BASE SIZES OF PRIMITIVE GROUPS: BOUNDS WITH EXPLICIT CONSTANTS

ZOLTÁN HALASI, MARTIN W. LIEBECK, AND ATTILA MARÓTI

ABSTRACT. We show that the minimal base size $b(G)$ of a finite primitive permutation group G of degree n is at most $2(\log |G|/ \log n) + 24$. This bound is asymptotically best possible since there exists a sequence of primitive permutation groups G of degrees n such that $b(G) = \lfloor 2(\log |G|/ \log n) \rceil - 2$ and $b(G)$ is unbounded. As a corollary we show that a primitive permutation group of degree n that does not contain the alternating group $\mathrm{Alt}(n)$ has a base of size at most $\max\{\sqrt{n},\ 25\}$.

1. INTRODUCTION

Let G be a permutation group acting on a finite set Ω of size n. A subset Σ of Ω is called a *base* for G if the pointwise stabilizer of Σ in G is trivial. The minimal size of a base for G on Ω is denoted by $b_\Omega(G)$ or by $b(G)$ in case Ω is clear from the context.

The minimal base size of a primitive permutation group has been much investigated. Already in the nineteenth century Bochert [6] showed that $b(G) \leq n/2$ for a primitive permutation group G of degree n not containing $\mathrm{Alt}(n)$. This bound was substantially improved by Babai to $b(G) < 4\sqrt{n} \log n$, for uniprimitive groups G, in [2], and to the estimate $b(G) < 2^{c\sqrt{\log n}}$ for a universal constant c, for doubly transitive groups G not containing $\mathrm{Alt}(n)$, in [3]. (Here and throughout the paper the base of the logarithms is 2 unless otherwise stated.) The latter bound was improved by Pyber [30] to $b(G) < c(\log n)^2$ where c is a universal constant. These estimates are elementary in the sense that their proofs do not require the Classification of Finite Simple Groups (CFSG). Using CFSG, Liebeck [23] classified all primitive permutation groups G of degree n with $b(G) \geq 9 \log n$.

It is easy to see that any permutation group G of degree n satisfies $|G| < n^{b(G)}$, and hence $b(G) > \log |G|/ \log n$. A well-known question of Pyber [31, Page 207], going back to 1993, asks whether there exists a universal constant c such that $b(G) < c(\log |G|/ \log n)$ for all *primitive* groups G. This question generalizes other conjectures in the area: for example, the Cameron-Kantor conjecture, which asserts that every almost simple primitive group in a non-standard action has base size bounded by a universal constant C; and Babai's conjecture, that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that any primitive group that has no alternating or classical composition factor of degree or dimension greater than d, has base size less than $f(d)$.

The Cameron-Kantor conjecture was proved in [25] (and in a strong form with $C = 7$ in [7], [10]). Babai's conjecture was proved in [17] with f a quadratic function (improved to a linear function in [25]).

Despite a great deal of attention, Pyber's conjecture remained open until very recently, when it was proved in [13]. It is shown in [13] that there exists a universal constant $c > 0$ such that for every primitive permutation group G of degree n we have

$$b(G) < 45(\log |G| / \log n) + c.$$

To obtain a more explicit, usable bound, one would like to reduce the multiplicative constant 45 in the above, and also to estimate the constant c.

In this paper we achieve this aim. Our main result is the following.

Theorem 1.1. *Let G be a primitive permutation group of degree n. Then the minimal base size $b(G)$ satisfies*

$$b(G) \leq 2\frac{\log |G|}{\log n} + 24.$$

The multiplicative constant 2 in Theorem 1.1 is best possible, as is shown by the following.

Proposition 1.2.

(i) *For every positive integer k there exists a sequence of finite primitive permutation groups G_n of degrees n such that as $n \to \infty$,*

$$(b(G_n) \log n) / \log |G_n| \to 2k/(k+1).$$

(ii) *There is an infinite sequence of primitive permutation groups H_n of degrees n such that $b(H_n) = \lfloor 2(\log |H_n| / \log n) \rfloor - 2$ for all n and $b(H_n)$ is unbounded.*

A corollary of Theorem 1.1 and its proof is the following.

Corollary 1.3. *Let G be a primitive permutation group of degree n not containing $\mathrm{Alt}(n)$. Then G has a base of size at most $\max\{\sqrt{n}, 25\}$.*

Theorem 1.1 is proved for almost simple groups in the next two sections (see Theorems 2.1, 3.1 and 4.2): alternating and symmetric groups are handled in §2, and classical groups in §3. The remaining non-affine primitive groups are covered in §4 (see Theorem 4.1), and affine groups in §5 (Theorem 5.1). Proposition 1.2 follows from Proposition 2.6 and Proposition 5.3. Finally, Corollary 1.3 is proved in Section 6.

2. Alternating and symmetric groups

In this section we consider the minimal base sizes of alternating and symmetric groups in primitive actions. Here is the main result.

Theorem 2.1. *Let G be a primitive permutation group of degree n with socle isomorphic to $\mathrm{Alt}(m)$ for some integer $m \geq 5$. Then*

$$b(G) \leq 2\frac{\log |G|}{\log n} + 16.$$

In the proof of Theorem 2.1, we may assume that $19 \leq b(G) \leq \log |G|$. In particular $m \geq 7$ and $G = \mathrm{Alt}(m)$ or $\mathrm{Sym}(m)$.

Let Ω be a set of size n permuted by G, let $\alpha \in \Omega$ and let $H = G_\alpha$, a maximal subgroup of G. There are three possiblities to consider, according to the action of H on the underlying set $\{1, \ldots, m\}$:

(1) H is intransitive: here $H = (\mathrm{Sym}(k) \times \mathrm{Sym}(m-k)) \cap G$ for some $k \le m/2$;
(2) H is transitive and imprimitive: here $H = (\mathrm{Sym}(b) \wr \mathrm{Sym}(a)) \cap G$, where $m = ab$;
(3) H is primitive on $\{1, \ldots, m\}$.

In case (1), the action of G on Ω is the action on k-element subsets of $\{1, \ldots, m\}$, and in case (2) the action is on partitions into a parts of size b. These actions are considered in Sections 2.1 and 2.2, and the proof of Theorem 2.1 is completed in Section 2.3.

2.1. **Action on subsets.** Here we prove Theorem 2.1 in the case when the action is on subsets (see Proposition 2.5). Let $\mathrm{Sym}(m)$ act on the set $\Omega(m, k)$ of all k-element subsets of the set $\{1, \ldots, m\}$, where $k \le m/2$. Set $n = |\Omega(m, k)| = \binom{m}{k}$. Let $b(m, k)$ denote the minimal size of a base for $\mathrm{Sym}(m)$ acting on $\Omega(m, k)$. For convenience set $t = m/k$.

A detailed study of the function $b(m, k)$ was carried out in [20]. Here are the main results from that paper that we need.

Theorem 2.2. ([20, Thm. 3.2, Cor. 4.3])

(i) *We have* $b(m, k) \le \left\lceil \log_{\lceil t \rceil}(m) \right\rceil (\lceil t \rceil - 1)$.
(ii) *If* $k^2 \le m$, *then*
$$b(m, k) = \left\lceil \frac{2m - 2}{k + 1} \right\rceil < \frac{2m}{k + 1} + 1 = \frac{2k}{k + 1} t + 1.$$

We shall need the following estimates for $\ln |\mathrm{Sym}(m)| / \ln |\Omega(m, k)|$.

Lemma 2.3. *We have*
$$\left(\frac{t}{\ln(t) + 1} \right)(\ln m - 1) < \frac{\ln |\mathrm{Sym}(m)|}{\ln |\Omega(m, k)|} < \left(\frac{t}{\ln(t)} \right) \ln m.$$

Proof. By the inequalities
$$(m/k)^k < \binom{m}{k} < (me/k)^k \text{ and } (m/e)^m < m! < m^m,$$
we have
$$\frac{m(\ln m - 1)}{k(\ln(m/k) + 1)} < \frac{\ln |\mathrm{Sym}(m)|}{\ln |\Omega(m, k)|} < \frac{m \ln m}{k \ln(m/k)} = \frac{m/k}{\ln(m/k)} \ln m.$$
From this the lemma follows. $\qquad\square$

The next result establishes the conclusion of Theorem 2.1 under the assumption that $k^2 \le m$.

Lemma 2.4. *Assume that* $k^2 \le m$. *Then*
$$b(m, k) < 2 \frac{\ln |\mathrm{Sym}(m)|}{\ln |\Omega(m, k)|} + 4.$$

Proof. Assume first that $k \ge 8 > e^2$. By Theorem 2.2(ii) and Lemma 2.3, we have
$$\frac{b(m, k) \ln n}{\ln |\mathrm{Sym}(m)|} < \left(\frac{2t + 1}{t} \right) \left(\frac{\ln(t) + 1}{\ln(m) - 1} \right).$$
Since $k \ge 8 > e^2$, it follows that $\frac{\ln(t) + 1}{\ln(m) - 1} < 1$. By this and Lemma 2.3,
$$b(m, k) < 2 \frac{\ln |\mathrm{Sym}(m)|}{\ln n} + \left(\frac{\ln(m)}{\ln(m) - 1} \right) \left(\frac{\ln(t) + 1}{\ln(t)} \right).$$
It is easy to see that the second term is less than 4, giving the conclusion in this case $(k \ge 8 > e^2)$.

Hence we may assume that $k \leq 7$. A GAP [15] computation shows that the bound in the conclusion of the lemma holds for $5 \leq m \leq 148 < e^5$. Thus assume also that $m \geq 149 > e^5$.

If $2 \leq k \leq 7$ then Theorem 2.2 gives $b(m,k) < \frac{2k}{k+1}t + 1$, and so by Lemma 2.3,

$$\frac{b(m,k)\ln n}{\ln |\mathrm{Sym}(m)|} < \Big(\frac{2k}{k+1} + \frac{k}{m}\Big)\Big(\frac{\ln m - \ln k + 1}{\ln m - 1}\Big).$$

This is less than 2 for $m \geq 149 > e^5$. □

Here is the main result of this subsection.

Proposition 2.5. *We have*

$$b(m,k) \leq 2\frac{\ln |\mathrm{Sym}(m)|}{\ln |\Omega(m,k)|} + 16.$$

Proof. By Lemma 2.4, we may assume that $k^2 > m$, which is equivalent to saying that $t^2 < m$.

Define r to be the integer $r \geq 2$ with $t^r < m \leq t^{r+1}$. Then by Theorem 2.2(i), we have $b(m,k) \leq (r+1)t$. By Lemma 2.3, this gives

$$\frac{b(m,k)\ln n}{\ln m!} < \frac{(r+1)(\ln t + 1)}{r \ln t - 1}.$$

A GAP [15] computation shows that the right hand side is less than 2 provided that $r = 2$ and $t \geq 149 > e^5$, or $r = 3$ and $t \geq 20 > e^3$, or $r \geq 4$ and $t \geq 11$.

If $r = 3$ and $t \leq 20 < e^3$, then $4t - 2(\frac{3\ln t - 1}{\ln t + 1})t \leq 16$, which gives the conclusion (using Lemma 2.3). Similarly, if $r \geq 4$ and $t < 11$, then $(r+1)t - 2(\frac{r\ln t - 1}{\ln t + 1})t \leq 11$, giving the conclusion.

This leaves the case where $r = 2$ and $t \leq 148 < e^5$. We first distinguish eleven different cases according to some possible ranges of values of $\ln t$. If $\ln t$ falls in any of the intervals $[\epsilon, \epsilon + 0.2]$ where $\epsilon = 2.8 + 0.2\ell$ and ℓ is a non-negative integer at most 10, then $3t - 2(\frac{2\ln t - 1}{\ln t + 1})t \leq 16$. Thus we may assume that $t < e^{2.8}$. But then $m \leq t^3 < e^{8.4} < 4500$. By a GAP [15] calculation, we see that if $5 \leq m \leq 4500$, then $3t - 2(\frac{2\ln t - 1}{\ln t + 1})t \leq 11$. This completes the proof. □

The final result of this subsection gives the first part of Proposition 1.2.

