal is to study a slightly different problem, from which we can deduce the results above. Our aim is to find for a given "skeleton" S and for a given nonempty compact set of centers C (instead of a given S and a given dim C) the smallest possible value of dim A, where A contains a scaled copy of S centered at each point of C.

We will study the case when S is the k-skeleton of a polytope, or more generally, the case when S is a countable union $S = \bigcup S_i$, where each S_i is contained in an affine subspace V_i . We will assume that C is compact and nonempty. Our aim is to show that, in the sense of Baire category, a typical set A that contains a scaled copy of S centered at each point of C has minimal dimension.

Let us make this more precise. Fix a nonempty compact set $C \subset \mathbb{R}^n$ and a non-degenerate closed interval $I \subset (0, \infty)$. In what follows, we view $C \times I$ as a parametrization of the space of certain scaled copies of a given set $S \subset \mathbb{R}^n$; in particular, $(x, r) \in C \times I$ corresponds to the copy centered at x and scaled by r. Let \mathcal{K} denote the space of all compact sets $K \subset C \times I$ that have full projection onto C. (That is, for each $x \in C$ there is an $r \in I$ with $(x, r) \in K$.) We equip \mathcal{K} with the Hausdorff metric. Clearly, \mathcal{K} is a closed subset of the space of all compact subsets of $C \times I$, and hence it is a complete metric space. In particular, the Baire category theorem holds for \mathcal{K} , so we can speak about a typical $K \in \mathcal{K}$ in the Baire category sense: a property P holds for a typical $K \in \mathcal{K}$ if $\{K \in \mathcal{K} : P$ holds for $K\}$ is residual in \mathcal{K} , or equivalently, if there exists a dense G_{δ} set $\mathcal{G} \subset \mathcal{K}$ such that the property holds for every $K \in \mathcal{G}$.

Let A be an arbitrary set that contains a scaled copy of $S \subset \mathbb{R}^n$ centered at each point of C. First we show an easy lower estimate on dim A, which in some important cases will turn out to be sharp. Let C' denote the orthogonal projection of C onto $W := \operatorname{span}\{S\}^{\perp}$. (As usual, we denote by $\operatorname{span}\{S\}$ the linear span of S, so it always contains the origin.) For every point $x' \in C'$ there exists an $x \in C$ such that the projection of x onto W is x', and there exists an r > 0 such that $x+rS \subset A$ and hence $x + rS \subset (x' + \operatorname{span}\{S\}) \cap A$. Since for any $x' \in C' \subset W = \operatorname{span}\{S\}^{\perp}$ the set $(x' + \operatorname{span}\{S\}) \cap A$ contains a scaled copy of S, we obtain by the general Fubini type inequality (see e.g. in [5] or [7])

(2.1)
$$\dim A \ge \dim C' + \dim S.$$

Now let $K \in \mathcal{K}$ and $S \subset \mathbb{R}^n$ and consider

(2.2)
$$A = A_{K,S} := \bigcup_{(x,r) \in K} x + rS.$$

Note that $A_{K,S}$ contains a scaled copy of S centered at each point of C, so by the previous paragraph,

(2.3)
$$\dim A_{K,S} \ge \dim C' + \dim S.$$

The following lemma shows that for a typical $K \in \mathcal{K}$ we have equality in (2.3) if S is an affine subspace.

Lemma 2.2. Let V be an affine subspace of \mathbb{R}^n , let $\emptyset \neq C \subset \mathbb{R}^n$ be compact, and let C' denote the projection of C onto span $\{V\}^{\perp}$. Then for a typical $K \in \mathcal{K}$, and

for $A_{K,V}$ defined by (2.2),

$$\dim A_{K,V} = \dim C' + \dim V.$$

We postpone the proof of this lemma and first study some of its corollaries. Suppose that S is a countable union $S = \bigcup S_i$, where each S_i is a subset of an affine subspace V_i . Let C'_i denote the orthogonal projection of C onto $W_i := \operatorname{span}\{V_i\}^{\perp}$. Since a countable intersection of residual sets is residual, and since the Hausdorff dimension of a countable union of sets is the supremum of the Hausdorff dimension of the individual sets, it follows that for a typical $K \in \mathcal{K}$,

$$\dim A_{K,S} = \dim \left(\bigcup_{i} A_{K,S_i} \right) \le \sup_{i} (\dim C'_i + \dim V_i).$$

On the other hand, if A contains a scaled copy of $S = \bigcup_i S_i$ centered at each $x \in C$, then applying (2.1) to each S_i , we get dim $A \ge \dim C'_i + \dim S_i$ for each i and thus dim $A \ge \sup_i (\dim C'_i + \dim S_i)$. Therefore, we obtain the following theorem:

Theorem 2.3. Let C be an arbitrary nonempty compact subset in \mathbb{R}^n , and let $S = \bigcup_{i=1}^{\infty} S_i$, where each S_i is a subset of an affine subspace V_i . Let C'_i denote the orthogonal projection of C onto span $\{V_i\}^{\perp}$. Then:

 (i) For every set A that contains a scaled copy of S centered at each point of C,

$$\dim A \ge \sup(\dim C'_i + \dim S_i).$$

(ii) For a typical $K \in \mathcal{K}$, the set $A = A_{K,S}$ defined by (2.2) contains a scaled copy of S centered at each point of C and

$$\dim A \le \sup(\dim C'_i + \dim V_i).$$

Furthermore, if S is compact then so is A.

Let $W_i = \operatorname{span}\{V_i\}^{\perp}$. Note that if $0 \notin V_i$ then $\dim W_i = n - \dim V_i - 1$. Therefore if $\dim C = n, k < n, \dim S = k$, and for every *i* we have $0 \notin V_i$ and $\dim V_i = k$, then $\sup_i \dim S_i = k$ and $\dim C'_i = n - k - 1$ for every *i*, so Theorem 2.3 gives $\dim A = n - 1$, which proves the general version of (1) of Corollary 1.1.

So far we studied the problem of finding the minimal Hausdorff dimension of a set A that contains a copy of a given set S centered at each point of a given set C. Now we turn to the problem when, instead of S and C, we are only given S and $d = \dim C$. We suppose that $\dim S_i = \dim V_i$ for each i, so the lower and upper estimates in (i) and (ii) agree.

Since clearly dim $C'_i \ge \max(0, \dim C - \operatorname{codim} W_i)$, where codim W_i denotes the co-dimension of the linear space W_i , therefore Theorem 2.3(i) gives

$$\dim A \ge \sup(\max(0, d - \operatorname{codim} W_i) + \dim S_i).$$

In order to show that this estimate is sharp when dim $S_i = \dim V_i$, by Theorem 2.3(ii), it is enough to find a compact set $C \subset \mathbb{R}^n$ for which dim $C'_i = \max(0, \dim C - \operatorname{codim} W_i)$ holds for each *i*. This can be done by the following claim, which we will prove later.

