

Volume of the space of qubit-qubit channels and state transformations under random quantum channels*

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Abstract

The simplest building blocks for quantum computations are the qubit-qubit quantum channels. In this paper, we analyze the structure of these channels via their Choi representation. The restriction of a quantum channel to the space of classical states (i.e. probability distributions) is called the underlying classical channel. The structure of quantum channels over a fixed classical channel is studied, the volume of general and unital qubit channels with respect to the Lebesgue measure is computed and explicit formulas are presented for the distribution of the volume of quantum channels over given classical channels. We study the state transformation under uniformly random quantum channels. If one applies a uniformly random quantum channel (general or unital) to a given qubit state, the distribution of the resulted quantum states is presented.

1 Introduction

In quantum information theory, a qubit is the non-commutative analogue of the classical bit. A qubit can be represented by a 2×2 complex self-adjoint positive semidefinite matrix with trace one [7, 11, 12]. The space of qubits is denoted by \mathcal{M}_2 and it can be identified with the unit ball in \mathbb{R}^3 via the Stokes parameterization. A linear map $Q : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ is called a qubit channel (or qubit quantum operation) if it is completely positive and trace preserving (CPT) [11]. A qubit channel is said to be *unital* (or equivalently identity preserving) if it leaves the maximally mixed state invariant.

Choi and Jamiolkowski has published a tractable representation for completely positive (CP) linear maps [4, 6]. To a superoperator $Q : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{2 \times 2}$ a block matrix

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \quad Q_{11}, Q_{12}, Q_{21}, Q_{22} \in \mathbb{C}^{2 \times 2} \quad (1)$$

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is associated, which is called Choi matrix, such that the action of Q is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto aQ_{11} + bQ_{12} + cQ_{21} + dQ_{22}.$$

Due to Choi's theorem, the linear map $Q : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{2 \times 2}$ is CP if and only if its Choi matrix is positive definite [4]. Hereafter, we will use the same symbol for the qubit channel and its Choi matrix. Clearly, a block matrix Q of the form (1) corresponds to a qubit channel if and only if $Q_{11}, Q_{22} \in \mathcal{M}_2$, $Q_{21} = Q_{12}^*$, $\text{Tr } Q_{12} = 0$ and $Q \geq 0$, thus the space of qubit channels can be identified with a convex subset of \mathbb{R}^{12} which is denoted by \mathcal{Q} . If we consider the set of unital qubit channels, identity preserving property requires that $Q_{11} + Q_{22} = I$ must hold in the Choi representation (1), hence the space of unital qubit channels (\mathcal{Q}^1) can be identified with a convex submanifold of \mathbb{R}^9 .

Investigation of the set \mathcal{Q} of all qubit channels play the key role in the field of quantum information processing [7], since any physical transformation of a qubit carrying quantum information has to be described by an element of this set. Although the classical analogues of \mathcal{Q} and \mathcal{Q}^1 are trivial objects, the geometric properties of qubit channels are widely studied [8, 10]. However, the volume of the sets \mathcal{Q} and \mathcal{Q}^1 is still unknown. Random quantum operations and especially random qubit channels are subject of a considerable scientific interest [2]. For example, an effect of external noise acting on qubits can be modeled by random qubit channels. Authors in [3] have studied the spectral properties of quantum channels and designed algorithms to generate random quantum maps. We should mention that transformations of the maximally mixed state have important applications in superdense coding [5] which provide motivation for research on the distance of the maximally mixed state and its image under the action of a random qubit channel.

Quantum channels are non-commutative analogues of classical stochastic maps, therefore it is natural to consider their actions on classical quantum states (i.e diagonal density matrices). For a qubit channel Q , the *underlying classical channel* is defined as the restriction of Q to the space of classical bits. For example, the following Markov chain transition matrix represents the underlying classical channel of $Q \in \mathcal{Q}$ given by (1)

$$P = \begin{pmatrix} \text{diag}(Q_{11}) \\ \text{diag}(Q_{22}) \end{pmatrix},$$

where $\text{diag}(Q_{ii})$ is a row vector that contains the diagonal of Q_{ii} .

The main aim of this paper is to compute the volume of general and unital qubit channels and investigate the distribution of the resulted quantum states if a general or unital uniformly random quantum channel was applied to a given state. To compute the volume, we follow a similar strategy to those that was introduced by Andai in [1] to calculate the volume of density matrices. This approach makes possible to gain information about the distribution of volume over classical states and to compute the effect of uniformly random quantum channels on the given state.

The paper is organized as follows. In the second Section, we fix the notations for further computations and we mention some elementary lemmas which will be used in the sequel. In Section 3, the volume of general and unital qubit channels with respect to the Lebesgue measure is computed and explicit formulas are

presented for the distribution of volume over classical channels. Section 4 deals with state transformations under uniformly random quantum channels.

2 Basic lemmas and notations

The following lemmas will be our main tools, we will use them frequently. We also introduce some notations which will be used in the sequel.

The first two lemmas are elementary propositions in linear algebra. For an $n \times n$ matrix A we set A_i to be the left upper $i \times i$ submatrix of A , where $i = 1, \dots, n$.

Lemma 1. *The $n \times n$ self-adjoint matrix A is positive definite if and only if the inequality $\det(A_i) > 0$ holds for every $i = 1, \dots, n$.*

Lemma 2. *Assume that A is an $n \times n$ self-adjoint, positive definite matrix with entries $(a_{ij})_{i,j=1,\dots,n}$ and the vector x consists of the first $(n-1)$ elements of the last column, that is $x = (a_{1,n}, \dots, a_{n-1,n})$. Then we have*

$$\det(A) = a_{nn} \det(A_{n-1}) - \langle x, Tx \rangle,$$

where $T = \det(A_{n-1})(A_{n-1})^{-1}$.