Proposition 2.6. *Fix a positive integer k. Then as $m \to \infty$,*

$$\frac{b(m,k)\log |\Omega(m,k)|}{\log |\mathrm{Sym}(m)|} \to 2k/(k+1).$$

Proof. Assume that $m \geq k^2$. Then, by Theorem 2.2(ii), $b(m,k) = \left\lceil \frac{2m-2}{k+1} \right\rceil$, and hence $b(m,k)/m \to \frac{2}{k+1}$ as $m \to \infty$. Also $(m\ln |\Omega(m,k)|/\ln |\mathrm{Sym}(m)|) \to k$ by Lemma 2.3. The result follows. □

2.2. **Action on partitions.** Now consider the minimal base size $f(a,b)$ of the group $\mathrm{Sym}(m)$ acting on the set Ω of all partitions of $\{1, \ldots, m\}$ into a parts each of size b, where $m = ab$ and $a, b \geq 2$. In this case $n = |\Omega| = m!/(b!^a a!)$. Bases for this action were studied in [5], where the following was proved.

Theorem 2.7. ([5]) *Suppose $b \geq 3$. Then one of the following holds:*

   (i) $a \geq b$ and $f(a,b) \leq 6$;
   (ii) $a < b$ and $f(a,b) \leq \log_a(b) + 4$.

We shall need the following bound.

Lemma 2.8. *Let a, b be integers with $2 \leq a < b$. Then*

$$\frac{\ln b}{\ln a} - 1 < \frac{\ln((ab)!)}{\ln\left(\frac{(ab)!}{(b!)^a a!}\right)}.$$

Proof. Write $g(a, b) = \ln((ab)!) / \ln\left(\frac{(ab)!}{(b!)^a a!}\right)$. Then using the bounds

$$\sqrt{2\pi} \cdot \ell^{1/2}\left(\frac{\ell}{e}\right)^\ell < \ell! < e \cdot \ell^{1/2}\left(\frac{\ell}{e}\right)^\ell$$

which hold for all positive integers ℓ, we have

$$
\begin{aligned}
g(a, b) &> \frac{ab(\ln(ab) - 1)}{\ln((ab)!) - a\ln(b!) - \ln(a!)} \\
&> \frac{ab(\ln(ab) - 1)}{\ln(e/\sqrt{2\pi}) + ab(\ln a) + \frac{1}{2}\ln(ab) - a\ln(\sqrt{2\pi}) - \frac{1}{2}a\ln b - \frac{1}{2}\ln a - a\ln a + a} \\
&= \frac{\ln(ab) - 1}{\ln a + \frac{1}{b}(1 - \ln a - \frac{1}{2}\ln(2\pi)) + \ln(e/\sqrt{2\pi})/(ab) + \frac{1}{2b}((\ln(b))/a - \ln b)} \\
&\geq \frac{\ln(ab) - 1}{\ln a + \frac{1}{b}(1 - \ln a - \frac{1}{2}\ln(2\pi) - \frac{\ln b}{4}) + \ln(e/\sqrt{2\pi})/(ab)} \geq \\
&\geq \frac{\ln(ab) - 1}{\ln a + \frac{1}{b}(1 - \ln 2 - \frac{1}{2}\ln(2\pi) - \frac{\ln 3}{4} + \ln(e/\sqrt{2\pi})/2)} \\
&> \frac{\ln(ab) - 1}{\ln a - (0.8)/b} > \frac{\ln b}{\ln a} - 1.
\end{aligned}
$$

$\square$

Here is the main result of this subsection.

Proposition 2.9. *With the above notation, we have $f(a, b) \leq \frac{\ln|\mathrm{Sym}(m)|}{\ln n} + 5$.*

Proof. If $b \geq 3$, this follows immediately from Theorem 2.7. And for $b = 2$, Remark 1.6(ii) of [8] gives $f(a, 2) \leq 3$. $\square$

2.3. Proof of Theorem 2.1. Let $G = \mathrm{Alt}(m)$ or $\mathrm{Sym}(m)$ act primitively on a set Ω, and let H be a point-stabilizer in G. The cases where Ω is a set of k-subsets or partitions of $\{1, \ldots, m\}$ have been dealt with in Propositions 2.5 and 2.9. Hence by the remarks at the beginning of the section, we may assume that H is primitive on $\{1, \ldots, m\}$. In this case, it is proved in [8, Cor. 2] that $b_\Omega(G) \leq 5$. This completes the proof of Theorem 2.1.

3. Classical groups

In this section we study base sizes of primitive actions of classical groups. Our main result is the following.

Theorem 3.1. *Let G be an almost simple primitive permutation group of degree n whose socle is a classical simple group. Then $b(G) \leq 2(\log|G|/\log n) + 16$.*

We shall divide the proof of this theorem into several subcases. First we give a definition, taken from [25]. Let G be an almost simple group with socle G_0, a classical group with natural module V, a vector space of dimension d over a field $\mathbb{F}_q$ of characteristic p. We call a maximal subgroup M of G a *subspace subgroup* if it is reducible on V, or is an orthogonal group on V embedded in a symplectic group with $p = 2$; more specifically, M is a subspace subgroup if one of the following holds:

(1) $M = G_U$ for some proper nonzero subspace U of V, where U is totally singular, non-degenerate, or, if G is orthogonal and $p = 2$, a nonsingular 1-space (U is any subspace if $G_0 = PSL(V)$);

(2) $G_0 = PSL(V)$, G contains a graph automorphism of G_0, and $M \cap G_0 = (G_0)_{U,W}$ where U, W are proper nonzero subspaces of V, $\dim V = \dim U + \dim W$ and either $U \subseteq W$ or $V = U \oplus W$;

(3) $G_0 = Sp_{2m}(q)$, $p = 2$ and $M \cap G_0 = O_{2m}^{\pm}(q)$.

Note that in (3), if we regard G_0 as the isomorphic orthogonal group $O_{2m+1}(q)$, then $M \cap G_0 = O_{2m}^{\pm}(q)$ is the stabilizer of a hyperplane of the natural module of dimension $2m + 1$.

If M is a subspace subgroup, we call the action of G on the coset space G/M a *subspace action*.

Bases for non-subspace actions of classical groups were studied in detail in [7], so our main task is to prove Theorem 3.1 for subspace actions. First we require the following general bound.

Proposition 3.2. *Let G be as above, and suppose M is as in* (1)*, so that the coset space $X = G/M$ is a G-orbit of k-dimensional subspaces of V, for some k. Assume also that $k \le d/2$. Then*

$$\frac{\log |G|}{\log |X|} \ge \frac{d}{tk} - 1,$$

where $t = 1$ if $G_0 = PSL(V)$, and $t = 2$ otherwise.

Proof. Observe that $|G| > q^{(d^2/t)-d}$, while

$$|X| \le \binom{d}{k}_q := \frac{(q^d - 1)(q^d - q) \cdots (q^d - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} \le \left(\frac{q^d}{q^{k-1}}\right)^k = q^{dk - k^2 + k}.$$

Hence

$$\frac{\log |G|}{\log |X|} \ge \frac{(d^2/t) - d}{dk - k^2 + k} \ge \frac{d}{tk} - 1.$$

$\square$

3.1. **Action on an orbit of subspaces.** In this subsection we prove Theorem 3.1 for subspace actions of classical simple groups as in case (1) in the list above. This is the main part of the proof of the theorem.

Theorem 3.3. *Let G be a simple classical group on V, a vector space of dimension d over $\mathbb{F}_q$. Let X be a G-orbit of k-dimensional subspaces of V with $k \le d/2$, on which G acts primitively. Then*

$$b_X(G) \le \frac{d}{k} + 11.$$

Proof. In every subcase, we will define a base $\mathcal{B} \subset X$ of G in a number of steps. We do this by starting with $\mathcal{B} = \emptyset$ and at each step adding some subspaces to $\mathcal{B}$. Throughout the proof, $G_{(\mathcal{B})}$ denotes the pointwise stabilizer of $\mathcal{B}$ – that is, the set of group elements that fix all the subspaces in $\mathcal{B}$.

Action on the set of all k-dimensional subspaces.

First, let us assume that X is the set of all k-dimensional subspaces. Let $d = ak + r$ for $a \ge 2$ and $0 \le r < k$. Take any direct sum decomposition $V = V_1 \oplus \ldots \oplus V_a \oplus U$ with $\dim V_i = k$ for $1 \le i \le a$, and let $V_1, \ldots, V_a \in \mathcal{B}$. Fix a basis $B_i = \{x_1^{(i)}, \ldots, x_k^{(i)}\} \in V_i$ for each i and define $W_1 = \langle \sum_{i=1}^a x_s^{(i)} \mid 1 \le s \le k \rangle \in X$ and put W_1 into $\mathcal{B}$. Then the matrix form of the restriction of a $g \in G_{(\mathcal{B})}$ to $V_1 \oplus \ldots \oplus V_a$ is a block diagonal matrix with equal blocks (with respect to the basis $B_1 \cup \ldots \cup B_a$). Define $M_g = [g_{V_1}]_{B_1} = [g_{V_2}]_{B_2} = \ldots =$

$[g_{V_a}]_{B_a}$ and let $\{\gamma, \delta\} \subset SL(V_2)$ be a generating set of $SL(V_2)$ and $C = [\gamma]_{B_2}$, $D = [\delta]_{B_2}$ be their matrix forms. Then $g \in G_{(\mathcal{B})}$ fixes the subspaces

$$W_2 = \langle x_s^{(1)} + \gamma(x_s^{(2)}) \mid 1 \leq s \leq k \rangle \in X, \quad W_3 = \langle x_s^{(1)} + \delta(x_s^{(2)}) \mid 1 \leq s \leq k \rangle \in X$$

if and only if M_g commutes with both C and D. Thus, putting W_2 and W_3 into $\mathcal{B}$, it follows that $G_{(\mathcal{B})}$ acts as scalars on $V_1 \oplus \ldots \oplus V_a$. Finally, if $r > 0$ then let $\{f_1, \ldots, f_r\}$ be a basis of U and define

$$W_4 = \langle f_1, \ldots, f_r, x_{r+1}^{(1)}, \ldots, x_k^{(1)} \rangle, \qquad W_5 = \langle f_1 + x_1^{(2)}, \ldots, f_r + x_r^{(2)}, x_{r+1}^{(2)}, \ldots, x_k^{(2)} \rangle.$$

Adding W_4 and W_5 to $\mathcal{B}$ it is easy to see that $G_{(\mathcal{B})}$ contains only scalar transformations. Thus, $b_X(G) \leq a + 5 \leq \frac{d}{k} + 5$ for this case.

Action on an orbit of non-degenerate subspaces.

Now we turn to the case when G is a group fixing some non-degenerate form $[\,,\,]$ on V and X is a G-orbit of non-degenerate subspaces. In the special case $d = 2k$, we also assume that the Witt index of elements of X is no more than the Witt index of elements of $X^\perp$ (this is only interesting in the orthogonal case, when k is even and V has Witt index $k - 1$). This will guarantee that the sums defining u_i and v_i below will have at least two terms.

Let $d = ak + r$ with $1 \leq r \leq k$ and take any orthogonal decomposition $V = V_1 \oplus \ldots \oplus V_a \oplus V_{a+1}$ with $V_1, \ldots, V_a \in X$. Put $V_1, \ldots, V_a$ into $\mathcal{B}$. Then any $g \in G_{(\mathcal{B})}$ also fixes $V_{a+1} = (\sum_{i=1}^{a} V_i)^\perp$.