Claim 2.4. For each $i \in \mathbb{N}$, let W_i be a linear subspace of \mathbb{R}^n of co-dimension $l_i \in \{0, 1, \ldots, n\}$. Then for every $d \in [0, n]$ there exists a d-dimensional compact set $C \subset \mathbb{R}^n$ whose projection onto W_i has dimension $\max(0, d - l_i)$ for each i.

Therefore Theorem 2.3 and Claim 2.4 give the following.

Corollary 2.5. Suppose that $S = \bigcup_{i=1}^{\infty} S_i$, where each S_i is a subset of an affine subspace V_i with dim $S_i = \dim V_i$. For each i, let $W_i = \operatorname{span}\{V_i\}^{\perp}$. Let $d \in [0, n]$ be arbitrary. Then the smallest possible dimension of a set A that contains a scaled copy of S centered at each point of some d-dimensional set C is $\sup_i (\max(0, d - \operatorname{codim} W_i) + \dim S_i)$.

Now we claim that Theorem 2.1 is a special case of Corollary 2.5. Indeed, if S is a k-skeleton of a polytope, then for each i we have dim $S_i = \dim V_i = k$, and W_i has co-dimension either k+1 if $0 \notin V_i$, or k if $0 \in V_i$. Thus max $(0, d-\operatorname{codim} W_i)+\dim S_i$ is either max(k, d-1) if $0 \notin V_i$, or max(k, d) if $0 \in V_i$.

It remains to prove Claim 2.4 and Lemma 2.2. The following simple proof is based on an argument that was communicated to us by K. J. Falconer.

Proof of Claim 2.4. We can clearly suppose that d > 0 and $l_i \in \{1, \ldots, n-1\}$. For $0 < s \le n$, Falconer [8] introduced \mathcal{G}_n^s as the class of those G_{δ} subsets $F \subset \mathbb{R}^n$ for which $\bigcap_{i=1}^{\infty} f_i(F)$ has Hausdorff dimension at least s for all sequences of similarity transformations $\{f_i\}_{i=1}^{\infty}$. Among other results, Falconer proved that \mathcal{G}_n^s is closed under countable intersection, and if $F_1 \in \mathcal{G}_n^s$ and $F_2 \in \mathcal{G}_m^t$ then $F_1 \times F_2 \in \mathcal{G}_{n+m}^{s+t}$. Examples of sets of \mathcal{G}_n^s with Hausdorff dimension exactly s are also shown in [8] for every $0 < s \le n$.

For l < d, let $E_l \in \mathcal{G}_{n-l}^{d-l}$ with dim $E_l = d - l$, and for $l \ge d$ let E_l be a dense G_{δ} subset of \mathbb{R}^{n-l} with dim $E_l = 0$. Let $F_l = E_l \times \mathbb{R}^l \subset \mathbb{R}^{n-l} \times \mathbb{R}^l$. Clearly, the projection of F_l onto \mathbb{R}^{n-l} has Hausdorff dimension max(0, d-l).

Now we show that $F_l \in \mathcal{G}_n^d$. This follows from the product rule mentioned above if l < d. In the case $l \ge d$, we need to prove that $\dim(\bigcap_{i=1}^{\infty} f_i(E_l \times \mathbb{R}^l)) \ge d$ for any sequence of similarity transformations $\{f_i\}_{i=1}^{\infty}$. Let V be an (n-l)-dimensional subspace of \mathbb{R}^n which is generic in the sense that it intersects all the countably many l-dimensional affine subspaces $f_i(\{0\} \times \mathbb{R}^l)$ in a single point. Then for each translate V + x of V, the set $f_i(E_l \times \mathbb{R}^l) \cap (V + x)$ is similar to the dense G_δ set E_l , hence $(\bigcap_{i=1}^{\infty} f_i(E_l \times \mathbb{R}^l)) \cap (V + x)$ is nonempty for each x, which implies that indeed $\dim(\bigcap_{i=1}^{\infty} f_i(E_l \times \mathbb{R}^l)) \ge l \ge d$.

For each *i*, let H_i be a rotated copy of F_{l_i} with projection of Hausdorff dimension $\max(0, d-l_i)$ onto W_i . Since each H_i is of class \mathcal{G}^d , the intersection $D := \bigcap_{i=1}^{\infty} H_i$ is of class \mathcal{G}^d . In particular, its Hausdorff dimension is at least *d*. It is also clear that the projection of *D* onto each W_i has Hausdorff dimension at most $\max(0, d-l_i)$.

Now D has all the required properties except that it might have Hausdorff dimension larger than d, and it is not compact but G_{δ} . If dim D > d, then let C be a compact subset of D with Hausdorff dimension d. Then for each i, the projection of C onto W_i is at most $\max(0, d - l_i)$, but it cannot be smaller since W_i has codimension l_i . If dim D = d then let D_j be compact subsets of D with dim $D_j \to d$ and let C be a disjoint union of shrunken converging copies of D_j and their limit point.

Proof of Lemma 2.2. By (2.3), it is enough to show that $\dim A_{K,V} \leq \dim C' + \dim V$ holds for a typical $K \in \mathcal{K}$. Write $V = v + V_0$ where V_0 is a k-dimensional linear subspace, $v \in \mathbb{R}^n$ and $v \perp V_0$. Without loss of generality we can assume that v = 0 or |v| = 1. Let x' denote the projection of a point x onto $\operatorname{span}\{V\}^{\perp}$, and let $\operatorname{proj} x \in \mathbb{R}$ denote the projection of x onto $\mathbb{R}v$. (Clearly, if v = 0, then $\operatorname{proj} x = 0$.)

Let \mathcal{K}^n denote the space of all nonempty compact subsets of \mathbb{R}^n , equipped with the Hausdorff metric. Then

$$A = A_{K,V} = \bigcup_{(x,r)\in K} x' + (\operatorname{proj} x + r)v + V_0,$$

SO

$$\dim A = \dim V_0 + \dim \left(\bigcup_{(x,r) \in K} x' + (\operatorname{proj} x + r)v \right) = k + \dim F(K),$$

where $F: \mathcal{K} \to \mathcal{K}^n$ is defined by

$$F(K) = \bigcup_{(x,r)\in K} x' + (\operatorname{proj} x + r)v.$$

It is easy to see that F is continuous.

Since for every open set $G \subset \mathbb{R}^n$ and for every compact set $K \subset G$ we have $\operatorname{dist}(\mathbb{R}^n \setminus G, K) > 0$, it follows that for any open set $G \subset \mathbb{R}^n$, $\{K \in \mathcal{K}^n : K \subset G\}$ is an open subset of \mathcal{K}^n . Consequently, for any $s, \delta, \varepsilon > 0$, the set of those compact sets $K \in \mathcal{K}^n$ that have an open cover $\bigcup G_i$ where $\sum_i (\operatorname{diam} G_i)^s < \varepsilon$ and $\operatorname{diam} G_i < \delta$ for each i is an open subset of \mathcal{K}^n . Therefore for any s > 0, $\{K \in \mathcal{K}^n : \dim K \leq s\}$ is a G_δ subset of \mathcal{K}^n . Since F is continuous, $\{K \in \mathcal{K} : \dim F(K) \leq s\}$ is a G_δ subset of \mathcal{K} .