Proof. Elementary matrix computation, one should expand $\det(A)$ by minors, with respect to the last row. \square

When we integrate on a subset of the Euclidean space we always integrate with respect to the usual Lebesgue measure. The Lebesgue measure on \mathbb{R}^n will be denoted by λ_n .

Lemma 3. *If T is an $n \times n$ self-adjoint, positive definite matrix and $k, \rho \in \mathbb{R}^+$, then*

$$\int_{\{x \in \mathbb{C}^n \mid \langle x, Tx \rangle < \rho\}} (\rho - \langle x, Tx \rangle)^k d\lambda_{2n}(x) = \frac{\pi^n \rho^{n+k} k!}{(n+k)! \det T}.$$

Proof. The set $\{x \in \mathbb{C}^n \mid \langle x, Tx \rangle < \rho\}$ is an n dimensional ellipsoid, so to compute the integral first we transform our canonical basis to a new one, which is parallel to the axes of the ellipsoid. Since this is an orthogonal transformation, its Jacobian is 1. When we transform this ellipsoid to a unit sphere, the Jacobian of this transformation is

$$\prod_{k=1}^n \frac{\rho}{\mu_k},$$

where $(\mu_k)_{k=1,\dots,n}$ are the eigenvalues of T . Then we compute the integral in spherical coordinates. The integral with respect to the angles gives the surface of the $2n$ dimensional sphere that is $\frac{2\pi^n}{(n-1)!} r^{2n-1}$. The integral of the radial part is

$$\int_0^1 \frac{2\pi^n}{(n-1)!} r^{2n-1} \frac{\rho^n}{\det T} (\rho - \rho r^2)^k dr = \frac{2\pi^n \rho^{n+k}}{(n-1)! \det T} \int_0^1 r^{2n-1} (1 - r^2)^k dr$$

which gives back the stated formula. \square

Lemma 4. *Assume that X is a spherically symmetric and continuous random variable which takes values in the unit ball $\{x \in \mathbb{R}^3 : \|x\| \leq 1\}$. If f denotes the probability density function of the z component of X , then for the density of $\|X\|$ we have*

$$\rho(r) = -2rf'(r) \quad r \in]0, 1[. \quad (2)$$

Proof. Let us denote by f_X the probability density function of X in the unit ball. The distribution of X is rotation invariant thus there exists a function $g : [0, 1] \rightarrow \mathbb{R}^+$ such that, for every x in the unit ball $f_X(x) = g(\|x\|)$. For the radial density we have

$$\rho(r) = \frac{d}{dr} \mathbb{P}(\|X\| < r) = \frac{d}{dr} 4\pi \int_0^r g(s)s^2 ds = 4\pi g(r)r^2 \quad r \in]0, 1[.$$

Now we compute the density function of the z component from the radial distribution. For every $z_0 \in]0, 1[$

$$\begin{aligned} f(z_0) &= \frac{d}{dz_0} \mathbb{P}(z < z_0) = -\frac{d}{dz_0} \mathbb{P}(z \geq z_0) \\ &= -\frac{d}{dz_0} \int_0^{2\pi} \int_y^1 \int_0^{\arccos(z_0/r)} g(r)r^2 \sin \phi d\phi dr d\theta = 2\pi \int_{z_0}^1 g(r)r dr \end{aligned}$$

holds and from this by derivation we get

$$f'(r) = -2\pi g(r)r = -\frac{\rho(r)}{2r},$$

which completes the proof. \square

3 The volume of qubit channels

To determine the volume of different qubit quantum channels we use the same method which consists of three parts. First, we use a unitary transformation to represent channels in a suitable form for further computations. Then we split the parameter space into lower dimensional parts such that the adequate applications of the previously mentioned lemmas lead us to the result.

3.1 General qubit channels

The following parametrization of $\mathcal{Q} \subset \mathbb{R}^{12}$ is considered

$$Q = \begin{pmatrix} a_1 & b & c & d \\ \bar{b} & a_2 & e & -c \\ \bar{c} & \bar{e} & f_1 & g \\ \bar{d} & -\bar{c} & \bar{g} & f_2 \end{pmatrix}, \quad (3)$$

where $a_1, f_1 \in [0, 1]$, $a_2 = 1 - a_1$, $f_2 = 1 - f_1$ and $b, c, d, e, g \in \mathbb{C}$. Let us define $a = a_1$ and $f = f_1$.

The underlying classical channel corresponding to these parameter values are given by $Q_{cl} = \begin{pmatrix} a & 1-a \\ f & 1-f \end{pmatrix}$. Let us choose the unitary matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

and define the matrix A as

$$A = U^* Q U = \begin{pmatrix} a & c & b & d \\ \bar{c} & f & \bar{e} & g \\ \bar{b} & e & a_2 & -c \\ \bar{d} & \bar{g} & -\bar{c} & f_2 \end{pmatrix}, \quad (5)$$

which is positive definite if and only if Q is positive definite.