Let l and $m \leq 2$ denote the Witt index and the Witt defect of the subspaces in X, so $k = 2l + m$. First, let us assume that $l \geq 1$. Then for every $1 \leq s \leq a$, the subspace V_s is a direct sum of orthogonal subspaces $V_s = V_s^{(h)} \oplus V_s^{(m)}$, where each $V_s^{(h)}$ contains a basis $B_s = \{x_1^{(s)}, \ldots, x_l^{(s)}, y_1^{(s)}, \ldots, y_l^{(s)}\}$ with $[x_i^{(s)}, x_j^{(s)}] = [y_i^{(s)}, y_j^{(s)}] = 0$, $[x_i^{(s)}, y_j^{(s)}] = \delta_{ij}$ for all i, j and $V_s^{(m)}$ has dimension and Witt defect m. (For orthogonal groups of characteristic 2, we also have $Q(x_i^{(s)}) = Q(y_i^{(s)}) = 0$ for all i, where Q is the underlying quadratic form.) Furthermore, let l' be the minimum of l and the Witt index of V_{a+1} and $m' = \dim(V_{a+1}) - 2l'$. Similarly to the above, choose an orthogonal decomposition $V_{a+1} = V_{a+1}^{(h)} \oplus V_{a+1}^{(m)}$ along with a basis $B_{a+1} = \{x_1^{(a+1)}, \ldots, x_{l'}^{(a+1)}, y_1^{(a+1)}, \ldots, y_{l'}^{(a+1)}\}$ of $V_{a+1}^{(h)}$ satisfying $[x_i^{(a+1)}, x_j^{(a+1)}] = [y_i^{(a+1)}, y_j^{(a+1)}] = 0$, $[x_i^{(a+1)}, y_j^{(a+1)}] = \delta_{ij}$ for $1 \leq i, j \leq l'$, and define $x_i^{(a+1)} = y_i^{(a+1)} = 0$ for $l' < i \leq l$.

For $1 \leq i \leq l$, define

$$u_i = \sum_{s=1}^{a+1} x_i^{(s)}, \; v_i = \sum_{s=1}^{(a+1)} y_i^{(s)}.$$

We define the subspaces

$$W_1^{(h)} = \langle u_1, \ldots, u_l, y_1^{(1)}, \ldots, y_l^{(1)} \rangle, \qquad W_2^{(h)} = \langle x_1^{(1)}, \ldots, x_l^{(1)}, v_1, \ldots, v_l \rangle,$$
$$W_3^{(h)} = \langle u_1, \ldots, u_l, y_1^{(2)}, \ldots, y_l^{(2)} \rangle, \qquad W_4^{(h)} = \langle x_1^{(2)}, \ldots, x_l^{(2)}, v_1, \ldots, v_l \rangle.$$

Then $W_t := W_t^{(h)} \oplus V_1^{(m)} \in X$ for each $1 \leq t \leq 4$. Adding W_1, W_2, W_3, W_4 to $\mathcal{B}$, we see that any $g \in G_{(\mathcal{B})}$ fixes each $V_s^{(h)}$ and, moreover, the matrix form of each restriction $g_{V_s^{(h)}}$ satisfies

$$\left[g_{V_1^{(h)}}\right]_{B_1} = \left[g_{V_2^{(h)}}\right]_{B_2} = \ldots = \left[g_{V_a^{(h)}}\right]_{B_a} = \begin{pmatrix} A_g & 0 \\ 0 & B_g \end{pmatrix}$$

for some $A_g, B_g \in GL(l, q)$. The use of the $x_i^{(a+1)}, y_i^{(a+1)}$ as summands in the u_i and v_i also implies that

$$\left[g_{V_{a+1}^{(h)}} \right]_{B_{a+1}} = \begin{pmatrix} A_g' & 0 \\ 0 & B_g' \end{pmatrix},$$

where A_g' and B_g' are left upper $l' \times l'$ submatrices of A_g and B_g, respectively.

Adding also the subspace $W_5 = W_5^{(h)} \oplus V_1^{(m)}$ with $W_5^{(h)} := \langle x_i^{(1)}, y_i^{(1)} + x_i^{(2)} \mid 1 \le i \le l \rangle$ we can also guarantee that $A_g = B_g$ holds for any $g \in G_{(\mathcal{B})}$.

Let $B_2^{(x)} = \{x_1^{(2)}, \dots, x_l^{(2)}\}$ and $V_2^{(x)}$ be the subspace spanned by $B_2^{(x)}$. Choose $\varphi, \psi \in \mathrm{End}(V_2^{(x)})$ generating $\mathrm{End}(V_2^{(x)})$ as an algebra. Define

$$W_6^{(h)} = \langle x_i^{(1)} + \varphi(x_i^{(2)}), y_i^{(1)} \mid 1 \le i \le l \rangle, \quad W_7^{(h)} = \langle x_i^{(1)} + \psi(x_i^{(2)}), y_i^{(1)} \mid 1 \le i \le l \rangle.$$

and let $W_6 := W_6^{(h)} \oplus V_1^{(m)}$, $W_7 := W_7^{(h)} \oplus V_1^{(m)}$. Adding W_6, W_7 to $\mathcal{B}$, we see that $g_{V_2^{(x)}}$ commutes with both φ and ψ for any $g \in G_{(\mathcal{B})}$. Thus $g_{V_2^{(x)}}$ is a scalar transformation. Therefore, any $g \in G_{(\mathcal{B})}$ acts as a scalar on $V_1^{(h)} \oplus \dots \oplus V_{a+1}^{(h)}$.

For every $1 \le s \le a+1$, choose other orthogonal decompositions $V_s = V_s^{(h)'} \oplus V_s^{(m)'}$ such that each $V_s^{(h)'}$ is isometric to $V_s^{(h)}$ and $V_s^{(h)} + V_s^{(h)'} = V_s$. Applying similar constructions as for $W_1, \dots, W_4$ before, by adding 4 further subspaces to $\mathcal{B}$ we can synchronize the action of any $g \in G_{(\mathcal{B})}$ on each $V_s^{(h)'}$ with its action on each $V_t^{(h)}$. Thus, now each $g \in G_{(\mathcal{B})}$ acts as a scalar on the whole vector space $V_1 \oplus \dots \oplus V_{a+1}$. Thus, $b_X(G) \le a + 7 + 4 \le \frac{d}{k} + 11$ in this case.

Now, let us assume that $l = 0$, so $k = m \le 2$. The case $k = 1$ is trivial. The case $k = m = 2$ implies that V is orthogonal. We can also assume that $a \ge 3$, since otherwise $d \le 6$. Then each V_s has a basis $x^{(s)}, y^{(s)}$ with $Q(x^{(s)}) = 1$, $Q(y^{(s)}) = \alpha$, $[x^{(s)}, y^{(s)}] = 1$, where $\alpha \in \mathbb{F}_q$ is such that the polynomial $t^2 + t + \alpha$ is irreducible over $\mathbb{F}_q$. Additionally, choose an arbitrary spanning set $\{x^{a+1}, y^{a+1}\}$ of V_{a+1}. Since $\{Q(z) \mid z \in V_s\} = \mathbb{F}_q$, we can define

$$u_1 = \sum_{s=2}^{a+1} x^{(s)} + z^{(1)}, \quad v_1 = \sum_{s=2}^{a+1} y^{(s)} + w^{(1)}, \quad u_2 = \sum_{s=1}^{a-1} x^{(s)} + z^{(a)}, \quad v_2 = \sum_{s=1}^{a-1} y^{(s)} + w^{(a)}$$

with $z^{(1)}, w^{(1)} \in V_1$, $z^{(a)}, w^{(a)} \in V_a$ such that $Q(u_1) = Q(u_2) = 1$, $Q(v_1) = Q(v_2) = \alpha$. Now, let

$$W_1 = \langle u_1, y^{(2)} \rangle, \quad W_2 = \langle u_1, y^{(3)} \rangle, \quad W_3 = \langle x^{(2)}, v_1 \rangle, \quad W_4 = \langle x^{(3)}, v_1 \rangle,$$
$$W_5 = \langle u_2, y^{(1)} \rangle, \quad W_6 = \langle u_2, y^{(2)} \rangle, \quad W_7 = \langle x^{(1)}, v_2 \rangle, \quad W_8 = \langle x^{(2)}, v_2 \rangle.$$

Adding each W_i to $\mathcal{B}$ we see that the restriction of any $g \in G_{(\mathcal{B})}$ to the subspaces $V_1, \dots, V_a$ has matrix form

$$\left[g_{V_1} \right]_{\{x^{(1)}, y^{(1)}\}} = \dots = \left[g_{V_a} \right]_{\{x^{(a)}, y^{(a)}\}} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \text{ for some } c, d \in \mathbb{F}_q.$$

Using that $Q(g(x^s)) = 1$, $Q(g(y^{(s)})) = \alpha$, $[g(x^{(s)}, g(y^{(s)})] = 1$ it follows that $c = m = \pm 1$, so any $g \in G_{(\mathcal{B})}$ acts on $V_1 \oplus \dots \oplus V_a$ as a scalar transformation. The use of $x^{(a+1)}, y^{(a+1)}$ guarantees that $g \in G_{(\mathcal{B})}$ is a scalar on the whole of V. Thus, $b_X(G) \le \frac{d}{k} + 8$ for this case.

Action on an orbit of totally singular subspaces.

From now on, let X be the set of k-dimensional totally singular subspaces of V. Again, we can assume that $k \ge 2$.

Let l be the Witt index and let $m \leq 2$ be the Witt defect of V, so $k \leq l$ (since otherwise $X = \emptyset$) and $d = 2l + m$. Let $l = ak + r$ for $0 \leq r < k$ and denote $w(s) = k$ for $1 \leq s \leq k$ and $w(a+1) = r$. Take an orthogonal decomposition $V = V_1 \oplus \ldots \oplus V_a \oplus V_{a+1} \oplus U$ such that V_s has dimension $2w(s)$ and Witt index $w(s)$ for each $1 \leq s \leq a+1$. For every $1 \leq s \leq a+1$ let $B_s = \{x_1^{(s)}, \ldots, x_{w(s)}^{(s)}, y_1^{(s)}, \ldots, y_{w(s)}^{(s)}\}$ be a basis of V_s such that $V_s^{(x)} = \langle x_1^{(s)}, \ldots, x_{w(s)}^{(s)} \rangle$, $V_s^{(y)} = \langle y_1^{(s)}, \ldots, y_{w(s)}^{(s)} \rangle$ are $w(s)$-dimensional singular subspaces, and $[x_i^{(s)}, y_j^{(s)}] = \delta_{ij}$ for every $1 \leq i, j \leq w(s)$. Furthermore, define $x_i^{(a+1)} = y_i^{(a+1)} = 0$ for $r < i \leq k$. Finally, take the additional k-dimensional singular subspaces

$$V_{a+1}^{(x)'} = \langle x_1^{(a+1)}, \ldots, x_r^{(a+1)}, x_{r+1}^{(1)}, \ldots, x_k^{(1)} \rangle,$$
$$V_{a+1}^{(y)'} = \langle y_1^{(a+1)}, \ldots, y_r^{(a+1)}, y_{r+1}^{(1)}, \ldots, y_k^{(1)} \rangle.$$

Let $u_i = \sum_{s=1}^{(a+1)} x_i^{(s)}$, $v_i = \sum_{s=1}^{(a+1)} y_i^{(s)}$ for $1 \leq i \leq k$ and define

$$W_1 = \langle u_1, \ldots, u_k \rangle, \qquad W_2 = \langle v_1, \ldots, v_k \rangle.$$

First, add each of the subspaces $V_1^{(x)}, V_1^{(y)}, \ldots, V_a^{(x)}, V_a^{(y)}, V_{a+1}^{(x)'}, V_{a+1}^{(y)'}, W_1, W_2$ to $\mathcal{B}$. Then the subspaces V_s for each $1 \leq s \leq a+1$ are fixed by any $g \in G_{(\mathcal{B})}$ and the restrictions g_{V_s} have matrix form

$$\left[g_{V_1}\right]_{B_1} = \ldots = \left[g_{V_a}\right]_{B_a} = \begin{pmatrix} A_g & 0 \\ 0 & (A_g)^{-T} \end{pmatrix},$$

$$\left[g_{V_{a+1}}\right]_{B_{a+1}} = \begin{pmatrix} A_g' & 0 \\ 0 & (A_g')^{-T} \end{pmatrix},$$

where $A_g \in GL(k, q)$ and A_g' is the left upper $r \times r$ submatrix of A_g.