We finish the proof by showing that $\{K \in \mathcal{K} : \dim F(K) \leq \dim C'\}$ is dense. To obtain this, for every compact set $L \in \mathcal{K}$ we construct another compact set $K \in \mathcal{K}$ arbitrary close to L, such that $\{\operatorname{proj} x + r : (x, r) \in K\}$ is finite and so F(K) is covered by a finite union of copies of C'. For a given $L \in \mathcal{K}$, such a $K \in \mathcal{K}$ can be constructed by choosing a sufficiently small $\varepsilon > 0$ and letting

$$K := \{ (x, r) : \exists r' \text{ s.t. } (x, r') \in L, \operatorname{proj} x + r \in \varepsilon \mathbb{Z}, |r - r'| \le \varepsilon \}. \quad \Box$$

3. Scaled and rotated copies

In this section, we study the problem when we are allowed to *scale and rotate* copies of S. That is, now our aim is to find for a given set $S \subset \mathbb{R}^n$ and a nonempty compact set of centers $C \subset \mathbb{R}^n$ the minimal possible value of dim A, where A contains a scaled and rotated copy of S centered at each point of C. (That is, for every $x \in C$, there exist r > 0 and $T \in SO(n)$ such that $x + rT(S) \subset A$.)

For a fixed nonempty compact set $C \subset \mathbb{R}^n$ and a closed interval $I \subset (0, \infty)$, let \mathcal{K}' denote the space of all compact sets $K \subset C \times I \times SO(n)$ that have full projection onto C. We fix a metric on SO(n) that induces the natural topology and equip \mathcal{K}' with the Hausdorff metric. Then \mathcal{K}' is also a complete metric space, so again we can talk about typical $K \in \mathcal{K}'$ in the Baire category sense. Now for $K \in \mathcal{K}'$ and $S \subset \mathbb{R}^n$, we let

(3.1)
$$A'_{K,S} := \bigcup_{(x,r,T)\in K} x + rT(S)$$

Note that $A'_{K,S}$ contains a scaled and rotated copy of S centered at each point of C.

Again, first we consider the case when S is an affine subspace, but we now exclude the case when S contains 0.

Lemma 3.1. Let V be an affine subspace of \mathbb{R}^n such that $0 \notin V$ and let $C \subset \mathbb{R}^n$ be an arbitrary nonempty compact set. Then for a typical $K \in \mathcal{K}'$, and for $A'_{K,V}$ defined by (3.1),

$$\dim A'_{K,V} = \dim V.$$

Proof. Clearly it is enough to show that $\dim A'_{K,V} \leq \dim V$ holds for a typical $K \in \mathcal{K}'$. For any $N \in \mathbb{N}$, we define $F'_N : \mathcal{K}' \to \mathcal{K}^n$ by $F'_N(K) = A'_{K,V} \cap [-N, N]^n$. It is easy to see that F'_N is continuous. Then exactly the same argument as in the proof of Lemma 2.2 gives that $\{K \in \mathcal{K}' : \dim F'_N(K) \leq s\}$ is a G_{δ} subset of \mathcal{K}' , which implies that $\{K \in \mathcal{K}' : \dim A'_{K,V} \leq s\}$ is also G_{δ} .

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So it remains to prove that $\{K \in \mathcal{K}' : \dim A'_{K,V} \leq \dim V\}$ is dense. Fix $\varepsilon > 0$. Then, by compactness and since $0 \notin V$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ and $(\dim V)$ -dimensional affine subspaces V_1, \ldots, V_N such that for any $(x, r, T) \in C \times I \times SO(n)$ there exists $(r', T') \in I \times SO(n)$ within ε distance of (r, T) such that $x + r'T'(V) = V_i$ for some $i \leq N$. Thus, given any compact set $L \in \mathcal{K}'$ and $\varepsilon > 0$, we can take

$$K = \{ (x, r', T') : x + r'T'(V) \in \{V_1, \dots, V_N\} \} \cap L_{\varepsilon},$$

where

$$L_{\varepsilon} = \{ (x, r', T') : \exists (r, T) \text{ s.t. } (x, r, T) \in L, \operatorname{dist}((r', T'), (r, T)) \leq \varepsilon \}.$$

It follows that $K \in \mathcal{K}'$ and the (Hausdorff) distance between K and L is at most ε . Furthermore, dim $A'_{K,V} = \dim V$, since $A'_{K,V}$ can be covered by finitely many (dim V)-dimensional affine spaces.

By taking a countable intersection of residual sets we obtain the following corollary of Lemma 3.1, which clearly implies the general form of (2) of Corollary 1.1.

Theorem 3.2. Let C be an arbitrary nonempty compact subset in \mathbb{R}^n , k < n and let $S \subset \mathbb{R}^n$ be a k-Hausdorff-dimensional set that can be covered by a countable union of k-dimensional affine subspaces that do not contain 0. Then for a typical $K \in \mathcal{K}'$, the set $A'_{K,S}$ contains a scaled and rotated copy of S centered at every point of C, and dim $A'_{K,S} = \dim S$.

Remark 3.3. If $0 \in V$ and V is k-dimensional then a scaled and rotated copy of V centered at x is a k-dimensional affine subspace that contains x. Therefore a set A that contains a scaled and rotated copy of V centered at every point of C is a set that contains a k-dimensional affine subspace through every point of C. The Lebesgue measure of such an A is clearly bounded below by the Lebesgue measure of C. By generalizing the planar result of Davies [4] to higher dimensions, Falconer [6] proved there is such an A which attains this lower bound. In Section 5 we show that the Lebesgue measure of a typical such A is in fact this minimum. On the other hand, to find the minimal dimension of such an A is closely related to the Kakeya problem, especially in the special case k = 1, and for some nontrivial C this problem is as hard as the Kakeya problem.

4. ROTATED COPIES: DIMENSION

Now we study what happens if we allow rotation but do not allow scaling. As we mentioned in the introduction, it is *not* true that for a general nonempty compact set of centers C, a typical construction has minimal dimension. However, we will show that this is true provided that C has full dimension.

The following lower estimate can be found in [9]:

Fact 4.1. Let $0 \leq k < n$ be integers, and let $S \subset \mathbb{R}^n$ be a k-Hausdorff-dimensional set that can be covered by a countable union of k-dimensional affine subspaces that do not contain 0. Let $\emptyset \neq C \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ be such that for every $x \in C$, there exists a rotated copy of S centered at x contained in A. Then dim $A \geq \max\{k, k + \dim C - (n-1)\}$.