Theorem 1. *The volume of the space \mathcal{Q} with respect to the Lebesgue measure is*

$$V(\mathcal{Q}) = \frac{2\pi^5}{4725}$$

and the distribution of volume over classical channels can be written as

$$V(a_1, f_1) = \frac{2^4 \pi^5}{45} \begin{cases} a_1^3 f_1^3 (a_1^2 f_1^2 - 5a_1 a_2 f_1 f_2 + 10a_2^2 f_2^2) & \text{if } a_1 + f_1 \leq 1, \\ a_2^3 f_2^3 (a_2^2 f_2^2 - 5a_1 a_2 f_1 f_2 + 10a_1^2 f_1^2) & \text{if } a_1 + f_1 > 1. \end{cases} \quad (6)$$

Proof. Since there is an unitary transformation (4) between the set of matrices of the form of (5) and the quantum channels their volumes are the same. We compute the volume of the set of matrices given by parameterization (5).

The volume element corresponding to the parametrization (3) is $2^7 d\lambda_{12}$. The matrix A in Equation (5) is positive definite if and only if $\det(A_i) > 0$ for $i = 1, 2, 3, 4$.

First we assume that the parameters a, f and the submatrix A_3 are given and consider the requirement $\det A_4 \geq 0$. Simple calculation shows that we have

$$\det A_4 = R_3 - \left\langle \begin{pmatrix} d' \\ g' \end{pmatrix}, T_3 \begin{pmatrix} d' \\ g' \end{pmatrix} \right\rangle,$$

where $d' = d + \frac{bc}{a_2}$, $g' = g + \frac{c\bar{e}}{a_2}$, $R_3 = (\det A_3) \left(f_2 - \frac{|c|^2}{a_2} \right)$ and

$$T_3 = \begin{pmatrix} a_2 f_1 - |e|^2 & be - a_2 c \\ \bar{b}\bar{e} - a_2 \bar{c} & a_1 a_2 - |b|^2 \end{pmatrix}.$$

In this case the inequality $\det A_4 \geq 0$ can be written in the form of

$$\left\langle \begin{pmatrix} d' \\ g' \end{pmatrix}, T_3 \begin{pmatrix} d' \\ g' \end{pmatrix} \right\rangle \leq R_3. \quad (7)$$

The matrix T_3 is positive, because $\det T_3 = a_2 \det A_3 \geq 0$ and $(T_3)_{11}$ is the determinant of the middle 2×2 submatrix of A . It means, that the Inequality (7) has solution if, and only if $f_2 a_2 \geq |c|^2$.

The transformation $(d, g) \mapsto (d', g')$ is a shift, therefore it does not change the volume element. We have by Lemma 3

$$V(a, f, b, c, e) = 2^7 \int_{\left\langle \begin{pmatrix} d' \\ g' \end{pmatrix}, T_3 \begin{pmatrix} d' \\ g' \end{pmatrix} \right\rangle \leq R_3} 1 \, d(d', g') = \frac{2^6 R_3^2 \pi^2}{\det T_3},$$

where $\det T_3 = a_2 \det A_3$, therefore

$$V(a, f, b, c, e) = \begin{cases} \frac{2^6 \pi^2}{a_2^3} (a_2 f_2 - |c|^2)^2 \det A_3 & \text{if } f_2 a_2 \geq |c|^2, \\ 0 & \text{if } f_2 a_2 < |c|^2. \end{cases}$$

In the second step we assume that the parameters a, f and the submatrix A_2 are given and consider the requirement $\det A_3 \geq 0$. We have

$$\det A_3 = R_2 - \left\langle \begin{pmatrix} b \\ \bar{e} \end{pmatrix}, T_2 \begin{pmatrix} b \\ \bar{e} \end{pmatrix} \right\rangle,$$

where $R_2 = (1 - a) \det A_2$ and

$$T_2 = \begin{pmatrix} f & -c \\ -\bar{c} & a \end{pmatrix}.$$

The inequality $\det A_3 \geq 0$ can be written in the form of

$$\left\langle \begin{pmatrix} b \\ \bar{e} \end{pmatrix}, T_2 \begin{pmatrix} b \\ \bar{e} \end{pmatrix} \right\rangle \leq R_2.$$

We now integrate with respect to b and e . To compute the integral

$$\begin{aligned} V(a, f, c) &= \int_{\left\langle \begin{pmatrix} b \\ \bar{e} \end{pmatrix}, T_2 \begin{pmatrix} b \\ \bar{e} \end{pmatrix} \right\rangle \leq R_2} V(a, f, b, c, e) \, d(b, e) \\ &= \int_{\substack{\left\langle \begin{pmatrix} b \\ \bar{e} \end{pmatrix}, T_2 \begin{pmatrix} b \\ \bar{e} \end{pmatrix} \right\rangle \leq R_2 \\ f_2 a_2 \geq |c|^2}} \frac{2^6 \pi^2}{a_2^3} (a_2 f_2 - |c|^2)^2 \det A_3 \, d(b, e) \end{aligned}$$

we substitute $\det A_3 = R_2 - \left\langle \begin{pmatrix} b \\ \bar{e} \end{pmatrix}, T_2 \begin{pmatrix} b \\ \bar{e} \end{pmatrix} \right\rangle$ and by Lemma 3 we have

$$\begin{aligned} V(a, f, c) &= \frac{2^6 \pi^2}{a_2^3} (a_2 f_2 - |c|^2)^2 \int_{\left\langle \begin{pmatrix} b \\ \bar{e} \end{pmatrix}, T_2 \begin{pmatrix} b \\ \bar{e} \end{pmatrix} \right\rangle \leq R_2} \left(R_2 - \left\langle \begin{pmatrix} b \\ \bar{e} \end{pmatrix}, T_2 \begin{pmatrix} b \\ \bar{e} \end{pmatrix} \right\rangle \right) d(b, e) \\ &= \frac{2^6 \pi^2}{a_2^3} (a_2 f_2 - |c|^2)^2 \times \frac{\pi^2 R_2^3}{6 \det T_2}, \end{aligned}$$

where $\det T_2 = \det A_2$, therefore

$$V(a, f, c) = \begin{cases} \frac{2^5 \pi^4}{3} (a_2 f_2 - |c|^2)^2 \times (\det A_2)^2 & \text{if } f_2 a_2 \geq |c|^2, \\ 0 & \text{if } f_2 a_2 < |c|^2. \end{cases}$$