Next, we define additional k-dimensional singular subspaces of the form

$$W^{(x)}(C) = \Big\langle \sum_{s=1}^{a+1} \Big(x_j^{(s)} + \sum_{i=1}^{k} c_{ij} y_i^{(s)} \Big) \,\Big|\, 1 \leq j \leq k \Big\rangle,$$

$$W^{(y)}(C) = \Big\langle \sum_{s=1}^{a+1} \Big(y_j^{(s)} + \sum_{i=1}^{k} c_{ij} x_j^{(s)} \Big) \,\Big|\, 1 \leq j \leq k \Big\rangle,$$

where $C = (c_{ij}) \in M(k, q)$. The subspaces $W^{(x)}(C)$ and $W^{(y)}(C)$ are singular if the matrix C is symmetric (when V is a symplectic space) or anti-symmetric (when V is an orthogonal or a unitary space). Furthermore, $g \in G_{(\mathcal{B})}$ fixes $W^{(x)}(C)$ (resp. $W^{(y)}(C)$) if and only if $A_g^T C = C A_g^{-1}$ (resp. $A_g C = C A_g^{-T}$) holds.

First, let us assume that V is a symplectic space and choose $C, D \in M(k, q)$ symmetric matrices, which generate the full matrix algebra $M(k, q)$ (as an algebra). Adding $W^{(y)}(I), W^{(y)}(C), W^{(y)}(D)$ to $\mathcal{B}$ we see that any $g \in G_{(\mathcal{B})}$ satisfies $A_g = A_g^{-T}$, and, therefore, $A_g C = C A_g$, $A_g D = D A_g$. It follows that A_g is a scalar matrix for any $g \in G_{(\mathcal{B})}$. Thus, g acts as a scalar on the whole $V = V_1 \oplus \ldots \oplus V_a \oplus V_{a+1}$.

Now, let V be an orthogonal or unitary space and choose antisymmetric matrices $C = E_{12} - E_{21}$, $D = \sum_{i=2}^{k-1}(E_{i,i+1} - E_{i+1,i})$. (Here $\{E_{ij} \,|\, 1 \leq i, j \leq k\}$ denotes the usual basis of the full matrix algebra $M(k, q)$.) Add the subspaces $W^{(x)}(C), W^{(y)}(C), W^{(x)}(D), W^{(y)}(D)$ to $\mathcal{B}$ and let $g \in G_{(\mathcal{B})}$. Then we have $A_g C A_g^T = A_g^T C A_g = C$ and $A_g D A_g^T = A_g^T D A_g = D$. Using the implication

$$AXA^T = X, \quad A^T Y A = Y \Rightarrow AXY = AXA^T YA = XYA,$$

we see that A_g commutes with every product P of C's and D's with an even number of terms. Similarly, $A_g P A_g^T = A_g^T P A_g = P$ holds for every product P of C's and D's with an odd number of terms. In particular, A_g commutes with $CD = E_{13}$, $-DC =$

$E_{31}, -CD^2C = E_{11}$ etc. Continuing this way, we see that $A_g = \lambda \cdot I$ is a scalar matrix. The equation $A_g C A_g^T = C$ also shows that $\lambda^2 = 1$ and so $A_g = A_g^{-T} = \pm I$. Thus, any $g \in G_{(\mathcal{B})}$ is a scalar transformation on $V_1 \oplus \ldots \oplus V_{a+1}$. If $U \neq 0$, we can choose a k-dimensional singular subspace $V_{a+2}^{(x)} \leq V_1 \oplus U$ satisfying $V_1 + V_{a+2}^{(x)} = V_1 + U$. Let $x_1^{(a+2)}, \ldots, x_k^{(a+2)}$ be any basis of $V_{a+2}^{(x)}$. Adding the subspaces $V_{a+2}^{(x)}$, $\langle x_1^{(a+2)} + y_1^{(1)}, \ldots, x_k^{(a+2)} + y_k^{(1)} \rangle$ to $\mathcal{B}$ gives the result. So, $b_X(G) \leq 2a + 10 \leq \frac{d}{k} + 10$.

The above argument works if X is the set of all totally singular subspaces, which is indeed a G-orbit in most cases. The only exception is when V is an orthogonal space, $d = 2k$, and $G = \Omega^+(V)$, so we assume this from now on. Then two totally singular k-dimensional subspaces V_1, V_2 are in the same G-orbit if and only if $\dim(V_1 \cap V_2) \equiv k$ (mod 2). Since the full orthogonal group $O(V)$ interchanges the two G-orbits, it does not matter, which orbit we choose. Note that in the above construction the subspaces $V_1^{(x)}, V_1^{(y)}, W^{(x)}(C), W^{(y)}(C), W^{(x)}(D), W^{(y)}(D)$ are in the same G-orbit provided that k is even (the further subspaces defined in the proof are now meaningless). So, $b_X(G) \leq 6$ in this case. Now, let k be odd, and choose an orthogonal decomposition $V = \langle x, y \rangle \oplus U$, where x, y is a hyperbolic pair. Then $\dim U = d - 2 = 2(k-1)$, so the above construction works for a G_U-orbit of $(k-1)$-dimensional totally singular subspaces of U, since $k-1$ is even. That is, there are 6 totally singular $(k-1)$-dimensional subspaces $U_1, \ldots, U_6$ of U, which form a base for the action of G_U on U. By construction,

$$U_1 = \langle x_1, \ldots, x_{k-1} \rangle, \quad U_2 = \langle y_1, \ldots, y_{k-1} \rangle$$

with $[x_i, y_j] = \delta_{ij}$ for $1 \leq i, j \leq k-1$. Define the subspaces $V_s = \langle x \rangle \oplus U_s$ for $1 \leq s \leq 6$. Furthermore, let

$$W_1 = \langle y, y_1, x_2, \ldots, x_{k-1} \rangle, \quad W_2 = \langle y, x_1, y_2, \ldots, y_{k-1} \rangle.$$

Then all of $V_1, \ldots, V_6, W_1, W_2$ are totally singular k-dimensional subspaces with pairwise odd-dimensional intersections, so they are in the same G-orbit X. Adding $V_1, \ldots, V_6, W_1$, W_2 to $\mathcal{B}$ we see that any $g \in G_{(\mathcal{B})}$ fixes the subspaces $\langle x \rangle = V_1 \cap V_2$, $\langle y \rangle = W_1 \cap W_2$ and $U = (V_1 + V_2) \cap (W_1 + W_2)$. Furthermore, $g \in G_{(\mathcal{B})}$ also fixes $U_s = V_s \cap U$ for each s, so g_U is a scalar transformation by the definition of the U_s. Adding also the subspace

$$W_3 = \langle x + x_1, y - y_1, x_2, \ldots, x_{k-1} \rangle \in X$$

to $\mathcal{B}$, we get that any $g \in G_{(\mathcal{B})}$ is a scalar transformation on the whole of V. Hence $b_X(G) \leq 9$ in this case. $\qquad\square$

Remark 3.4. By a more detailed argument, Burness, Guralnick and Saxl were able to calculate the exact base size for a classical group over an algebraically closed field acting on an orbit of subspaces of its natural module [9, Section 4]. While part of their constructions could be translated to the finite case, we had to give new constructions for other cases. (This is especially true for the orthogonal case, since, in contrast to the finite case discussed above, in any dimension there is just one type of non-degenerate orthogonal space over an algebraically closed field.)

3.2. **Action on pairs of subspaces.** In this subsection we handle the subspace actions arising from case (2) in the list after the statement of Theorem 3.1.

Proposition 3.5. *Let $G = PSL(V) = PSL_d(q)$, and let M be the stabilizer in G of a pair U, W of nonzero subspaces, where $\dim U = k < d/2$, $\dim W = d - k$, and either $U \subseteq W$ or $V = U \oplus W$. Let X be the coset space G/M. Then*

$$b_X(G) \leq \frac{d}{k} + 11 \leq 2\frac{\log|G|}{\log|X|} + 12.$$

Proof. Let X_k be the set of all k-dimensional subspaces of V. A straightforward computation shows that $|X| < |X_k|^2$. Clearly $b_X(G) \leq b_{X_k}(G)$. Now the result follows from Theorem 3.3 and Proposition 3.2. □

3.3. **Proof of Theorem 3.1.** Let G be an almost simple group with socle G_0, a classical group on V, a vector space of dimension d over $\mathbb{F}_q$. Suppose G acts faithfully and primitively on a set Ω.

If the action of G on Ω is not a subspace action, then $b(G) \leq 5$ by [7]. Hence we may assume that the action is a subspace action, so that one of the cases (1), (2), (3) listed after the statement of Theorem 3.1 holds.

In case (1), $\Omega = U^G$ is an orbit of G on k-dimensional subspaces, for some k, and we can assume that $k \leq d/2$ (by replacing U with $U^\perp$ if necessary, in the case where $G_0 \neq PSL(V)$, and by considering the equivalent action of G on $(d-k)$-spaces, when $G_0 = PSL(V)$). Now Theorem 3.3 and Proposition 3.2 give

$$b(G_0) \leq \frac{d}{k} + 11 \leq 2\frac{\log|G|}{\log|\Omega|} + 13.$$

Hence we can choose a set $\mathcal{B}$ of at most $\frac{d}{k} + 11$ points of Ω such that $G_{(\mathcal{B})} \cap G_0 = 1$, so that $G_{(\mathcal{B})}$ is isomorphic to a subgroup of G/G_0. This is a soluble group possessing a normal series of length at most 3 with cyclic factor groups. Since the base size of a cyclic linear group is 1, by [33], it follows that $b(G) \leq 2\frac{\log|G|}{\log|\Omega|} + 16$, as required.

Now consider case (2): here $G_0 = PSL(V)$ and $\Omega = \{U, W\}^G$ where U, W are subspaces of dimensions k, $d-k$ and either $U \subseteq W$ or $V = U \oplus W$. In the latter case, if $k = d/2$ then G_0 has an element interchanging U and W, and (G, Ω) is not a subspace action (it is a $\mathcal{C}_2$-action in the terminology of [7]). Hence we may assume that $k < d/2$. Now Proposition 3.5 implies that $b(G_0) \leq 2\frac{\log|G|}{\log|\Omega|} + 12$, and this yields the result as above.

Finally, consider case (3): here $G_0 = Sp_{2m}(q)$, $p = 2$ and $M \cap G_0 = O_{2m}^\pm(q)$, where M is a point-stabilizer in G. Regarding G_0 as the isomorphic orthogonal group $O_{2m+1}(q)$, the set Ω is an orbit of G_0 on hyperplanes of the natural module $V_{2m+1}(q)$. Hence $b_\Omega(G_0) \leq 2m+1$, which is less than $2\frac{\log|G|}{\log|\Omega|} + 3$, and the conclusion follows again.

This completes the proof of Theorem 3.1.

4. NON-AFFINE PRIMITIVE PERMUTATION GROUPS

In this section we prove the main Theorem 1.1 for primitive groups which are not of affine type.

Theorem 4.1. *Let G be a primitive permutation group of degree n. Assume that G is not of affine type. Then $b(G) \leq 2(\log|G|/\log n) + 24$.*

According to the O'Nan-Scott theorem (see for example [24]), non-affine primitive groups are of the following types: almost simple, diagonal type, product type, and twisted wreath type. We shall deal with these types separately in the following subsections.