In particular, if dim C = n then dim $A \ge k + 1$.

Remark 4.2. If instead of fixing C, we fix only the dimension d of C, and S can be covered by one k-dimensional affine subspace V, then the following simple examples show that the estimate in Fact 4.1 is sharp. Without loss of generality we can assume that V is at unit distance from 0. For $d \leq n-1$, we can take $A = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ and take C to be a d-dimensional subset of $\mathbb{R}^k \times S^{n-k-1}$, where S^m denotes the unit sphere in \mathbb{R}^{m+1} centered at 0. For d = n-1+s, where

 $s \in [0,1]$, let $E \subset \mathbb{R}^{n-k}$ be an s-dimensional subset of a line and let $F \subset \mathbb{R}^{n-k}$ be the set with a copy of S^{n-k-1} centered at every point of E. It is easy to show that dim F = n - k - 1 + s. Let $C = \mathbb{R}^k \times F$ and $A = \mathbb{R}^k \times E$. In both cases A contains a rotated copy of S centered at every point of C, dim C = d and dim $A = \max\{k, k + \dim C - (n-1)\}$.

If S can be covered by two distinct k-dimensional affine subspaces but cannot be covered by one, then this question becomes much more difficult. Consider, for example, the case when S consists of two points, both at distant 1 from 0, so now A contains two distinct points at distance 1 from every point of a 1-dimensional set $C \subset \mathbb{R}^2$. The discussion in the introduction implies that if we take $C = S^1$, then $\dim A \ge 1$. We do not know if there exists a set C with $\dim C = 1$ for which there is such a set A with $\dim A < 1$.

Our goal is to show that for every fixed C with dim C = n, the estimate dim $A \ge k+1$ in Fact 4.1 is always sharp. Moreover, we construct sets of Hausdorff dimension k+1 that contain the k-skeleton of an n-dimensional rotated polytope of every size centered at every point. More precisely, we want to construct a set A that contains a rotated copy of every positive size of a given set $S \subset \mathbb{R}^n$ centered at every point of a given nonempty compact set C. (That is, for every $x \in C$ and r > 0 there exists $T \in SO(n)$ such that $x + rT(S) \subset A$.) Instead of every $x \in C$ and r > 0 we will guarantee only every (x, r) from each fixed nonempty compact set $J \subset \mathbb{R}^n \times (0, \infty)$. By taking countable unions, we get the desired construction for every $(x, r) \in \mathbb{R}^n \times (0, \infty)$.

For a fixed nonempty compact set $J \subset \mathbb{R}^n \times (0, \infty)$, let \mathcal{K}'' denote the space of all compact sets $K \subset J \times SO(n)$ that have full projection onto J. Again, by taking a metric on SO(n) that induces the natural topology and equipping \mathcal{K}'' with the Hausdorff metric, \mathcal{K}'' is also a complete metric space, so again we can talk about typical $K \in \mathcal{K}''$ in the Baire category sense.

Now for any $K \in \mathcal{K}''$ and $S \subset \mathbb{R}^n$, the set

(4.1)
$$A_{K,S}'' := \bigcup_{(x,r,T)\in K} x + rT(S)$$

contains a rotated copy of S of scale r centered at x for every $(x, r) \in J$. Note that taking $J = C \times \{1\}$ gives us the special case when only rotation is used.

Again, we start with the case when S is a k-dimensional $(0 \le k < n)$ affine subspace of \mathbb{R}^n that does not contain the origin. Note that if $d = \operatorname{dist}(S, 0)$ then x + rT(S) is at distance rd from x. This motivates the following easy deterministic (k + 1)-dimensional construction.

Proposition 4.3. For any integers $0 \le k < n$ there exists a Borel set $B \subset \mathbb{R}^n$ of Hausdorff dimension k + 1 that contains a k-dimensional affine subspace at every positive distance from every point of \mathbb{R}^n .

Proof. Let W_1, W_2, \ldots be a countable collection of (k + 1)-dimensional affine subspaces of \mathbb{R}^n such that $B := \bigcup_i W_i$ is dense. Then B is clearly a Borel set $B \subset \mathbb{R}^n$ of Hausdorff dimension k + 1, so all we need to show is that for any fixed $x \in \mathbb{R}^n$ and r > 0 the set B contains a k-dimensional affine subspace at distance r from x. Choose i such that W_i intersects the interior of the ball B(x, r). Then the intersection of W_i and the sphere S(x, r) is a sphere in the (k + 1)-dimensional affine subspace W_i , and any k-dimensional affine subspace of $W_i \subset B$ that is tangent to this sphere is at distance r from x.

The proof of the following lemma is based on the same idea as in the construction above.

Lemma 4.4. Let $0 \le k < n$ be integers and V be a k-dimensional affine subspace of \mathbb{R}^n such that $0 \notin V$. Let $J \subset \mathbb{R}^n \times (0, \infty)$ be an arbitrary nonempty compact set. Then for a typical $K \in \mathcal{K}''$, and for $A''_{K,V}$ defined by (4.1),

$$\dim A_{K,V}'' \le k+1.$$

Proof. Without loss of generality we can assume that V is at distance 1 from the origin.

Let A(n, k + 1) be the space of all (k + 1)-dimensional affine subspaces of \mathbb{R}^n , equipped with a natural metric (for example the metric defined in [17, 3.16]), and let W_1, W_2, \ldots be a countable dense set in A(n, k + 1). Let $B = \bigcup_i W_i$.

Exactly the same argument as in the proof of Lemma 3.1 gives that $\{K \in \mathcal{K}'' : \dim A''_{K,V} \leq s\}$ is G_{δ} for any s, so again it remains to prove that $\{K \in \mathcal{K}'' : \dim A''_{K,V} \leq k+1\}$ is dense in \mathcal{K}'' . Since dim B = k+1, it is enough to show that $\{K \in \mathcal{K}'' : A''_{K,V} \subset B\}$ is dense in \mathcal{K}'' .

First we show that for any $(x, r, T) \in J \times SO(n)$ and $\varepsilon > 0$, there exist $i \in \mathbb{N}$ and $T' \in SO(n)$ such that $\operatorname{dist}(T, T') < \varepsilon$ and $x + rT'(V) \subset W_i$. We will also see from the proof that for the given $\varepsilon > 0$ and the above chosen i, there exists a neighborhood of (x, r, T) such that for any (x^*, r^*, T^*) from that neighborhood, there exists $T^{*'} \in SO(n)$ such that $\operatorname{dist}(T^*, T^{*'}) < \varepsilon$ and $x^* + r^*T^{*'}(V) \subset W_i$. Hence, by the compactness of $J \times SO(n)$, for a given $\varepsilon > 0$, there exists an N such that we can choose an $i \leq N$ for every $(x, r, T) \in J \times SO(n)$.