In the final step we assume that the parameters a, f are given and consider the requirement $\det A_2 \geq 0$. It means that $|c|^2 \leq af$, therefore if

$$|c|^2 \leq \min \{af, (1-a)(1-f)\}$$

then

$$V(a, f, c) = \frac{2^5 \pi^4}{3} (a_2 f_2 - |c|^2)^2 \times (af - |c|^2)^2.$$

If $a + f \leq 1$, then $af \leq (1-a)(1-f)$. In this case using polar coordinates for c we have

$$\begin{aligned} V(a, f) &= 2\pi \int_0^{\sqrt{af}} \frac{2^5 \pi^4}{3} (a_2 f_2 - r^2)^2 (af - r^2)^2 \times r dr \\ &= \frac{2^4 \pi^5}{45} a_1^3 f_1^3 (a_1^2 f_1^2 - 5a_1 a_2 f_1 f_2 + 10a_2^2 f_2^2). \end{aligned} \quad (8)$$

If $a + f \geq 1$, then $af \geq (1-a)(1-f)$. In this case using polar coordinates for c we have

$$\begin{aligned} V(a, f) &= 2\pi \int_0^{\sqrt{(1-a)(1-f)}} \frac{2^5 \pi^4}{3} (a_2 f_2 - r^2)^2 (af - r^2)^2 \times r dr \\ &= \frac{2^4 \pi^5}{45} a_2^3 f_2^3 (a_2^2 f_2^2 - 5a_1 a_2 f_1 f_2 + 10a_1^2 f_1^2). \end{aligned} \quad (9)$$

Equations (8) and (9) give back Equation (6). The volume of the space of quantum channels is

$$V = \int_0^1 \int_0^1 V(a, f) da df = \frac{2\pi^5}{4725}. \quad (10)$$

□

3.2 Unital qubit channels

The following parametrization of $\mathcal{Q}^1 \subset \mathbb{R}^9$ is considered

$$Q = \begin{pmatrix} a_1 & b & c & d \\ \bar{b} & a_2 & e & -c \\ \bar{c} & \bar{e} & a_2 & -b \\ \bar{d} & -\bar{c} & -\bar{b} & a_1 \end{pmatrix}, \quad (11)$$

where $a_1 \in [0, 1]$, $a_2 = 1 - a_1$ and $b, c, d, e \in \mathbb{C}$. Let us define $a = a_1$. The underlying classical channel corresponding to these parameter values are given

by $Q_{\text{cl}}^1 = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$. Let us choose the unitary matrix

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (12)$$

and define the matrix A as

$$A = U^* Q U = \begin{pmatrix} a_2 & e & \bar{b} & -c \\ \bar{e} & a_2 & \bar{c} & -b \\ b & c & a_1 & d \\ -\bar{c} & -\bar{b} & \bar{d} & a_1 \end{pmatrix}, \quad (13)$$

which is positive definite if and only if Q is positive definite.

Theorem 2. *The volume of the space \mathcal{Q}^1 with respect to the Lebesgue measure is*

$$V(\mathcal{Q}) = \frac{8\pi^4}{945}$$

and the distribution of volume over classical channels can be written as

$$V(a) = \frac{2^4 \pi^4}{3} a^4 (1-a)^4. \quad (14)$$

Proof. Since there is an unitary transformation (12) between the set of matrices of the form of (13) and the quantum channels, their volumes are the same. We compute the volume of the set of matrices given by parameterization (13).

The volume element corresponding to the parametrization (11) is $2^7 d\lambda_{12}$. The matrix A in Equation (13) is positive definite if and only if $\det(A_i) > 0$ for $i = 1, 2, 3, 4$.

First we assume that the parameter a and the submatrix A_3 are given and consider the requirement $\det A_4 \geq 0$. Simple calculation shows that we have

$$\det A_4 = \frac{(\det A_3)^2}{\det A_2} - |d'|^2 \det A_2,$$

where

$$d' = d + \frac{2bc(1-a) - \bar{e}c^2 - b^2e}{\det A_2}.$$

In this case the inequality $\det A_4 \geq 0$ can be written in the form of

$$|d'| \leq \frac{\det A_3}{\det A_2}.$$

The transformation $d \mapsto d'$ is a shift, therefore it does not change the volume element. So we have

$$V(a, b, c, e) = \int_{|d'| \leq \frac{\det A_3}{\det A_2}} 2^7 d(d') = 2^7 \pi \left(\frac{\det A_3}{\det A_2} \right)^2.$$

In the next step we assume that the parameter a and the submatrix A_2 are given and consider the requirement $\det A_3 \geq 0$. We have

$$\det A_3 = R_2 - \left\langle \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}, T_2 \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle,$$

where

$$R_2 = A_{33} \det A_2 \quad \text{and} \quad T_2 = \begin{pmatrix} a_2 & -e \\ -\bar{c} & a_2 \end{pmatrix}.$$