4.1. **Almost simple groups.** For this case we prove

Theorem 4.2. *Let G be an almost simple primitive permutation group of degree n. Then $b(G) \leq 2(\log|G|/\log n) + 16$.*

Proof. Theorems 2.1 and 3.1 give the result when the socle of G is an alternating or classical group. For the remaining cases, the socle of G is a group of exceptional Lie type or a sporadic group. In these cases we have $b(G) \leq 7$ by [10] and [11]. □

4.2. **Diagonal type groups.** Work of Fawcett [14] (and also Gluck, Seress, Shalev [17, Remark 4.3]) implies that, in the diagonal type case, we have

$$b(G) \leq (\log |G| / \log n) + 3. \tag{1}$$

4.3. **Product type groups.** Bases for primitive groups of product type were studied by Burness and Seress in [12]. We will use their notation. Let $\Omega = \Gamma^k$ for some set Γ and integer $k \geq 2$. There exists a primitive group $H \leq \mathrm{Sym}(\Gamma)$ of almost simple type or of diagonal type such that the following holds. Let the socle of H be T. Let P be the (transitive) action of G on the set of the k direct factors of $\mathrm{Soc}(G) = T^k$. We have $T^k \leq G \leq H \wr P$.

We recall two definitions. A *distinguishing partition* for a finite group X acting on a finite set Σ is a coloring of the points of Σ in such a way that every element of X fixing this coloring is contained in the kernel of the action of X on Σ. The minimal number of parts (or colors) of a distinguishing partition is called the *distinguishing number* of X and is denoted by $d(X)$.

Let $d(P)$ be the distinguishing number of the transitive permutation group P. By [13, Theorem 1.2], we have $d(P) \leq 48 \sqrt[k]{|P|}$.

4.3.1. *The case when H is almost simple.* Assume that $H \leq \mathrm{Sym}(\Gamma)$ is an almost simple group with socle T. We follow not only [12] here but [13, §4]. However we avoid the use of the bound $|\mathrm{Out}(T)| \leq |T|^\alpha$, since this is expensive. Instead we use the estimate $|\mathrm{Out}(T)| \leq |\Gamma|$ found in [1, Lemma 2.7]. Thus $|G| \geq |T|^k |P| \geq (|H|^k |P|)/|\Gamma|^k$. This gives $\log(|H|^k |P|) / \log |\Omega| \leq (\log |G| / \log |\Omega|) + 1$.

By using the idea of [12, Lemma 3.8] combined with Lemma 2.1 of [13], we see that

$$b(G) < \frac{\log d(P)}{\log |\Gamma|} + 1 + b(H) < \frac{\log |P|}{\log |\Omega|} + b(H) + 4, \tag{2}$$

since $|\Gamma| \geq 5$. By Theorem 4.2, this gives

$$b(G) < \frac{\log |P|}{\log |\Omega|} + 2 \frac{\log |H|}{\log |\Gamma|} + 20 < 2 \frac{\log(|H|^k |P|)}{\log |\Omega|} + 20 \leq 2 \frac{\log |G|}{\log |\Omega|} + 22.$$

4.3.2. *The case when H is of diagonal type.* Now assume that H is of diagonal type. Here $\mathrm{Soc}(H) = T = S^\ell$, where S is a non-abelian simple group and $\ell \geq 2$. We have $S^\ell \leq H \leq S^\ell.(\mathrm{Out}(S) \times Q)$ where $Q \leq \mathrm{Sym}(\ell)$ is the permutation group induced by the conjugation action of H on the ℓ factors of S^ℓ.

The set Γ can be thought of as the set of right cosets in H of the subgroup $H_0 = (D \times Q) \cap H$ where D denotes the diagonal subgroup of $\mathrm{Aut}(S)^\ell$. In particular, $|\Gamma| = |S|^{\ell-1}$. By (1), $b(H) \leq (\log |H| / \log |\Gamma|) + 3 \leq 8$, provided that $\ell \leq |S|$. Thus, in view of (2), we may assume that $\ell \geq 3$.

Let $\mathcal{C}$ be the set of complete representatives of the right cosets in H of the subgroup H_0 consisting of the elements of S^ℓ where the first coordinate is 1. Let s_1 and s_2 be elements of S such that they together generate S. Let $\gamma_0, \gamma_1, \gamma_2 \in \mathcal{C}$ be those elements for which every coordinate of γ_0 is 1, all but the first coordinate of γ_1 is s_1, and all but the first coordinate of γ_2 is s_2. Consider the pointwise stabilizer Q_0 of $\{H_0\gamma_0, H_0\gamma_1, H_0\gamma_2\}$. This group Q_0 is contained in the stabilizer H_0 of $H_0\gamma_0$. For any element $h_0 \in H_0$ and any index i in $\{1, 2\}$, we have $H_0\gamma_i h_0 = H_0\gamma$ for some $\gamma \in \mathcal{C}$ with 1 or $\ell - 1 > 1$ entries equal to 1. Moreover if h_0 is in Q_0, then the first case must hold. Since the only automorphism of S fixing both s_1 and s_2 is the identity, we see that Q_0 is a subgroup of Q leaving $\mathcal{C}$ invariant. Therefore, whenever $q_0 \in Q_0$ and $\gamma \in \mathcal{C}$, we have $(H_0\gamma)q_0 = H_0\gamma^{q_0}$ for $\gamma^{q_0} \in \mathcal{C}$.

Let ω_0, ω_1, ω_2 be those elements of Ω for which all k coordinates of Ω are $H_0\gamma_0$, $H_0\gamma_1$, $H_0\gamma_2$, respectively. By the previous paragraph and the fact that $G \leq H \wr P$, the pointwise stabilizer in G of $\{\omega_0, \omega_1, \omega_2\}$ is a permutation group R permuting the $k\ell$ coordinates of the vectors in $S^{k\ell}$. More precisely, if the coordinates are labelled by the integers $1, \ldots, k\ell$, then R is a permutation group on $\{1, \ldots, k\ell\}$ such that $\{j\ell + 1 \mid j \in \{0, \ldots, k-1\}\}$ is R-invariant. Since R is a subgroup of a transitive group on $k\ell$ points which has order at most $|G|$, we see, by [13, Theorem 1.2], that $d(R) \leq 48 \sqrt[k\ell]{|G|}$.

Consider a distinguishing partition $\mathcal{P}$ with $d(R)$ colors for the action of R on $\{1, \ldots, k\ell\}$. Define a new coloring of the R-invariant subset $\{1, \ldots, k\ell\} \setminus \{j\ell + 1 \mid j \in \{0, \ldots, k-1\}\}$ using no more than $d(R)^2$ colors in the following way. For any integers j and u with $0 \leq j \leq k-1$ and $1 < u \leq \ell$ color $j\ell + u$ with the color (α, β) where α is the color of $j\ell + 1$ in $\mathcal{P}$ and β is the color of $j\ell + u$ in $\mathcal{P}$. Clearly, no non-identity element of R preserves this new coloring. For $\ell \geq 3$, we see, by Lemma 2.1 of [13], that G has a base B containing $\{\omega_0, \omega_1, \omega_2\}$ such that

$$b(G) \leq |B| = 2\frac{\log d(R)}{\log |S|} + 4 < 2\frac{\log |G|}{k\ell \log |S|} + 6 = 2\frac{\log |G|}{\log n} + 6.$$

4.4. **Twisted wreath product type groups.** This type was treated in Burness and Seress [12, Section 4]. We follow their discussion. By the previous section we know that if L is a primitive permutation group of product type acting on a set Ω, then we have $b(L) \leq 2(\log |L|/\log |\Omega|) + 22$. Let G be a primitive permutation group of twisted wreath product type acting on the set Ω. Then G contains a regular normal subgroup T^k isomorphic to the direct product of k copies of a non-abelian simple group T. We may write $G = T^k P$ where P is a transitive permutation group acting on k points. As explained in [29, Section 3.6], we may embed G in a group of product type L which is of the form $(T^2)^k.P$. Thus $b(G) \leq b(L) \leq 2(\log |L|/\log |\Omega|) + 22 = 2(\log |G|/\log |\Omega|) + 24$.

This completes the proof of Theorem 4.1.

5. AFFINE PRIMITIVE PERMUTATION GROUPS

The main result of this section is

Theorem 5.1. *Let G be an affine primitive permutation group of degree n. Then $b(G) \leq 2(\log |G|/\log n) + 16$.*

Let G be an affine primitive permutation group of degree n with a point-stabilizer H. Then $G = VH \leq AGL(V)$, where V is a finite vector space of order $n = p^k$ (p prime), and the stabilizer H of the zero vector is an irreducible subgroup of $GL(V)$. Since $b(G) = b(H) + 1$, Theorem 5.1 follows immediately from

Theorem 5.2. *Let H be a subgroup of $GL(V)$ acting irreducibly on the finite vector space V. Then $b_V(H) \leq 2(\log |H|/\log |V|) + 17$.*

In the above theorems, the multiplicative constant 2 is best possible, as shown by the following example (which completes the proof of Proposition 1.2).

Proposition 5.3. *Let V be a d-dimensional (d even) non-degenerate symplectic space over the finite field $\mathbb{F}_q$ and let $H = Sp(V)$ with its natural action on V.*

(i) *Then $b_V(H) = d$.*
(ii) *If $G = VH \leq AGL(V)$ is the corresponding affine primitive permutation group, then for sufficiently large values of q, we have*

$$b(G) = \lfloor 2(\log |G|/\log n) \rfloor - 2.$$

Proof. (i) Clearly, any basis of V (as a vector space) is also a base for H, so $b_V(H) \leq d$. For the equality, let $\{b_1, \ldots, b_l\} \in V$ be any set of vectors with $l \leq d - 1$. Then there is a subspace $U \leq V$ containing $\{b_1, \ldots, b_l\}$ with $\dim U = d - 1$. Hence it is enough to show that for every such subspace U, there exists a non-identity $g \in H$ that acts trivially on U.

Let $U \leq V$ be a subspace of dimension $d - 1$ and let $[\ ,\ ]$ denote the non-degenerate symplectic bilinear form on V preserved by H. Then the restriction of $[\ ,\ ]$ to U is degenerate: there exists $0 \neq x \in U$ such that $\langle x \rangle = U^\perp$. Let $y \in V \setminus U$ be arbitrary. We claim that the map

$$A : cy + u \mapsto c(y + x) + u \ (c \in \mathbb{F}_q, \ u \in U)$$

is an element of H, which acts trivially on U. To see this, let $c, d \in \mathbb{F}_q$ and $u, v \in U$. Then we have $[A(cy + u), A(dy + v)] = [cy + u + cx, dy + v + dx] = [cy + u, dy + v]$, proving the claim.

(ii) This follows from a simple computation using the order formula for $|Sp(V)|$. $\square$

It remains to prove Theorem 5.2. We do this in the following two subsections.

5.1. **Primitive linear groups.** In this subsection we prove Theorem 5.2 in the case where $H \leq GL(V)$ acts primitively on V as a linear group. In fact we prove the following stronger bound for this case.

Theorem 5.4. *Let V be a finite vector space, and let $H \leq GL(V)$ be an irreducible, primitive linear group on V. Then one of the following holds:*

(i) $b(H) \leq 15$;
(ii) $b(H) \leq 2 \frac{\log |H|}{\log |V|} + 9$.

A version of Theorem 5.4 was proved in [26, 27] with a much worse multiplicative constant, and unspecified constants in place of the constants 15 and 9. The proof of Theorem 5.4 will be along the lines of that proof. However, in order to make our constants explicit (and small), we need to improve several of the results in [26, 27].

For a field $\mathbb{F}_q$ and a positive integer m, by the *natural* module for the symmetric or alternating group $\mathrm{Sym}(m)$ or $\mathrm{Alt}(m)$ over $\mathbb{F}_q$, we mean the fully deleted permutation module of dimension $m' = m - \delta$, where $\delta \in \{1, 2\}$.