So fix $(x, r, T) \in J \times SO(n)$ and $\varepsilon > 0$. Let W be a (k + 1)-dimensional affine subspace of \mathbb{R}^n that contains V such that $0 < \operatorname{dist}(W, 0) < \operatorname{dist}(V, 0) = 1$. We denote by v be the point of V closest to the origin, and let $V_0 = x + rT(V)$, $v_0 = x + rT(v)$ and $W_0 = x + rT(W)$. Then $S_0 := W_0 \cap S(x, r)$ is a sphere in W_0 , and V_0 is the tangent of S_0 at the point v_0 . If W_i is sufficiently close to W_0 , then we can pick a point $v'_0 \in S'_0 := W_i \cap S(x, r)$ close to v_0 , and a k-dimensional affine subspace $V'_0 \subset W_i$ close to V_0 that is the tangent of S'_0 at v'_0 . Then V'_0 is at distance r from x and it is as close to $V_0 = x + rT(V)$ as we wish, so $V'_0 = x + rT'(V)$ for some $T' \in SO(n)$ and T' can be chosen arbitrarily close to T, which completes the proof of the claim of the previous paragraph.

Thus, for a given $L \in \mathcal{K}''$ and $\varepsilon > 0$, if we let

 $K = \{(x, r, T') : \exists i \leq N, \exists T \text{ s.t. } (x, r, T) \in L, \text{ dist}(T, T') \leq \varepsilon, \ x + rT'(V) \subset W_i\},$ then $K \in \mathcal{K}''$ and the Hausdorff distance between K and L is at most ε . Furthermore, $A''_{K,V} \subset \bigcup_{i=1}^{N} W_i \subset B$, which completes the proof. \Box

The same statements hold if, instead of S = V, we consider any subset $S \subset V$. By taking a countable intersection of residual sets we obtain the following.

Theorem 4.5. Let $0 \leq k < n$ be integers and let $S = \bigcup_{i=1}^{\infty} S_i$, where each S_i is a subset of a k-dimensional affine subspace V_i with $0 \notin V_i$. Let $J \subset \mathbb{R}^n \times (0, \infty)$ be an arbitrary nonempty compact set. Recall that \mathcal{K}'' denotes the space of all compact sets $K \subset J \times SO(n)$ that have full projection onto J.

Then for a typical $K \in \mathcal{K}''$, the set $A''_{K,S}$ defined by (4.1) is a closed set with $\dim A''_{K,S} \leq k+1$, and for every $(x,r) \in J$, there exists a $T \in SO(n)$ such that $x + rT(S) \subset A''_{K,S}$.

We can see from Fact 4.1 that the estimate k + 1 above is sharp, provided that $\dim S = k$ and $J \supset C \times \{r\}$ for some r > 0 and $C \subset \mathbb{R}^n$ with $\dim C = n$. This gives the general version of (3) and (4) of Corollary 1.1.

Remark 4.6. In Theorem 4.5 we obtain a rotated and scaled copy of S for every $(x, r) \in J$ inside a set of Hausdorff dimension k + 1. We claim that using a similar argument as in [11, Remark 1.6] we can also move S continuously inside

a set of Hausdorff dimension k + 1 so that during this motion we get S in every required position. Indeed, let K be a fixed (typical) element of \mathcal{K}'' guranteed by Theorem 4.5 such that $\dim A''_{K,S} \leq k + 1$. Since K is a nonempty compact subset of the metric space $\operatorname{conv}(J) \times SO(n)$, where conv denotes the convex hull, there exists a continuous function $g: C_{1/3} \to \operatorname{conv}(J) \times SO(n)$ on the classical Cantor set $C_{1/3}$ such that $g(C_{1/3}) = K$. All we need to do is to extend this map continuously to [0, 1] such that $\dim A''_{g([0,1]),S} \leq k + 1$. For each complementary interval (a, b)of the Cantor set, we define g on (a, b) in such a way that g is smooth on [a, b]and that the diameter of g([a, b]) is at most a constant multiple of the distance between g(a) and g(b). This gives the desired extension since the union of the sets of the form x + rT(S) $((x, r, T) \in g((a, b)))$ will be a countable union of smooth k + 1-dimensional manifolds, so $\dim A''_{g((a,b)),S} = k + 1$.

Note that if $J = C \times \{1\}$ then we get only congruent copies. So in particular, for any k < n, the k-skeleton of a unit cube can be continuously moved by rigid motions in \mathbb{R}^n within a set of Hausdorff dimension k + 1 in such a way that the center of the cube goes through every point of C, or by joining such motions, through every point of \mathbb{R}^n .

5. ROTATED COPIES: MEASURE

In this section, we study what happens when we place a rotated punctured hyperplane through every point. We show that typical arrangements of this kind have Lebesgue measure zero and are hence Nikodym sets. Using similar methods, we also show that typical arrangements of placing a rotated hyperplane at every positive distance from every point have measure zero. We use $|\cdot|$ to denote the Lebesgue measure.

Let $e_1 = (1, 0, ..., 0) \in \mathbb{R}^n$ and $H = \{(y_1, ..., y_n) \in \mathbb{R}^n : y_1 = 0\}$. By a rotated hyperplane at distance $r \in [0, \infty)$ from $x \in \mathbb{R}^n$, we mean a set of the form $x + rT(e_1) + T(H)$ for some $T \in SO(n)$. Note that we now allow r to be 0, and that $x + rT(e_1) + T(H)$ differs from $x + rT(e_1 + H)$ when r = 0.

Fix a nonempty compact set $J \subset \mathbb{R}^n \times [0, \infty)$. As in Section 4, we let \mathcal{K}'' denote the space of compact sets $K \subset J \times SO(n)$ that have full projection onto J.

In this section we prove the following result.

Theorem 5.1. For a typical $K \in \mathcal{K}''$, the set

$$\bigcup_{(x,r,T)\in K} (x+rT(e_1)+T(H))\setminus \{x\}$$

has measure zero.

Note that if r = 0, then $(x + rT(e_1) + T(H)) \setminus \{x\}$ is $x + T(H \setminus \{0\})$, so we are placing a rotated copy of the *punctured hyperplane* $H \setminus \{0\}$ through x. Thus, if we consider the case $J = C \times \{0\}$ for some compact set $C \subset \mathbb{R}^n$, we see that typical arrangements give rise to Nikodym sets. We also obtain our claim in Remark 3.3 that if we place an un-punctured hyperplane through every point in C, the typical arrangement of this kind has Lebesgue measure equal to |C|.

By taking countable unions of sets of the form in Theorem 5.1, we obtain the following:

Corollary 5.2. There is a set of measure zero in \mathbb{R}^n which contains a hyperplane at every positive distance from every point as well as a punctured hyperplane through every point.