The inequality $\det A_3 \geq 0$ can be written in the form of

$$\left\langle \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}, T_2 \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle \leq R_2.$$

We now integrate with respect to b and c . To compute the integral

$$\begin{aligned} V(a, e) &= \int_{\left\langle \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}, T_2 \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle \leq R_2} V(a, b, c, e) \, d(b, c) \\ &= \int_{\left\langle \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}, T_2 \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle \leq R_2} 2^7 \pi \left(\frac{\det A_3}{\det A_2} \right)^2 \, d(b, c) \end{aligned}$$

we substitute $\det A_3 = R_2 - \left\langle \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}, T_2 \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle$ and by Lemma 3 we have

$$\begin{aligned} V(a, e) &= \frac{2^7 \pi}{(\det A_2)^2} \int_{\left\langle \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}, T_2 \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle \leq R_2} \left(R_2 - \left\langle \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}, T_2 \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle \right)^2 \, d(b, c) \\ &= \frac{2^5 \pi^3 a^4}{3} \times \det A_2. \end{aligned}$$

Finally we assume that the parameter a is given and consider the requirement $\det A_2 \geq 0$. The condition $\det A_2 \geq 0$ means that $|e| \leq 1 - a$, therefore using polar coordinates for e we have

$$V(a) = 2\pi \int_0^{1-a} \frac{2^5 \pi^3 a^4}{3} \times ((1-a)^2 - r^2) \times r \, dr = \frac{2^4 \pi^4}{3} a^4 (1-a)^4,$$

which gives back Equation (14). The volume of the space of unital quantum channels is

$$V = \int_0^1 V(a) \, da = \frac{8\pi^4}{945}.$$

□

One might think about the generalization of the presented results, although in a more general setting several complications occur. For example, in the case of unital qubit channels one should integrate on the Birkhoff polytope, which would cause difficulties since even the volume of the polytope is still unknown [9].

4 State transformations under random channels

In this point, we study how qubits transform under uniformly distributed random quantum channels with respect to the Lebesgue measure. For simplification in this Section uniformly means that uniformly with respect to the Lebesgue measure.

For further calculations, we need the Pauli basis representation of qubit channels. The Pauli matrices are the following.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We use the Stokes representation of qubits which gives a bijective correspondence between qubits and the unit ball in \mathbb{R}^3 via the map

$$\{x \in \mathbb{R}^3 \mid \|x\|_2 \leq 1\} \rightarrow \mathcal{M}_2 \quad x \mapsto \frac{1}{2}(I + x \cdot \sigma),$$

where $x \cdot \sigma = \sum_{i=1}^3 x_i \sigma_i$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The vector x , which describes the state called Bloch vector and the unit ball in this setting is called Bloch sphere.

Any trace-preserving linear map $Q : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{2 \times 2}$ can be written in this basis as

$$Q\left(\frac{1}{2}(I + x \cdot \sigma)\right) = \frac{1}{2}(I + (v + Tx) \cdot \sigma),$$

where $v \in \mathbb{R}^3$ and T is a 3×3 real matrix. Necessary and sufficient condition for complete positivity of such maps are presented in [12]. If the Choi matrix of qubit channel is given by Equation (3), then the Pauli basis representation has the following form.

$$v = \begin{pmatrix} \operatorname{Re}(b+g) \\ -\operatorname{Im}(b+g) \\ a+f-1 \end{pmatrix} \quad T = \begin{pmatrix} \operatorname{Re}(d+e) & \operatorname{Im}(d+e) & \operatorname{Re}(b-g) \\ \operatorname{Im}(e-d) & \operatorname{Re}(d-e) & \operatorname{Im}(g-b) \\ 2\operatorname{Re}(c) & 2\operatorname{Im}(c) & a-f \end{pmatrix} \quad (15)$$

The next lemma expresses the simple fact that uniformly distributed qubit-qubit channels have no preferred direction according to the Stokes parameterization of the state space.

Lemma 5. *An orthogonal orientation preserving transformation O in \mathbb{R}^3 induces maps $\alpha_O, \beta_O : \mathcal{Q} \rightarrow \mathcal{Q}$ via Stokes parametrization $\alpha_O(Q) = O \circ Q$ and $\beta_O(Q) = Q \circ O$. The Jacobian of these transformations are 1. The Jacobian of the restricted transformations $\alpha'_O = \alpha_O|_{\mathcal{Q}_1}$ and $\beta'_O = \beta_O|_{\mathcal{Q}_1}$ are 1.*

Proof. We used a computer algebra program to verify this lemma. We considered three different kind of rotations according to the plane of rotations (xy , xz and yz plane). It is enough to prove that the Jacobian of the generated α, β transformation is 1, since every orthogonal orientation preserving transformation can be written as a suitable product of these elementary rotations. We present the calculations for β_0 , where

$$O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \quad \alpha \in \mathbb{R}.$$

If we consider a quantum channel given by parameters as in Equation (3), then the effect of β_O can be computed

$$\begin{aligned}\beta_O(a, f, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, g_1, g_2) \\ = (a', f', b'_1, b'_2, c'_1, c'_2, d'_1, d'_2, e'_1, e'_2, g'_1, g'_2),\end{aligned}$$

where all the parameters are real numbers and subscript 1 refers to the real part and 2 to the imaginary part. We list some of the new parameters.