The first result is a version of Proposition 2.2 of [26] with an explicit constant:

Proposition 5.5. *Let $V = V_d(q)$ $(q = p^e)$ and $G \leq GL(V)$, and suppose that $E(G)$ is quasisimple and absolutely irreducible on V. Then one of the following holds:*

(i) $E(G) = \mathrm{Alt}(m)$ *and V is the natural $\mathrm{Alt}(m)$-module over $\mathbb{F}_q$, of dimension $d = m - \delta$ $(\delta \in \{1, 2\})$;*
(ii) $E(G) = Cl_d(q_0)$, *a classical group with natural module of dimension d over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$;*
(iii) $b(G) \leq 6$.

Proof. This is proved in [22]. $\square$

The next result is an explicit version of [26, Proposition 3.6].

Proposition 5.6. *Let $V = V_d(q)$ $(q = p^e)$ and $G \leq GL(V)$, and suppose that the Fitting subgroup $F(G)$ is absolutely irreducible on V. Then $b(G) \leq 13$.*

Proof. We begin by arguing exactly as in the proof of [26, 3.6] that $F = F(G)$ can be taken to be the central product of an extraspecial group s^{1+2m} and the group $Z = \mathbb{F}_q^*$ of scalars, where s is a prime and $d = s^m$. We can also assume that $G = N_{GL(V)}(F)$, so that

G/F is isomorphic to either $Sp_{2m}(s)$ or $O_{2m}^{\pm}(2)$, with $s = 2$ in the latter case. Moreover, $q \equiv 1 \bmod s$, and also $q \equiv 1 \bmod 4$ if $s = 2$ and $G/F \cong Sp_{2m}(2)$.

Define $F^0 = F.Z(G/F)$, an extension of F by a group of order at most 2. By [32], there are three vectors $v_1, v_2, v_3 \in V$ such that $F_{v_1 v_2 v_3}^0 = 1$. So if we let $J = G_{v_1 v_2 v_3}$, then $J \cap F^0 = 1$ and $J \cong JF^0/F^0$ is isomorphic to a subgroup of $PSp_{2m}(s)$ or $O_{2m}^{\pm}(2)$.

Obviously $b(G) \leq 3 + b(J)$. Assume for a contradiction that

$$b(J) > 10.$$

Then clearly $d = \dim V > 10$, and $V^{10} = \bigcup_{h \in J \setminus 1} C_{V^{10}}(h)$. Hence

$$|V|^{10} \leq \sum_{h \in J \setminus 1} |C_V(h)|^{10}. \tag{3}$$

For $h \in J \setminus 1$, Theorem 4.1 of [19] shows that there are $2m+1$ conjugates of h that generate G modulo F, and hence there are $2m + 2$ conjugates of h generating G. It follows that

$$\dim C_V(h) \leq \left(1 - \frac{1}{2m + 2}\right) \dim V.$$

Hence (3) gives $|V|^{10/(2m+2)} \leq |J|$, and so as $|V| = q^d = q^{s^m}$, we have

$$q^{10s^m/(2m+2)} \leq |J| \leq \begin{cases} |PSp_{2m}(s)|, & s \text{ odd}, q \equiv 1 \bmod s \\ |Sp_{2m}(2)|, & s = 2, q \equiv 1 \bmod 4 \\ |O_{2m}^{\pm}(2)|, & s = 2, q \text{ odd}. \end{cases} \tag{4}$$

Straightforward computation shows that the only possible values satisfying (4) are $s = 2$, $q = 3$ and $m = 4$ or 5. For $m = 5$ we have $G/F \cong O_{10}^{\pm}(2)$, and in the above argument, [19, 4.1] shows that we may replace $2m + 2$ by $2m + 1$ in (4), yielding a contradiction. And if $m = 4$ then $d = 2^4 = 16$ and it is very easy to argue directly that $b(G) \leq 13$, as required. $\square$

The next result is an improvement of Lemma 3.7 of [26].

Proposition 5.7. (i) *Let $\mathbb{F}_{q_0}$ be a subfield of $\mathbb{F}_q$, let $q = q_0^r$, and let $M = \mathbb{F}_q^* GL_d(q_0) \leq GL_d(q) = GL(V)$, where $\mathbb{F}_q^*$ denotes the group of scalars. Then for the action of M on V we have*

$$b(M) \leq \frac{d}{r} + 2.$$

(ii) *Let $q = p^r$ with p prime, and let $M = \mathbb{F}_q^* \mathrm{Sym}(m) \leq GL_{m'}(q) = GL(V)$, where V is the natural module for $\mathrm{Sym}(m)$ over $\mathbb{F}_q$, of dimension $m' = m - \delta$ ($\delta \in \{1, 2\}$). Then*

$$b(M) \leq \frac{\log_p m}{r} + 4.$$

Proof. (i) Let $\lambda_1, \ldots, \lambda_r$ be an $\mathbb{F}_{q_0}$-basis for $\mathbb{F}_q$, and let $e_1 \ldots, e_d$ be the standard basis for $V = \mathbb{F}_q^d$ (that is, $e_i = (0, \ldots, 1, \ldots 0)$ where the 1 is in the ith coordinate). Write $d = kr + l$ with $k, l \in \mathbb{Z}$ and $0 \leq l < r$, and define

$$\begin{aligned} v_i &= \sum_{j=(i-1)r+1}^{ir} \lambda_j e_j \quad (1 \leq i \leq k), \\ v_{k+1} &= \sum_{j=kr+1}^{d} \lambda_j e_j. \end{aligned}$$

If we write $M_0 = GL_d(q_0)$, it is easy to see that $(M_0)_{v_1 \ldots v_{k+1}} = 1$. Hence if $J = M_{v_1 \ldots v_{k+1}}$ then $J \cong JM_0/M_0$ is cyclic, and so by [33, 3.1], J has a base of size 1. Thus $b(M) \leq k + 2 \leq \frac{d}{r} + 2$.

(ii) This follows directly from the proof of [26, 3.7(ii)]. $\square$

As in [26], for $H \leq GL(V)$ define $b^*(H)$ to be the minimal size of a set B of vectors such that any element of H that fixes every 1-space $\langle v \rangle$ with $v \in B$ is necessarily a scalar multiple of the identity. We call such a set B a *strong* base for H. By [26, 3.1],

$$b(H) \leq b^*(H) \leq b(H) + 1.$$

Next we give an improvement of Lemma 3.3(iii) of [26].

Lemma 5.8. *Let V_1, V_2 be vector spaces over $\mathbb{F}_q$ with $\dim V_i = n_i$ and $n_1 \leq n_2$, and let $H_i \leq GL(V_i)$ for $i = 1, 2$. Denote by $H_1 \otimes H_2$ the image of $H_1 \times H_2$ acting in the natural way on the tensor product $V_1 \otimes V_2$.*

If $n_1 \leq b^*(H_2)$, then

$$b(H_1 \otimes H_2) \leq \frac{b^*(H_2)}{n_1} + 3.$$

Proof. We follows the proof of [26, Lemma 3.3(iii)]. Let $b = b^*(H_2)$. Assume $n_1 \leq b$, and let $y_1, \ldots, y_b$ be a linearly independent strong base for H_2 in V_2. Let $x_1, \ldots, x_{n_1}$ be a basis of V_1.

Write $b = rn_1 + s$ with r, s integers and $0 \leq s < n_1$ For $1 \leq i \leq r$ define

$$v_i = \sum_{k=1}^{n_1} x_k \otimes y_{(i-1)n_1+k}, \quad W_i = \langle x_k \otimes y_{(i-1)n_1+k} : 1 \leq k \leq n_1 \rangle,$$

and set $v_{r+1} = \sum_{k=1}^{s} x_k \otimes y_{rn_1+k}$, $W_{r+1} = \langle x_k \otimes y_{rn_1+k} : 1 \leq k \leq s \rangle$.

Consider the stabilizer $L = (H_1 \otimes H_2)_{v_1 \ldots v_{r+1}}$. By Lemma 3.3(i) of [26], L stabilizes $V_1 \otimes W_i$ for all $1 \leq i \leq r + 1$.

Next choose $C, D \in SL_{n_1}(q)$ generating $SL_{n_1}(q)$, and for each i, define $\gamma_i = C \otimes 1, \delta_i = D \otimes 1 \in GL(V_1 \otimes W_i)$. Let

$$v = \sum_{i=1}^{r+1} v_i \gamma_i, \quad w = \sum_{i=1}^{r+1} v_i \delta_i.$$

At this point the argument at the end of the proof of [26, 3.3(iii)] shows that $L_{vw} = 1$. Hence $b(H_1 \otimes H_2) \leq r + 3 \leq \frac{n_1}{b} + 3$, as required.   □

The next result is Theorem 1 of [27], with an explicit constant $C = 14$. The proof is identical to that in [27], but using Propositions 5.5 and 5.6 at the end to justify that $C = 14$ works.

Proposition 5.9. *Let $V = V_d(q)$, and let H be a subgroup of $\Gamma L(V)$ such that H acts primitively on V and $H^0 := H \cap GL(V)$ is absolutely irreducible on V. Suppose that $b^*(H^0) > 14$. Then*

$$H^0 \leq H_0 \otimes \bigotimes_{i=1}^{s} \mathrm{Sym}(m_i) \otimes \bigotimes_{i=1}^{t} \mathrm{Cl}_{d_i}(q_i),$$

where $s + t \geq 1$ and the following hold:

(i) *$H_0 \leq GL_{d_0}(q)$ with $b^*(H_0) \leq 14$*
(ii) *each factor $\mathrm{Sym}(m_i) < GL_{m_i'}(q)$ and $\mathrm{Cl}_{d_i}(q_i) \leq GL_{d_i}(q)$ is acting on the natural module over $\mathbb{F}_q$, where $m_i' = m_i - \delta_i$, $\delta_i \in \{1, 2\}$*
(iii) *$d = d_0 \cdot \prod_1^s m_i' \cdot \prod_1^t d_i$*
(iv) *$F^*(H^0)$ contains $\prod_1^s \mathrm{Alt}(m_i) \cdot \prod_1^t \mathrm{Cl}_{d_i}(q_i)^{(\infty)}$.*

The next result is an improvement of [27, Proposition 2].

Proposition 5.10. *Let H, H^0 be as in Proposition 5.9, with $b^*(H^0) > 14$. Take $m_s' = \max(m_i' : 1 \leq i \leq s)$ and $d_t = \max(d_i : 1 \leq i \leq t)$ (define these to be 0 if $s = 0$ or $t = 0$, respectively).*

(i) *Suppose $t \geq 1$ and $m'_s \leq d_t$, and let $q = q_t^r$. Then $d < d_t^2$, $d_t \geq 14$, and*

$$b(H^0) \leq b(GL_{d/d_t}(q) \otimes GL_{d_t}(q_t)) \leq \frac{d_t^2}{dr} + 5.$$

(ii) *Suppose $s \geq 1$ and $m'_s > d_t$, and let $q = p^r$. Then $d < (m'_s)^2$, $m'_s \geq 14$, and*

$$b(H^0) \leq b(GL_{d/m'_s}(q) \otimes \mathrm{Sym}(m_s)) \leq \frac{m_s \log_p m_s}{dr} + 6.$$

Proof. We follow the proof of [27, Proposition 2], but as the constants are different we give a few details.

We proceed by induction on $s + t$. For the base case $s + t = 1$, we have $H^0 \leq H_0 \otimes M$, where $M = \mathrm{Cl}_{d_1}(q_1)$ or $\mathrm{Sym}(m_1)$. Consider the first case, and write $q = q_1^r$. Proposition 5.7 gives $b(M) \leq \frac{d_1}{r} + 2$, hence also $b^*(M) \leq \frac{d_1}{r} + 3$. If $d_0 = 1$ the conclusion in (i) is immediate, so assume $d_0 \geq 2$. As in the proof of [27, Proposition 2], we see that $d_0 \leq d_1$. Then Lemma 5.8 gives

$$
\begin{aligned}
b(H^0) \quad &\leq \frac{b^*(M)}{d_0} + 3 \\
&\leq \frac{d_1}{rd_0} + \frac{3}{d_0} + 3 \\
&< \frac{d_1}{rd_0} + 5,
\end{aligned}
$$

so that (i) holds (note that $d_1 \geq 14$ by [26, 3.3(ii)]). Similarly (ii) holds when $M = \mathrm{Sym}(m_1)$.