5.1. Translating cones. In this section we introduce the main geometric construction for proving Theorem 5.1. This construction is done in \mathbb{R}^2 and we will later see how to apply it to the *n*-dimensional problem. Our geometric arguments are similar to those used to construct Kakeya needle sets of arbitrarily small measure, see e.g. [1].

For $-\frac{\pi}{2} < \phi_1 < \phi_2 < \frac{\pi}{2}$, we define

$$D(\phi_1, \phi_2) = \{ (r \sin \theta, r \cos \theta) : r \in \mathbb{R}, \theta \in [\phi_1, \phi_2] \}.$$

In other words, $D(\phi_1, \phi_2) \subset \mathbb{R}^2$ denotes the double cone bounded by the lines through the origin of signed angles ϕ_1, ϕ_2 with respect to the *y*-axis. (Note in particular that our sign convention measures the angles in the clockwise direction.)

Our geometric construction begins by partitioning $D = D(\phi_1, \phi_2)$ into finitely many double cones $\{D_i\}$. Next, we translate each D_i downwards to a new vertex $v_i \in D_i \cap \{y_2 < 0\}$ to obtain $\widetilde{D}_i := v_i + D_i$. Our goal is to choose the $\{D_i\}$ and the $\{v_i\}$ so that the resulting double cones $\{\widetilde{D}_i\}$ satisfy the following three properties.

First, the $\{D_i\}$ should have considerable overlap (and hence small measure) in a strip below the x-axis.

Second, we would like our construction to preserve certain distances to lines. To be more precise, first let

$$D^{\perp}(\phi_1, \phi_2) = \{ (r\sin\theta, r\cos\theta) : r \ge 0, \theta \in [\phi_2 - \frac{\pi}{2}, \phi_1 + \frac{\pi}{2}] \}.$$

Our second desired property is that for any point $p \in D^{\perp}(\phi_1, \phi_2)$ and any line $\ell \subset D$, there is a line in some \widetilde{D}_i which has the same distance to p as ℓ does.

For a non-horizontal line $\ell \subset \mathbb{R}^2$ and $p \in \mathbb{R}^2$, we define $d(p, \ell)$ to be the signed distance from p to ℓ . The sign is positive if p is on the left of ℓ , and negative if p is on the right. In our construction, we will always consider only lines whose direction belongs to the original cone $D(\phi_1, \phi_2)$. In particular, they are never horizontal so the signed distance is defined. The essential property of $D^{\perp}(\phi_1, \phi_2)$ is that for any $p \in D^{\perp}(\phi_1, \phi_2)$, the map $\ell \mapsto d(p, \ell)$ is an increasing function as ℓ rotates from one boundary line ℓ_1 of D to the other boundary line ℓ_2 . Hence,

$$\{d(p,\ell): \ell \subset D\} = [d(p,\ell_1), d(p,\ell_2)].$$

Before stating the third and final property, we observe that since $v_i \in D_i \cap \{y_2 < 0\}$ for all i, we have $D \cap \{y_2 \ge 0\} \subset (\bigcup_i \widetilde{D}_i) \cap \{y_2 \ge 0\}$. The third desired property is that the reverse containment holds if we thicken D slightly. That is, $(\bigcup_i \widetilde{D}_i) \cap \{y_2 \ge 0\}$ should be contained in a small neighborhood of $D \cap \{y_2 \ge 0\}$.

The following lemma asserts that it is indeed possible to partition D and translate the pieces to achieve the three desired properties above.

Lemma 5.3. Let $-\frac{\pi}{2} < \phi_1 < \phi_2 < \frac{\pi}{2}$, $D = D(\phi_1, \phi_2)$, R > 0, and $\varepsilon > 0$. Then we can choose the partition $D = \bigcup D_i$ and the translates $\widetilde{D}_i = v_i + D_i$ so that

- (1) $|(\bigcup_i \widetilde{D}_i) \cap \{-R \le y_2 \le 0\}| < \varepsilon.$
- (2) If $p \in D^{\perp}$ and $\ell_0 \subset D$ is a line, then there is a line $\tilde{\ell}$ in some \tilde{D}_i such that $d(p,\tilde{\ell}) = d(p,\ell_0)$.
- (3) $(\bigcup_i \widetilde{D}_i) \cap \{y_2 \ge 0\}$ is contained in the ε -neighborhood of $D \cap \{y_2 \ge 0\}$.

To prove this lemma, we first need a more elementary construction, in which we translate each D_i downwards by only a small amount δ .

Lemma 5.4. Let $-\frac{\pi}{2} < \phi_1 < \phi_2 < \frac{\pi}{2}$, $D = D(\phi_1, \phi_2)$, $\delta > 0$, and $\varepsilon > 0$. Then we can choose the partition $D = \bigcup D_i$ and the translates $\widetilde{D}_i = v_i + D_i$ so that

(1) $|(\bigcup_i \widetilde{D}_i) \cap \{-\delta \le y_2 \le 0\}| \le c\delta^2$, where $c = |D \cap \{0 \le y_2 \le 1\}|$.

- (2) If $p \in D^{\perp}$ and $\ell_0 \subset D$ is a line, then there is a line $\tilde{\ell}$ in some \tilde{D}_i such that $d(p,\tilde{\ell}) = d(p,\ell_0)$.
- (3) For each $i, v_i \in \{y_2 = -\delta\}$.
- (4) For each $i, D^{\perp} \subset \widetilde{D}_i^{\perp}$.
- (5) $(\bigcup_i \widetilde{D}_i) \cap \{y_2 \ge 0\}$ is contained in the ε -neighborhood of $D \cap \{y_2 \ge 0\}$.

 $(If D_i = D(\psi_1, \psi_2), then D_i^{\perp} := D^{\perp}(\psi_1, \psi_2) and \widetilde{D}_i^{\perp} := v_i + D_i^{\perp}.)$

Proof. We claim that for any partition $D = \bigcup_i D_i$, if we choose any $v_i \in \{y_2 = -\delta\} \cap D_i \cap (-D_i^{\perp})$, then we have (1), (2), (3), and (4). Indeed, (3) is immediate. Since $-v_i \in D_i^{\perp}$, we have $D^{\perp} \subset D_i^{\perp} \subset \widetilde{D}_i^{\perp}$, so (4) holds. And (3) implies (1) since $|(\bigcup_i \widetilde{D}_i) \cap \{-\delta \leq y_2 \leq 0\}| \leq \sum_i |\widetilde{D}_i \cap \{-\delta \leq y_2 \leq 0\}| = \sum_i |D_i \cap \{0 \leq y_2 \leq \delta\}| = c\delta^2$.