$$\begin{aligned}a' &= \frac{a+f}{2} + \frac{(a-f)\cos\alpha}{2} + c_2\sin\alpha \\ f' &= \frac{a+f}{2} - \frac{(a-f)\cos\alpha}{2} - c_2\sin\alpha \\ b'_1 &= \frac{b_1(1+\cos\alpha) + g_1(1-\cos\alpha)}{2} + \frac{(e_2+d_2)\sin\alpha}{2} \\ b'_2 &= \frac{b_2(1+\cos\alpha) + g_2(1-\cos\alpha)}{2} + \frac{(e_2-d_2)\sin\alpha}{2} \\ c'_1 &= c_1 \\ c'_2 &= c_2\cos\alpha - \frac{(a-f)\sin\alpha}{2}\end{aligned}$$

Next we computed the 12×12 coefficient matrix, which is the derivative of the function β_0 , and the computed determinant of the coefficient matrix turned to be 1. The similar computation was done for the other rotations. The Jacobian of transformations α_O , α'_O and β'_O was checked similarly. \square

The idea of calculations about state transformations under random quantum channel is presented by the following simpler case.

Theorem 3. *Applying uniformly random channels to the most mixed state, the radii distribution of the resulted quantum states is the following.*

$$\kappa(r) = 40r^2(1-r)^6(r^3 + 6r^2 + 12r + 2) \quad r \in [0, 1] \quad (16)$$

Proof. Applying a quantum channel of the form of given by Equation (3) to the most mixed state gives z component $z' = a + f - 1$. If $z \geq 0$ then take the (not normalized) distribution from Equation (6)

$$\tilde{V}(a, f) = (1-a)^3(1-f)^3((1-a)^2(1-f)^2 - 5a(1-a)f(1-f) + 10a^2f^2).$$

The density function of the z component comes from the integral

$$\eta(z) \sim \int_z^1 \tilde{V}(a, z+1-a) da.$$

The $z < 0$ case can be handled in a similar way. After normalization we have the following formula for the density function.

$$\eta(z) = \frac{20}{11} \left(z^4 + 7|z|^3 + 17z^2 + 7|z| + 1 \right) (1-|z|)^7 \quad z \in [-1, 1] \quad (17)$$

The distribution of quantum channels is invariant for orthogonal transformations (Lemma 5, the Jacobian of α_O is 1). This means, that for every orthogonal

basis the distribution of the z component of the image of the maximally mixed state is given by Equation (17). Using Lemma 4 we have

$$\kappa(r) = -2r\eta'(r),$$

which gives the desired formula for κ immediately. \square

It is worth to note that contrary to the classical case in quantum setting the entropy of the most mixed state will decrease after a random quantum channel is applied, since the Bloch radius of the resulted quantum state is $\frac{50}{143}$ in average.

Now we study the effect of unital uniformly distributed quantum channels.

Theorem 4. *Assume that uniformly distributed unital quantum channel is applied to a given state with Bloch radius r_0 . The radii distribution of the resulted quantum states is the following.*

$$\kappa_1(r, r_0) = \frac{315}{16} \times \frac{r^2(r_0^2 - r^2)^3}{r_0^9} \chi_{[0, r_0]}(r) \quad r \in [0, 1]. \quad (18)$$

Proof. Since the distribution of unital uniform quantum channels is invariant for orthogonal transformations (Lemma 5, the Jacobian of β'_O is 1), we can assume that the initial state was given by the vector $(0, 0, r_0)$ ($r_0 \in]0, 1]$). Applying a unital quantum channel of the form of (11) to the initial state, we get a state with z component $z' = r_0(2a - 1)$. The density function of the parameter a of uniformly distributed unital quantum channels is a normalized form of (14)

$$\tilde{V}(a) = 630a^4(1 - a)^4 \quad a \in [0, 1].$$

If $z \in [-1, 1]$ arbitrary, then

$$P(z' < z) = P\left(a < \frac{z + r_0}{2r_0}\right) = \begin{cases} 0 & \text{if } z \leq -r_0, \\ \int_0^{(z+r_0)/(2r_0)} \tilde{V}(a) da & \text{if } -r_0 < z < r_0, \\ 1 & \text{if } z \geq r_0. \end{cases}$$

We have for the density function of the z component

$$f_{r_0}(z) = \frac{dP(z' < z)}{dz} = \frac{315}{256} \times \frac{(r_0^2 - z^2)^4}{r_0^9} \chi_{[-r_0, r_0]}(z),$$

where χ denotes the characteristic function. If the distribution of the z component is known then by Lemma 4 we can compute the radial distribution which gives us Equation (18). \square

The transition probability between different Bloch radii under uniformly distributed unital quantum channels $\kappa_1(r, r_0)$ is shown in Figure 1. As it is expected, a unital quantum channel decreases the initial Bloch radius r_0 , the new Bloch radius is $\frac{63}{128}r_0$ in average. Since $r' \sim \frac{r}{2}$, repeated application of uniformly distributed unital quantum channels maps every initial state to the most mixed state and the convergence is exponential.

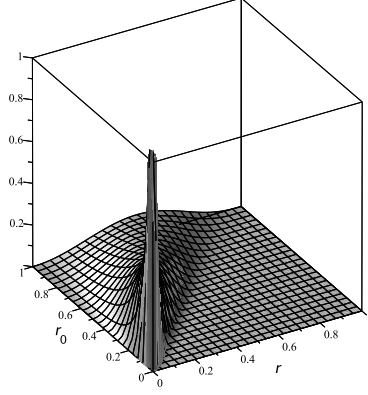


Figure 1: The function $\kappa_1(r, r_0)$. That is the radii distribution (r) of the resulted quantum states if uniformly distributed unital quantum channels were applied to a given state with Bloch radius r_0 .

