POISSON CENTRALIZER OF THE TRACE

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Abstract. The Poisson centralizer of the trace element $\sum_i x_{i,i}$ is determined in the coordinate ring of SL_n endowed with the Poisson structure obtained as the semiclassical limit of its quantized coordinate ring. It turns out that this maximal Poisson-commutative subalgebra coincides with the subalgebra of invariants with respect to the adjoint action.

1. Introduction

The semiclassical limit Poisson structure on $\mathcal{O}(SL_n)$ received considerable attention recently because of the connection between the primitive ideals of the quantized coordinate ring $\mathcal{O}_q(SL_n)$ and the symplectic leaves of the Poisson manifold SL_n (see for example [HL2],[G],[Y]). In this paper, we present another relation between $\mathcal{O}(SL_n)$ endowed with the semiclassical limit Poisson structure and $\mathcal{O}_q(SL_n)$.

In [M] it was shown that if $q \in \mathbb{C}^\times$ is not a root of unity then the centralizer of the trace element $\overline{\sigma}_1 = \sum_i x_{i,i}$ in $\mathcal{O}_q(SL_n)$ (resp. in $\mathcal{O}_q(M_n)$ and $\mathcal{O}_q(GL_n)$) is a maximal commutative subalgebra, generated by certain sums of principal quantum minors. By Theorem 2.4 and 5.1 in [DL2], this subalgebra coincides with the subalgebra of cocommutative elements in $\mathcal{O}_q(SL_n)$ and also with the subalgebra of invariants of the adjoint coaction. (This result is generalized in [AZ] for arbitrary characteristic and q being a root of unity.)

On the Poisson algebra side, the corresponding Poisson-subalgebra of $\mathcal{O}(SL_n)$ is generated by the coefficients of the characteristic polynomial $\overline{c}_1, \dots, \overline{c}_{n-1}$. We prove the following:

Theorem 1.1. *For $n \geq 1$ the subalgebra $\mathbb{C}[\overline{c}_1, \dots, \overline{c}_{n-1}]$ (resp. $\mathbb{C}[c_1, \dots, c_n]$ and $\mathbb{C}[c_1, \dots, c_n, c_n^{-1}]$) is maximal Poisson-commutative in $\mathcal{O}(SL_n)$ (resp. $\mathcal{O}(M_n)$ and $\mathcal{O}(GL_n)$) with respect to the semiclassical limit Poisson structure.*

It is easy to deduce from [DL1] or [DL2] that $\{c_i, c_j\} = 0$ $(1 \leq i, j \leq n)$ in $\mathcal{O}(M_n)$ (see Proposition 5.1 below). Therefore, Theorem 1.1 is a direct consequence of the following statement:

Theorem 1.2. *For $n \geq 1$ the Poisson-centralizer of $\overline{c}_1$ in $\mathcal{O}(SL_n)$ (resp. $c_1 \in \mathcal{O}(M_n)$ and $\mathcal{O}(GL_n)$) equipped with the semiclassical limit Poisson bracket is generated as a subalgebra by*

- *$\overline{c}_1, \dots \overline{c}_{n-1}$ in the case of $\mathcal{O}(SL_n)$,*
- *$c_1, \dots, c_n$ in the case of $\mathcal{O}(M_n)$, and*
- *$c_1, \dots, c_n, c_n^{-1}$ in the case of $\mathcal{O}(GL_n)$.*

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The proof is based on modifying the Poisson bracket of the algebras that makes an induction possible. A similar idea is used in the proof of the analogous result in the quantum setup (see [M]).

It is well known that the coefficient functions $c_1, \ldots, c_n \in \mathcal{O}(M_n)$ of the characteristic polynomial generate the subalgebra $\mathcal{O}(M_n)^{GL_n}$ of GL_n-invariants with respect to the adjoint action. This implies that the subalgebra coincides with the Poisson center of the coordinate ring $\mathcal{O}(M_n)$ endowed with the Kirillov-Kostant-Souriau (KKS) Poisson bracket. Hence, Theorem 1.1 for $\mathcal{O}(M_n)$ can be interpreted as an interesting interplay between the KKS and the semiclassical limit Poisson structure. Namely, while the subalgebra $\mathcal{O}(M_n)^{GL_n}$ is contained in every maximal Poisson-commutative subalgebra with respect to the former Poisson bracket, it is contained in only one maximal Poisson-commutative subalgebra (itself) with respect to the latter Poisson bracket.

A Poisson-commutative subalgebra is also called an involutive (or Hamiltonian) system, while a maximal one is called a complete involutive system (see Section 2 or [V]). Such a system is integrable if the (Krull) dimension of the generated subalgebra is sufficiently large. In our case, the subalgebra generated by the elements $c_1, \ldots, c_{n-1}$ is not integrable, as its dimension is $n-1$ (resp. n for GL_n) instead of the required $\binom{n+1}{2} - 1$ (resp. $\binom{n+1}{2}$ for GL_n), see Remark 5.3.

The article is organized as follows: First, we introduce the required notions, and in Section 3 we prove that the three statements in Theorem 1.2 are equivalent. In Section 3.1, we prove Theorem 1.2 for $n = 2$ as a starting case of an induction presented in Section 5 that completes the proof of the theorem. In the article, every algebra is understood over the field $\mathbb{C}$.