To show (2) holds, let $p \in D^{\perp}$ and $\ell_0 \subset D$. Then ℓ_0 is in some D_i . Let ℓ_1, ℓ_2 be the two boundary lines of D_i with $d(p, \ell_1) < d(p, \ell_2)$. Recall that $\ell \subset D_i$ if and only if $v_i + \ell \subset \widetilde{D}_i$. Since $p \in D_i^{\perp}$ and $p \in \widetilde{D}_i^{\perp}$, we have $\{d(p, \ell) : \ell \subset D_i\} = [d(p, \ell_1), d(p, \ell_2)]$ and $\{d(p, \ell) : \ell \subset \widetilde{D}_i\} = [d(p, v_i + \ell_1), d(p, v_i + \ell_2)]$. Since $-v_i \in D_i \cap \{y_2 \ge 0\}$, we have $[d(p, \ell_1), d(p, \ell_2)] \subset [d(p, v_i + \ell_1), d(p, v_i + \ell_2)]$. Thus,

$$d(p,\ell_0) \in \{d(p,\ell) : \ell \subset D_i\} \subset \{d(p,\ell) : \ell \subset D_i\},\$$

so there is some $\tilde{\ell} \subset \tilde{D}_i$ such that $d(p, \tilde{\ell}) = d(p, \ell_0)$, which completes the proof of (2) and hence our claim. Finally, by making the partition $\bigcup_i D_i$ sufficiently fine and choosing v_i as above, we can ensure that (5) holds.

Proof of Lemma 5.3. We fix a large N and repeatedly apply Lemma 5.4 with $\delta = R/N$ until the vertex of each double cone lies in $\{y_2 = -R\}$. That is, we apply Lemma 5.4 once on D to get E_1 , a union of double cones with vertices in $\{y_2 = -\delta\}$ and such that $|E_1 \cap \{-\delta \leq y_2 \leq 0\}| < c'\delta^2$, where $c' = 2|D \cap \{0 \leq y_2 \leq 1\}|$. Next, we apply Lemma 5.4 to every double cone in E_1 to get E_2 , a union of double cones with vertices in $\{y_2 = -\delta\}$ and such that $|E_2 - \{-\delta \leq y_2 \leq -\delta\}| < c'\delta^2$. By Lemma 5.4(5), we can also ensure that $|E_2 \cap \{-\delta \leq y_2 \leq 0\}| < c'\delta^2$.

We continue in this way to obtain E_1, \ldots, E_N , such that $|E_k \cap \{-j\delta \leq y_2 \leq -(j-1)\delta\}| < c'\delta^2$ for $1 \leq j \leq k \leq N$. Because of Lemma 5.4(5), we can also ensure that $E_k \cap \{y_2 \geq 0\}$ is in the ε -neighborhood of $D \cap \{y_2 \geq 0\}$ for each k. Ultimately, we have $|E_N \cap \{-R \leq y_2 \leq 0\}| \leq Nc'\delta^2 = c'R^2/N$. By choosing N sufficiently large, we can make this quantity as small as we wish. Writing E_N as $\bigcup_i \widetilde{D}_i$, we obtain (1) and (3). Furthermore, every time we translate downwards by δ , Lemma 5.4(4) allows us to apply Lemma 5.4(2). Thus, (2) holds.

5.2. **Proof of Theorem 5.1.** First we apply our main geometric construction from the previous section to prove the following lemma.

Lemma 5.5. Let $n \geq 2$, $(x_0, r_0, T_0) \in \mathbb{R}^n \times [0, \infty) \times SO(n)$, let $B \subset \mathbb{R}^n$ be a closed ball, and suppose that $x_0 + r_0T_0(e_1) \notin B$. Let $\eta > 0$ be arbitrary. Then there is a (relatively) open neighborhood \widetilde{U} of (x_0, r_0) in $\mathbb{R}^n \times [0, \infty)$ such that for each $\varepsilon > 0$, there is a set $\widetilde{D} \subset \mathbb{R}^n$ such that:

- (1) For all $(x,r) \in \widetilde{U}$, there is an affine hyperplane $V \subset \widetilde{D}$ of distance r from x and such that the angle between V and $T_0(H)$ is at most η .
- (2) $|B \cap D| < \varepsilon$.

Proof. First we show the lemma for n = 2. Since $x_0 + r_0T_0(e_1) \notin B$, without loss of generality, we may assume that $T_0 \in SO(2)$ is the identity, that $x_0 + r_0e_1 \in \{y_1 = 0, y_2 > 0\}$, and that B does not intersect $\{y_1 = 0, y_2 \ge 0\}$. We can also assume that B lies in $\{y_2 \ge -2 \operatorname{diam} B\}$. It follows that x_0 lies in the upper half-plane $\{y_2 > 0\}$.

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Using the notation from Section 5.1, let $D = D(-\phi, \phi)$ be a double cone, where $\phi \in (0, \eta)$ is small enough so that $x_0 \in D^{\perp}$ and $B \cap D \cap \{y_2 \ge 0\} = \emptyset$. The boundary of D is made up of two lines, ℓ_1, ℓ_2 , with $d(x_0, \ell_1) < r_0 < d(x_0, \ell_2)$. Let $\rho > 0$ be sufficiently small so that $d(x_0, \ell_1) < r_0 - \rho$ and $r_0 + \rho < d(x_0, \ell_2)$. Let \widetilde{U} be a (relatively) open neighborhood of (x_0, r_0) contained in

$$\{y \in D^{\perp} : d(y, \ell_1) < r_0 - \rho \text{ and } r_0 + \rho < d(y, \ell_2)\} \times (r_0 - \rho, r_0 + \rho).$$

Then for any $(x,r) \in \widetilde{U}$, there is a line $\ell \subset D$ of signed distance r from x. Given $\varepsilon > 0$, we apply Lemma 5.3 to get $\widetilde{D} := \bigcup_i \widetilde{D}_i$ with $|\widetilde{D} \cap \{-2 \operatorname{diam} B \leq y_2 \leq 0\}| < \varepsilon$ and $B \cap \widetilde{D} \cap \{y_2 \geq 0\} = \emptyset$. It follows that $|B \cap \widetilde{D}| < \varepsilon$. By Lemma 5.3(2), for every $(x,r) \in \widetilde{U}$, there is some line $\ell \subset \widetilde{D}$ of distance r from x. Every line $\ell \subset \widetilde{D}$ is a translate of some line in D, so the angle between ℓ and H is at most $\phi < \eta$. This completes the proof in dimension n = 2.

For an arbitrary $n \geq 2$, we can assume without loss of generality that T_0 is the identity, that x_0 (and hence also $x_0+r_0e_1$) is contained in the two-dimensional plane $\mathbb{R}^2 \subset \mathbb{R}^n$ defined by the first two coordinate axes, and that the same assumptions hold as in the first paragraph of our proof. Then, if we project the ball B into \mathbb{R}^2 , take the sets $\widetilde{U}, \widetilde{D} \subset \mathbb{R}^2$ constructed above, and multiply them by \mathbb{R}^{n-2} , the resulting sets satisfy the requirements of the statement of Lemma 5.5 with ε replaced by $\varepsilon \operatorname{diam}(B)^{n-2}$.