Theorem 5. Assume that uniformly distributed quantum channel is applied to a given state with Bloch radius r_0 . The radii distribution of the resulted quantum states is the following.

$$\kappa(r, r_0) = \begin{cases} \text{If } 0 < r \leq r_0: \\ \frac{40r^2}{r_0(1+r_0)^6} (21r^4 - 6r^2r_0^2 - 36r^2r_0 + r_0^4 + 6r_0^3 + 12r_0^2 + 2r_0), \\ \text{if } r_0 < r \leq 1: \\ \frac{40r(r-1)^6}{(1-r_0^2)^6} (21r_0^4 - 6r^2r_0^2 - 36rr_0^2 + r^4 + 6r^3 + 12r^2 + 2r). \end{cases} \quad (19)$$

Proof. Since the distribution of unital quantum channels is invariant for orthogonal transformations (Lemma 5, the Jacobian of β'_O is 1) we can assume that the initial state was given by the vector $(0, 0, r_0)$ ($r_0 \in]0, 1]$). Applying a quantum channel of the form of (3) to the initial state, we get a state with z component $z' = a + f - 1 + r_0(a - f)$. The density function of parameters a, f of uniformly distributed quantum channels is a normalized form of (6)

$$\tilde{V}(a, f) = \begin{cases} V_u(a, f) & \text{if } 1 \leq a_1 + f_1, \\ V_l(a, f) & \text{if } 1 > a_1 + f_1, \end{cases}$$

where

$$\begin{aligned} V_u(a, f) &= 840(1-a)^3(1-f)^3((1-a)^2(1-f)^2 - 5a(1-a)f(1-f) + 10a^2f^2) \\ V_l(a, f) &= 840a^3f^3(a^2f^2 - 5a(1-a)f(1-f) + 10(1-a)^2(1-f)^2). \end{aligned}$$

First, we compute the probability $P(z' < \xi)$, where $\xi \in [-1, 1]$ is an arbitrary parameter. To determine the probability $P(z' < \xi)$ the solution of the inequality

$$z' = a + f - 1 + r_0(a - f) < \xi$$

is needed for every parameter $r_0 \in]0, 1]$ and $\xi \in [-1, 1]$, taking into account the constraints $0 \leq a, f \leq 1$.

To simplify this computation we define temporarily

$$\begin{aligned} a_1 &= \frac{1+\xi}{1+r_0}, & a_2 &= \frac{\xi+r_0}{1+r_0}, & f_1 &= \frac{1+\xi}{1-r_0}, & f_2 &= \frac{\xi-r_0}{1-r_0} \\ q &= \frac{1+r_0}{1-r_0}, & \text{and} & & x_0 &= \frac{\xi+r_0}{2r_0}. \end{aligned}$$

In the $\xi < -r_0$ case to compute the probability $P(z' < \xi)$ we have to integrate the density function $\tilde{V}(a, f)$ over the marked area shown in Figure 2.

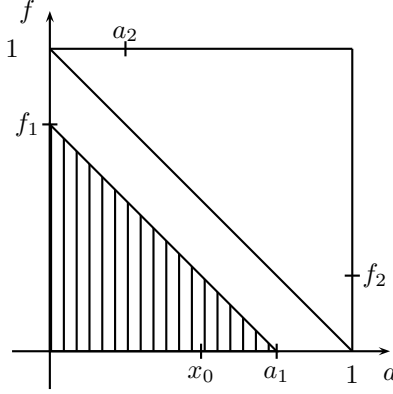


Figure 2: Solution of the inequality $z' < \xi$ in the $\xi < -r_0$ case.

$$\begin{aligned} P(z' < \xi) &= \int_0^{a_1} \int_0^{f_1-aq} V_l(a, f) df da = \\ &= \frac{(10\xi^4 - 88\xi^2 r_0^2 + 495r_0^4 - 80\xi^3 + 704\xi r_0^2 + 228\xi^2 - 198r_0^2 - 144\xi + 33)(1+\xi)^8}{66(1-r_0^2)^6} \end{aligned}$$

In the $-r_0 \leq \xi \leq r_0$ case to compute the probability $P(z' < \xi)$ we have to integrate the density function $\tilde{V}(a, f)$ over the four marked areas shown in Figure 3. That is

$$\begin{aligned} P(z' < \xi) &= \int_0^{x_0} \int_0^{1-a} V_l(a, f) df da + \int_{x_0}^{a_1} \int_0^{f_1-aq} V_l(a, f) df da \\ &+ \int_0^{a_2} \int_{1-a}^1 V_u(a, f) df da + \int_{a_2}^{x_0} \int_{1-a}^{f_1-aq} V_u(a, f) df da \end{aligned}$$

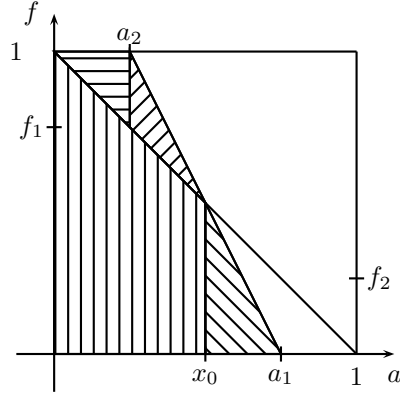


Figure 3: Solution of the inequality $z' < \xi$ in the $-r_0 \leq \xi \leq r_0$ case.

which gives us

$$\begin{aligned}
 P(z' < \xi) = & \frac{-1}{66r_0(1+r_0)^6} (660\xi^7 - 396\xi^5r_0^2 + 220\xi^3r_0^4 - 100\xi r_0^6 - 33r_0^7 \\
 & - 2376\xi^5r_0 + 1320\xi^3r_0^3 - 600\xi r_0^5 - 198r_0^6 + 2640\xi^3r_0^2 - 1440\xi r_0^4 - 495r_0^5 \\
 & + 440\xi^3r_0 - 1640\xi r_0^3 - 660r_0^4 - 720\xi r_0^2 - 495r_0^3 - 120\xi r_0 - 198r_0^2 - 33r_0).
 \end{aligned}$$

Finally in the $\xi > r_0$ case to compute the probability $P(z' < \xi)$ we have to integrate the density function $\tilde{V}(a, f)$ over the marked area shown in Figure 4.