Now assume $s + t \geq 2$. Let m be the maximum of d_t and m'_s, and write M for the corresponding group $\mathrm{Cl}_{d_t}(q_t)$ or $\mathrm{Sym}(m_s)$. Note that $m \geq 14$, since otherwise [26, 3.3(ii)] implies that $b^*(H^0) \leq 14$. Let N be the tensor product of H_0 and the other factors $\mathrm{Cl}_{d_i}(q_i)$, $\mathrm{Sym}(m_i)$, so that $H^0 \leq N \otimes M$. If $b^*(N) \leq 14$ the conclusion follows as in the $s + t = 1$ case, so assume $b^*(N) > 14$.

Let m' be the largest among the dimensions d_i, m'_i omitting m, and write N_1 for the corresponding group $\mathrm{Cl}_{d_i}(q_i)$ or $\mathrm{Sym}(m_i)$.

Consider the case where $N_1 = \mathrm{Cl}_{d_i}(q_i)$. Let $q = q_i^u$. By induction we have

$$b^*(N) \leq b(N) + 1 \leq \frac{d_i^2 m}{du} + 6 \leq \frac{d_i}{u} + 6.$$

Suppose $d \geq m^2$. Then $b^*(N) \geq m$ by [26, 3.3(iv)], so Lemma 5.8 implies that

$$b(H^0) \leq \frac{b^*(N)}{m} + 3 \leq \frac{d_i}{um} + \frac{6}{m} + 3.$$

Since $m \geq d_i$ and $m > 14$, this yields $b(H^0) < 5$, a contradiction. Hence $d < m^2$ in this case. Now the conclusion of the proposition follows by the argument given for the $s+t = 1$ case.

Finally, consider the case where $N_1 = \mathrm{Sym}(m_i)$. Let $q = p^r$. By induction,

$$b^*(N) \leq b(N) + 1 \leq \frac{(m_i \log_p m_i) \cdot m}{dr} + 7 \leq \frac{\log_p m_i}{r} + 8.$$

Now the argument of the previous paragraph gives the conclusion.   □

Proof of Theorem 5.4

We are now in a position to prove Theorem 5.4.

We begin just as in the proof of [27, Corollary 3]. Suppose $H \leq GL(V)$ acts primitively and irreducibly on a finite vector space V defined over a field of size q_0. Choose $q = q_0^r$ maximal such that $H \leq \Gamma L_d(q) \leq GL_{dr}(q_0)$. Write $H^0 = H \cap GL_d(q)$ and $V = V_d(q)$. By [16, 12.1], H^0 is absolutely irreducible on V.

If $b^*(H^0) \le 14$ then $b(H) \le 15$, and the conclusion of Theorem 5.4 holds. So assume now that $b^*(H^0) > 14$. Then H^0 is given by Proposition 5.9, and (i) or (ii) of Proposition 5.10 holds.

Consider case (i) of Proposition 5.10. Write $m = d_t$ and $q = q_t^r$. Then $d < m^2$, $H^0 \triangleleft Cl_m(q_t)$, $m \ge 14$, and

$$b(H^0) \le \frac{m^2}{dr} + 5. \tag{5}$$

From the order formulae for classical groups, we see that

$$\log_{q_t}(|H^0|) \ge \log_{q_t} |\Omega_m^\pm(q_t)| \ge \frac{1}{2}m(m-1) - 1.$$

Hence

$$\frac{\log |H|}{\log |V|} \ge \frac{\frac{1}{2}m(m-1) - 1}{rd},$$

and so (5) gives

$$b(H^0) \le 2\frac{\log |H|}{\log |V|} + 5 + \frac{m+2}{rd} < 2\frac{\log |H|}{\log |V|} + 7.$$

This completes the proof in case (i) of Proposition 5.10.

To conclude, consider case (ii) of Proposition 5.10. Write $m = m_s$, $m' = m'_s$ and $q = p^r$. Then $d < m^2$, $H^0 \triangleleft \mathrm{Alt}(m)$, $m' \ge 14$, and

$$b(H^0) \le \frac{m \log_p m}{dr} + 6. \tag{6}$$

Now $|H^0| \ge \frac{1}{2}m! > \frac{1}{2}\left(\frac{m}{e}\right)^m$, and so

$$\frac{\log |H|}{\log |V|} \ge \frac{m(\log_p m - \log_p e) - 1}{rd}.$$

Hence (6) gives

$$b(H^0) \le \frac{\log |H|}{\log |V|} + 6 + \frac{m \log_2 e + 3}{rd} < \frac{\log |H|}{\log |V|} + 8,$$

giving the conclusion of Theorem 5.4 (actually, a stronger version, with the constant 2 replaced by 1).

This completes the proof of Theorem 5.4.

5.2. **Imprimitive linear groups.** It remains to prove Theorem 5.2 in the case where the irreducible linear group $H \le GL(V)$ acts imprimitively on V. Assume that H preserves the direct sum decomposition $V = V_1 \oplus \cdots \oplus V_t$ where $V_1, \ldots, V_t$ are subspaces of V, and $t > 1$ is chosen maximally. Let H_1 be the stabilizer of V_1 in H. Let us denote the minimal base size of the action K_1 of H_1 on V_1 by $b(K_1)$. By the choice of t, the action of K_1 on V_1 is primitive, so by Theorem 5.4, we have $b(K_1) \le 15$ or

$$b(K_1) \le 2(\log |K_1| / \log |V_1|) + 9.$$

If $b(K_1) \le 15$, then, by [13, Theorem 3.4] and its proof, we have

$$b(H) \le \log |H| / \log |V| + b(K_1) + 1 + (\log 48 / \log(2^{b(K_1)})) \le \log |H| / \log |V| + 17.$$

So assume now that $b(K_1) \ge 16$ and $b(K_1) \le 2(\log |K_1| / \log |V_1|) + 9$. In that case our proof strictly follows the arguments of [13], but in order to be able to prove Theorem 5.2 we need to give more precise estimates for the constants appearing there. In the proof we will freely use the concepts and notation of [13].

A main step in proving [13, Theorem 3.17] was to reduce the problem for linear groups which do not preserve any tensor product decomposition of the vector space (possibly over

a proper field extension of the base field). In order for the reduction argument to work, a generalisation of the problem was needed.

Let us view H not just as a subgroup of $GL(V)$ but also as an abstract group. We will define certain maps $X : H \to GL(V)$. For this let T_V denote the group

$$T_V = \{g \in GL(V) \mid g(V_i) = V_i \text{ and } g|_{V_i} \in Z(GL(V_i)) \quad \forall 1 \leq i \leq t\} \simeq (\mathbb{F}_q^\times)^t$$

where $\mathbb{F}_q$ is the field of definition for V. According to [13, Definition 3.5] we say that a map $X : H \to GL(V)$ is a $(\text{mod } T_V)$-representation of H if the following two properties hold: (1) $X(g)$ normalizes T_V for every $g \in H$; and (2) $X(gh)T_V = X(g)X(h)T_V$ for every $g, h \in H$. We will always consider $(\text{mod } T_V)$-representations X of H such that the group $X(H)T_V$ acts transitively on the set of factors $\Pi = \{V_1, \ldots, V_t\}$ of the above direct sum decomposition of V. Let $b_X(H)$ denote the minimal base size of the group $X(H)T_V$.

In the rest of this subsection we will show that if X is an imprimitive irreducible linear representation of H preserving the decomposition $V = V_1 \oplus \cdots \oplus V_t$, then $b_X(H) \leq 2(\log |H| / \log |V|) + 17$, or $b(H) \leq 2(\log |H| / \log |V|) + 17$. Theorem 5.2 will follow by specializing to the case when X is the identity.

If the representation of H is alternating-induced in the sense of [13, Section 3.2], then $b(H) \leq 2(\log |H| / \log |V|) + 17$, by [13, Theorem 3.9].

By [13, Section 3.4] (and especially by [13, Corollary 3.15]), it is sufficient to establish the proposed bound in the claim for $b(H)$ or for $b_X(H)$ in case X is a (not necessarily linear) $(\text{mod } T_V)$-representation of H which is classical-induced satisfying the multiplicity-free condition, in the sense of [13, Section 3.3]. Let X be such a representation of H.

Let N be the kernel of the action of H on Π. Let $\mathfrak{X} : H \to N_{GL(V(p))}(T_V)/T_V$ be the homomorphism defined by $\mathfrak{X}(h) := X(h)T_V/T_V$ where $V(p)$ denotes the vector space V defined over the prime field (of size p) of $\mathbb{F}_q$ and where $h \in H$. In [13, Section 3.3] a bound is given for $b_X(H)$. By the argument after [13, Theorem 3.11] (from the second to the fifth paragraphs), we get $b_X(H) < (\log |H| / \log |V|) + 12$ when $\mathfrak{X}(N) = 1$. So assume that $\mathfrak{X}(N) \neq 1$.

We use the notation (and a minor modification of the argument) of the paragraph following [13, Theorem 3.11]. In this case $\text{Soc}(\mathfrak{X}(N))$ is a subdirect product of isomorphic simple classical groups $S_1, \ldots, S_t$. The linking factor, denoted by r, is at most 2. We have $|N| \geq |S_1|^{t/r}$. Since $b(K_1) \geq 16$, the center of K_1 has size less than $|V_1|^{1/16}$ and $|\text{Out}(S_1)| \leq |V_1|^{6/16}$, (the latter by [18, Lemma 7.8]). Thus $|S_1| \geq |K_1| / |V_1|^{1/2}$, which implies $|N| \geq |K_1|^{t/r} / |V|^{1/2}$.

Assume that $r = 1$. By Theorem 5.4, we have

$$b(K_1) \leq 2(\log |K_1| / \log |V_1|) + 9 = 2(\log(|K_1|^t) / \log |V|) + 9 \leq 2(\log |N| / \log |V|) + 10.$$

By [13, Theorem 3.4] and its proof, we then get $b(H) \leq 2(\log |H| / \log |V|) + 12$ since $\log |V_1| \geq 16$.

Now assume that $r = 2$, so $t = 2k$ for some integer k. By changing the order (if necessary) of the summands in the direct sum $V = \oplus_{i=1}^t V_i$, we can assume that the sets $\Delta_i := \{V_{2i-1}, V_{2i}\}$, for all i with $1 \leq i \leq k$, form a system of H-blocks with $S_{\Delta_i} := \text{Soc}(\mathfrak{X}_{\Delta_i}(N))$ a full diagonal subgroup of $S_{2i-1} \times S_{2i}$. Here $\mathfrak{X}_{\Delta_i}$ is defined as follows. Put $V_{\Delta_i} := V_{2i-1} \oplus V_{2i}$ and let $T_{V_{\Delta_i}}$ be defined analogously as T_V. Let $X_i : N_H(\Delta_i) \to GL(V_{\Delta_i})$ be the $(\text{mod } T_{V_{\Delta_i}})$-representation of $N_H(\Delta_i)$ obtained naturally from X by restricting first from H to $N_H(\Delta_i)$ and then from V to V_{Δ_i}. Now define $\mathfrak{X}_{\Delta_i}$ to be the homomorphism given by $\mathfrak{X}_{\Delta_i}(h) := X_i(h)T_{V_{\Delta_i}}/T_{V_{\Delta_i}}$ for $h \in N_H(\Delta_i)$. The multiplicity-free condition (and the definition of Δ_i) guarantees that there are no functions $\varphi_i : V_{2i} \to V_{2i-1}$ and

$\lambda_i : N_H(\Delta_i) \to \mathbb{F}_q^\times$ such that φ_i is a semilinear invertible map and

$$X_{2i-1}(g) = \lambda_i(g) \cdot \varphi_i X_{2i}(g) \varphi_i^{-1}$$

for every $g \in C_H(\Delta_i)$. By using [21, Theorem 2.1.4], it follows that $S_1 \simeq PSL(V_1)$. Thus, $|N| \geq (q^{\dim^2 V_1 - \dim V_1})^{t/2} / |V|^{1/2}$, which means that

$$b_{X_{\Delta_1}}(N_H(\Delta_1)) \leq \dim V_1 \leq 2(\log |N| / \log |V|) + 2.$$

Now we can apply [13, Theorem 3.4] to the H-invariant direct sum decomposition $V = \oplus_{i=1}^k V_{\Delta_i}$ to deduce that $b_X(H) \leq 2(\log |H| / \log |V|) + 3 + \log 48 < 2(\log |H| / \log |V|) + 9$.