Lemma 5.6. Let $(x_0, r_0, T_0) \in \mathbb{R}^n \times [0, \infty) \times SO(n)$, and let $B \subset \mathbb{R}^n$ be a closed ball. Let G be a (relatively) open neighborhood of (x_0, r_0, T_0) in $\mathbb{R}^n \times [0, \infty) \times SO(n)$. Then there is an open neighborhood $U \subset \mathbb{R}^n \times [0, \infty)$ of (x_0, r_0) such that for each $\varepsilon > 0$, there is a compact set $K \subset G$ with full projection onto U and such that

(5.1)
$$B \cap \bigcup_{\substack{(x,r,T) \in K \\ x+rT(e_1) \notin 2B}} (x+rT(e_1)+T(H))$$

has measure less than ε . (Here 2B denotes the closed ball with the same center as B and with twice the radius.)

Proof. Without loss of generality, we may assume $G = G_1 \times G_2$, where G_1 and G_2 are open sets in $\mathbb{R}^n \times [0, \infty)$ and SO(n), respectively.

If $x_0 + r_0T_0(e_1) \in B$, then we can choose $K \subset G$ to contain a neighborhood of (x_0, r_0, T_0) and such that $x + rT(e_1) \in 2B$ for all $(x, r, T) \in K$. Then the set (5.1) is empty, so the lemma holds trivially.

Now suppose $x_0 + r_0T_0(e_1) \notin B$. We can apply the previous lemma with η sufficiently small (depending on G_2) to get a set \tilde{U} . We take U to be an open neighborhood of (x_0, r_0) inside \tilde{U} and compactly contained in G_1 . Then for each $\varepsilon > 0$, the previous lemma gives a set \tilde{D} . We take K to be the closure of

$$\{(x, r, T) \in U \times SO(n) : x + rT(e_1) + T(H) \subset D\}$$

and by the properties of \tilde{D} given by the previous lemma, this K has the desired properties.

For $B \subset \mathbb{R}^n$ a closed ball, let $\mathcal{A}(B)$ be the set of all $K \in \mathcal{K}''$ such that

$$B \cap \bigcup_{\substack{(x,r,T) \in K \\ x+rT(e_1) \notin 2B}} (x+rT(e_1)+T(H))$$

has measure zero.

Lemma 5.7. $\mathcal{A}(B)$ is residual in \mathcal{K}'' .

Proof. For $\varepsilon > 0$, let $\mathcal{A}(B, \varepsilon)$ be the set of those $K \in \mathcal{K}''$ for which there is an $\eta > 0$ such that

$$B \cap \bigcup_{\substack{(x,r,T) \in K^{\eta} \\ x+rT(e_1) \notin 2B}} (x+rT(e_1)+T(H))$$

has measure less than ε , where K^{η} denotes the open η -neighborhood of K. Since $\mathcal{A}(B) = \bigcap_{m=1}^{\infty} \mathcal{A}(B, \frac{1}{m})$, it is enough to show that $\mathcal{A}(B, \varepsilon)$ is open and dense in \mathcal{K}'' for each $\varepsilon > 0$.

Fix $\varepsilon > 0$. $\mathcal{A}(B,\varepsilon)$ is clearly open in \mathcal{K}'' . To show that it is dense, let $L \in \mathcal{K}''$ be arbitrary. Our aim is to find a $K \in \mathcal{A}(B,\varepsilon)$ arbitrarily close to L. For each $(x,r,T) \in L$, we take a neighborhood $G_{(x,r,T)}$ of (x,r,T), which we choose sufficiently small (to be specified later). Then we apply Lemma 5.6 to (x,r,T) to get a neighborhood $U_{(x,r,T)} \subset \mathbb{R}^n \times [0,\infty)$ of (x,r). By compactness, there is a finite collection $\{(x_i,r_i,T_i)\} \subset L$ such that $\{U_{(x_i,r_i,T_i)}\}$ covers J.

Choose ε_i so that $\sum_i \varepsilon_i < \varepsilon$. We apply Lemma 5.6 to each $U_{(x_i,r_i,T_i)}$ with ε_i in place of ε to get a compact $K_i \subset G_{(x_i,r_i,T_i)}$ with full projection onto $U_{(x_i,r_i,T_i)}$. Let $\widetilde{K}_i = K_i \cap (J \times SO(n))$. Let K be the union of $\bigcup_i \widetilde{K}_i$ together with a finite δ -net of L. Then $K \in \mathcal{A}(B, \varepsilon)$. By choosing δ and all the $G_{(x,r,T)}$ sufficiently small, we can make K and L arbitrarily close to each other in the Hausdorff metric. \Box

Now we are ready to prove Theorem 5.1. It follows easily from Lemma 5.7 that, for a typical $K \in \mathcal{K}''$,

(5.2)
$$\bigcup_{(x,r,T)\in K} (x+rT(e_1)+T(H\setminus\{0\}))$$

has measure zero. Indeed, let $\{B_i\}$ be a countable collection of balls such that every point in \mathbb{R}^n is covered by a ball of arbitrarily small diameter, and suppose that $K \in \bigcap_i \mathcal{A}(B_i)$. For every $(x, r, T) \in K$ and for every $y \in H \setminus \{0\}$ there is a B_i which contains $x + rT(e_1) + T(y)$ and has diameter less than |y|/2. Then $x + rT(e_1) \notin 2B_i$, so $x + rT(e_1) + T(y)$ belongs to the null set

$$B_i \cap \bigcup_{\substack{(x,r,T) \in K\\ x+rT(e_1) \notin 2B_i}} (x+rT(e_1)+T(H)).$$

To complete the proof of Theorem 5.1, we need to show that we can remove the puncture from $H \setminus \{0\}$ when the distance r is nonzero. By adapting the argument in the proof of Lemma 4.4, we can show that for any $r_0 > 0$, for a typical $K \in \mathcal{K}''$, the set

(5.3)
$$\bigcup_{\substack{(x,r,T)\in K\\r>r_0}} x+rT(e_1)$$

has dimension at most 1, hence measure zero. By taking a countable intersection of r_0 tending to 0, we see that for a typical $K \in \mathcal{K}''$, the set

$$\bigcup_{\substack{(x,r,T)\in K\\r\neq 0}} x + rT(e_1)$$

has measure zero. This completes the proof of Theorem 5.1.

Remark 5.8. The argument in the proof of Lemma 4.4 cannot be applied directly to show that the set (5.3) has dimension at most 1 for a typical $K \in \mathcal{K}''$. There is a slight complication due to the fact that for $r_0 > 0$, the function $\mathcal{K}'' \to \mathcal{K}^n$ defined by $K \mapsto \bigcup_{(x,r,T)\in K, r\geq r_0} x + rT(e_1)$ is not necessarily continuous. However, the technical modifications required are straightforward, so we leave this to the reader.

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