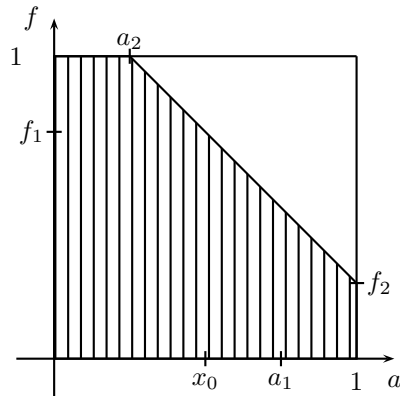


Figure 4: Solution of the inequality $z' < \xi$ in the $\xi > r_0$ case.

$$P(z' < \xi) = 1 - \int_{a_2}^1 \int_{f_1 - aq}^1 V_u(a, f) df da = 1 - \frac{(10\xi^4 - 88\xi^2 r_0^2 + 495r_0^4 + 80\xi^3 - 704\xi r_0^2 + 228\xi^2 - 198r_0^2 + 144\xi + 33)(1 - \xi)^8}{66(1 - r_0^2)^6}$$

Now we can compute the density function of the z component as

$$f_z(\xi) = \frac{dP(z' < \xi)}{d\xi}.$$

Since the density function is even ($f_z(\xi) = f_z(-\xi)$), we consider only the $\xi \geq 0$ case. If $\xi > r_0$, we have

$$f_z(\xi) = \frac{20(1 - \xi)^7}{33(1 - r_0)^6} (3\xi^4 - 22\xi^2 r_0^2 + 99r_0^4 + 21\xi^3 - 154\xi r_0^2 + 51\xi^2 - 22r_0^2 + 21\xi + 3)$$

and if $0 \leq x \leq r_0$, then

$$f_z(\xi) = \frac{-10}{33r_0(1 + r_0)^6} (231\xi^6 - 99\xi^4 r_0^2 + 33\xi^2 r_0^4 - 5r_0^6 - 594\xi^4 r_0 + 198\xi^2 r_0^3 - 30r_0^5 + 396\xi^2 r_0^2 - 72r_0^4 + 66\xi^2 r_0 - 82r_0^3 - 36r_0^2 - 6r_0).$$

Now we have the distribution of the z component and by Lemma 4 we can get the radial distribution κ (19). \square

The transition probability between different Bloch radii under uniformly distributed channel $\kappa(r, r_0)$ is shown in Figure 5.

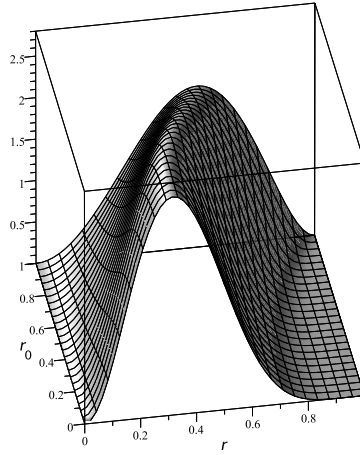


Figure 5: The function $\kappa(r, r_0)$. That is the radii distribution (r) of the resulted quantum states if uniformly distributed quantum channels were applied to a given state with Bloch radius r_0 .

Note that the function $\kappa(r, 0)$ gives back the formula (16) in Theorem 3. In Figure 6 the average Bloch radius is shown after uniformly distributed random quantum channel applied to a state with Bloch radius r_0 . From this figure it is clear that if the initial Bloch radius is small then a quantum channel likely increases the Bloch radius and if r_0 is big then decreases. Repeated application of such kind of random quantum channels will send initial states to the Bloch radius $r \approx 0.388$.

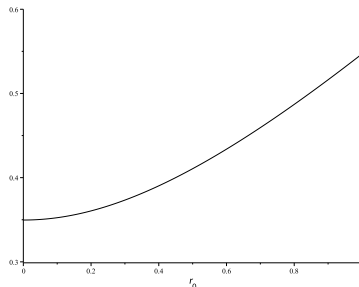


Figure 6: The average Bloch radius after uniformly distributed random quantum channel applied to a state with Bloch radius r_0 .

5 Concluding remarks

In this work we considered the Choi representation of quantum channels and the Lebesgue measure on matrix elements. We computed the volume of quantum channels and studied the effect of uniformly randomly distributed (with respect the Lebesgue measure) general and unital qbit-qbit quantum channels using the Choi's representation. It was shown that the chosen measure on the space of qbit-qbit channels is unitary invariant with respect to the initial and final qbit spaces separately. We presented the Bloch radii distributions of states after a uniformly random general or unital quantum channel was applied to a given state. This gives opportunity to study the distribution of different information theoretic quantities (for example different channel capacities, entropy gain, entropy of channels etc.) and the effect of repeated applications of uniformly random channels.

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