This completes the proof of Theorem 5.2.

6. Proof of Corollary 1.3

In this section we will prove Corollary 1.3. Let G be a primitive permutation group of degree n. For later use, we recall the definition of *standard* actions of almost simple primitive groups: these occur for groups with socle an alternating group $\mathrm{Alt}(m)$ or a classical group. In the former case they are actions on an orbit of subsets or partitions of $\{1, \ldots, m\}$; and in the latter, they are subspace actions (as defined in Section 3).

Assume first that G is $\mathrm{Sym}(m)$ or $\mathrm{Alt}(m)$ for some integer $m \geq 5$. We consider standard actions of G.

If the action of G is on a set of partitions of the underlying set of size m, then $b(G) \leq \log n + 4$ by Theorem 2.7. The right hand side is less than $\sqrt{n}$ for $n \geq 256$, and $b(G) < 12$ otherwise.

Now assume that G acts on k-element subsets of $\{1, \ldots, m\}$ for some integer k with $2 \leq k \leq m/2$ and $n = \binom{m}{k}$. Assume also that $b(G) \geq 26$.

Let $k^2 \leq m$. Then, by Theorem 2.2, $b(G) \leq \frac{2m}{k+1} + 1$. Since $b(G) \geq 26$, we have $m \geq 38$ and $n \geq 625$. For $k = 2$ and $n \geq 625$ we get $b(G) \leq \frac{2m}{3} + 1 < \sqrt{n}$. For $k \geq 3$ and $n \geq 625$ we find that $m^2 \leq \binom{m}{k} = n$ and so $b(G) < \sqrt{n}$.

Let $k^2 > m$. Then $k \geq 3$ since $m \geq 5$. By Theorem 2.2, $b(G) \leq \left\lceil \log_{\lceil t \rceil}(m) \right\rceil (\lceil t \rceil - 1)$ where $t = m/k$. Since $b(G) \geq 26$, this forces $m \geq 18$. In particular $k \geq 5$ and $n \geq 625$. Thus assume that $m \geq 18$, $k \geq 5$, and $n \geq 625$. Let $k \geq 8$. By Theorem 2.2 and the assumption $k^2 > m$, we have $b(G) \leq (\log m + 1)t < (2 \log k + 1)t$. Since $t \geq 2$, it follows that $(2 \log k + 1)t < t^{k/2} = (\frac{m}{k})^{k/2} < \sqrt{\binom{m}{k}}$. If $5 \leq k \leq 7$, then $b(G) \leq 6(\log m + 1)$, by Theorem 2.2 (recall that $m < k^2$). The right hand side is less than $\sqrt{\binom{m}{k}}$ provided that $m \geq 18$ and $5 \leq k \leq m/2$.

Next assume that G is a group as in Section 4.3.1. Let us use the notation and results of that section. By (2), we have $b(G) < \log k + 1 + n^{1/k}$. For $k \geq 3$ this is less than $\sqrt{n}$ since $n \geq 5^k \geq 125$. Now let $k = 2$. Then G is a subgroup of $\mathrm{Sym}(t) \wr \mathrm{Sym}(2)$ with $n = t^2$. In this case $b(G) \leq t$, that is, $b(G) \leq \sqrt{n}$.

At this point, by [28, Theorem 1.1], we may assume that $|G| \leq n^{1+\log n}$ (and $n \geq 26$).

If G is as in Section 4.3.2 or as in Section 4.4, then $n > 2500$, and so by Theorem 1.1,

$$b(G) \leq 2\frac{\log |G|}{\log n} + 24 \leq 2 \log n + 26 < \sqrt{n}.$$

If G is as in Section 4.2, we use Fawcett's [14] bound $b(G) \leq (\log |G| / \log n) + 3$ to obtain $b(G) \leq \log n + 4 < \sqrt{n}$ provided $n \neq 60$. If $n = 60$, then $|G| \leq 4n^2$, and so $b(G) \leq 5$.

Assume that G is almost simple. If the action of G is non-standard, then $b(G) \leq 7$ by [7] and [10]. Thus assume that the action of G is standard. In particular the socle of G is

a simple classical group or an alternating group. The case when the socle is an alternating group was treated before.

Let G be an almost simple group with socle a classical group with natural module a vector space of dimension d over some finite field. Since $|G| \leq n^{1+\log n}$ and $b(G) \leq 2(\log |G| / \log n) + 16$, the latter by Theorem 3.1, we see that $b(G) < \sqrt{n}$ for $n \geq 1600$. Thus assume that $n < 1600$. By Theorem 3.3 and Section 3.3, we have $b(G) \leq d + 14$. By [21, Table 5.2.A], we find that d must be at most 11, and so $b(G) \leq 25$.

Finally, assume that G is of affine type with $n \geq 4$. Put $n = p^d$ where p is a prime and d is an integer. Then $b(G) \leq 1 + d$. This is at most $p^{d/2}$ unless $p = 2$ and $2 \leq d \leq 5$. In particular $b(G) \leq 6$ and $n \leq 32$.

This completes the proof of Corollary 1.3.

References

[1] M. Aschbacher and R.M. Guralnick, On abelian quotients of primitive groups, *Proc. Amer. Math. Soc.* **107** (1989), no. 1, 89–95.

[2] L. Babai, On the order of uniprimitive permutation groups, *Ann. of Math.* (2) **113** (1981), 553–568.

[3] L. Babai, On the order of doubly transitive permutation groups, *Invent. Math.* **65** (1981/82), 473–484.

[4] C. Benbenishty, On actions of primitive groups, Ph.D. thesis, Hebrew University, Jerusalem, 2005.

[5] C. Benbenishty, J.A. Cohen and A.C. Niemeyer, The minimum length of a base for the symmetric group acting on partitions, *European J. Combin.* **28** (2007), 1575–1581.

[6] A. Bochert, Über die Transitivitätsgrenze der Substitutionengruppen, welche die alternirende ihres Grades nicht enthalten, *Math. Ann.* **33** (1889), 572–583.

[7] T.C. Burness, On base sizes for actions of finite classical groups, *J. Lond. Math. Soc.* (2) **75** (2007), 545–562.

[8] T.C. Burness, R.M. Guralnick and J. Saxl, On base sizes for symmetric groups, *Bull. Lond. Math. Soc.* **43** (2011), 386–391.

[9] T.C. Burness, R.M. Guralnick and J. Saxl, On base sizes for algebraic groups, *J. Eur. Math. Soc. (JEMS)* **19** (2017), no. 8, 2269–2341.

[10] T.C. Burness, M.W. Liebeck and A. Shalev, Base sizes for simple groups and a conjecture of Cameron, *Proc. Lond. Math. Soc.* (3) **98** (2009), 116–162.

[11] T.C. Burness, E.A. O'Brien and R.A. Wilson, Base sizes for sporadic simple groups, *Israel J. Math.* **177** (2010), 307–333.

[12] T.C. Burness and Á. Seress, On Pyber's base size conjecture, *Trans. Amer. Math. Soc.* **367** (2015), 5633–5651.

[13] H. Duyan, Z. Halasi and A. Maróti, A proof of Pyber's base size conjecture, arXiv:1611.09487. Submitted for publication.

[14] J.B. Fawcett, The base size of a primitive diagonal group, *J. Algebra* **375** (2013), 302–321.

[15] The GAP Group, *Groups, Algorithms, and Programming*, 2014, (http://www.gap-system.org).

[16] M. Giudici, M.W. Liebeck, C.E. Praeger, J. Saxl and P.H. Tiep, Arithmetic results on orbits of linear groups, *Trans. Amer. Math. Soc.* **368** (2016), 2415–2467.

[17] D. Gluck, Á. Seress, A. Shalev, Bases for primitive permutation groups and a conjecture of Babai, *J. Algebra* **199** (1998), 367–378.

[18] R.M. Guralnick, A. Maróti, L. Pyber, Normalizers of primitive permutation groups, *Adv. Math.* **310** (2017), 1017–1063.

[19] R.M. Guralnick and J. Saxl, Generation of finite almost simple groups by conjugates, *J. Algebra* **268** (2003), 519–571.

[20] Z. Halasi, On the base size for the symmetric group acting on subsets, *Studia Sci. Math. Hungar.* **49** (2012), 492–500.

[21] P. Kleidman and M. W. Liebeck, The subgroup structure of the finite classical groups, LMS Lecture Note Series, 129. Cambridge University Press, Cambridge, 1990.

[22] M. Lee and M.W. Liebeck, Bases of quasisimple linear groups, preprint.

[23] M.W. Liebeck, On minimal degrees and base sizes of primitive permutation groups, *Arch. Math. (Basel)* **43** (1984), 11–15.

[24] M.W. Liebeck, C.E. Praeger and J. Saxl, On the O'Nan-Scott theorem for finite primitive permutation groups, *J. Austral. Math. Soc. Ser. A* **44** (1988), 389–396.

[25] M.W. Liebeck and A. Shalev, Simple groups, permutation groups, and probability, *J. Amer. Math. Soc.* **12** (1999), 497–520.

[26] M.W. Liebeck and A. Shalev, Bases of primitive linear groups, *J. Algebra* **252** (2002), 95–113.

[27] M.W. Liebeck and A. Shalev, Bases of primitive linear groups II, *J. Algebra* **403** (2014), 223–228.

[28] A. Maróti, On the orders of primitive groups, *J. Algebra* **258** (2002), no. 2, 631–640.

[29] C.E. Praeger, The inclusion problem for finite primitive permutation groups, *Proc. London Math. Soc.* (3) **60** (1990), 68–88.

[30] L. Pyber, On the orders of doubly transitive permutation groups, elementary estimates, *J. Combin. Theory Ser. A* **62** (1993), 361–366.

[31] L. Pyber, Asymptotic results for permutation groups, Groups and computation, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 11 (ed. L. Finkelstein and W. M. Kantor) Amer. Math. Soc. Providence RI 1993, pp. 197–219.

[32] Á. Seress, The minimal base size of primitive solvable permutation groups, *J. London Math. Soc.* **53** (1996), 243–255.

[33] J. Siemons and Z. Zalesskii, Intersections of matrix algebras and permutation representations of $PSL(n, q)$, *J. Algebra* **226** (2000), 451–478.

DEPARTMENT OF ALGEBRA AND NUMBER THEORY, EÖTVÖS UNIVERSITY, PÁZMÁNY PÉTER SÉTÁNY 1/C, H-1117, BUDAPEST, HUNGARY AND ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, REÁLTANODA UTCA 13-15, H-1053, BUDAPEST, HUNGARY

E-mail address: zhalasi@cs.elte.hu and halasi.zoltan@renyi.mta.hu

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON SW7 2BZ, UNITED KINGDOM

E-mail address: m.liebeck@imperial.ac.uk

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, REÁLTANODA UTCA 13-15, H-1053, BUDAPEST, HUNGARY

E-mail address: maroti.attila@renyi.mta.hu