2. Preliminaries

2.1. **Poisson algebras.** First, we collect the basic notions about Poisson algebras we use in the article. For further details about Poisson algebras, see [V].

A commutative Poisson algebra $\big(A, \{.,.\}\big)$ is a unital commutative associative algebra A together with a bilinear operation $\{.,.\} : A \times A \to A$ called the Poisson bracket such that it is antisymmetric, satisfies the Jacobi identity, and for any $a \in A$, $\{a, .\} : A \to A$ is a derivation. For commutative Poisson algebras A and B, the map $\varphi : A \to B$ is a morphism of Poisson algebras if it is both an algebra homomorphism and a Lie-homomorphism.

There is a natural notion of Poisson subalgebra (i.e. a subalgebra that is also a Lie-subalgebra), Poisson ideal (i.e an ideal that is also a Lie-ideal) and quotient Poisson algebra (as the quotient Lie-algebra inherits the bracket). The Poisson centralizer $C(a)$ of an element $a \in A$ is defined as $\{b \in A \mid \{a, b\} = 0\}$. Clearly, it is a Poisson subalgebra. Analogously, $a \in A$ is called Poisson-central if $C(a) = A$. One says that a subalgebra $C \leq A$ is Poisson-commutative (or involutive) if $\{c, d\} = 0$ for all $c, d \in C$ and it is maximal Poisson-commutative (or maximal involutive) if there is no Poisson-commutative subalgebra in A that strictly contains C.

The Poisson center (or Casimir subalgebra) of A is $Z(A) := \{a \in A \mid C(a) = A\}$. Let A be a reduced, finitely generated commutative Poisson algebra. The rank $\mathrm{Rk}\{.,.\}$ of the Poisson structure $\{.,.\}$ is defined by the rank of the matrix $\big(\{g_i, g_j\}\big)_{i,j} \in A^{N \times N}$ for a generating system $g_1, \ldots, g_N \in A$. (One can prove that it is independent of the chosen generating system.) A maximal Poisson-commutative subalgebra C is called integrable if

$$\dim C = \dim A - \frac{1}{2}\mathrm{Rk}\{.,.\}$$

The inequality $\leq$ holds for any Poisson-commutative subalgebra (Proposition II.3.4 in [V]), hence integrability is a maximality condition on the size of C that does not necessarily hold for every maximal involutive system.

2.2. **Filtered Poisson algebras.**

Definition 2.1. A filtered Poisson algebra is a Poisson algebra together with an ascending chain of subspaces $\{\mathcal{F}^d\}_{d\in\mathbb{N}}$ in A such that

- $A = \cup_{d\in\mathbb{N}}\mathcal{F}^d$,
- $\mathcal{F}^d \cdot \mathcal{F}^e \subseteq \mathcal{F}^{d+e}$ for all $d, e \in \mathbb{N}$, and
- $\{\mathcal{F}^d, \mathcal{F}^e\} \subseteq \mathcal{F}^{d+e}$ for all $d, e \in \mathbb{N}$.

Together with the filtration preserving morphisms of Poisson algebras, they form a category.

For a filtered Poisson algebra A, we may define its associated graded Poisson algebra gr$A$ as

$$\mathrm{gr}(A) := \bigoplus_{d\in\mathbb{N}} \mathcal{F}^d/\mathcal{F}^{d-1}$$

where we used the simplifying notation $\mathcal{F}^{-1} = \{0\}$. The multiplication of $\mathrm{gr}(A)$ is defined the usual way:

$$\mathcal{F}^d/\mathcal{F}^{d-1} \times \mathcal{F}^e/\mathcal{F}^{e-1} \to \mathcal{F}^{d+e}/\mathcal{F}^{d+e-1}$$

$$\left(x + \mathcal{F}^{d-1}, y + \mathcal{F}^{e-1}\right) \mapsto xy + \mathcal{F}^{d+e-1}$$

Analogously, the Poisson structure of $\mathrm{gr}(A)$ is defined by $\left(x + \mathcal{F}^{d-1}, y + \mathcal{F}^{e-1}\right) \mapsto \{x, y\} + \mathcal{F}^{d+e-1}$. One can check that this way $\mathrm{gr}(A)$ is a Poisson algebra.

Let $(S, +)$ be an abelian monoid. (We will only use this definition for $S = \mathbb{N}$ and $S = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$.) An S-graded Poisson algebra R is a Poisson algebra together with a fixed grading

$$R = \oplus_{d\in S} R_d$$

such that R is both a graded algebra (i.e. $R_d \cdot R_e \subseteq R_{d+e}$ for all $d, e \in S$) and a graded Lie algebra (i.e. $\{R_d, R_e\} \subseteq R_{d+e}$ for all $d, e \in S$) with respect to the given grading.

The above construction $A \mapsto \mathrm{gr}(A)$ yields an $\mathbb{N}$-graded Poisson algebra. In fact, $\mathrm{gr}(.)$ can be turned into a functor: for a morphism of filtered Poisson algebras $f : \left(A, \{\mathcal{F}^d\}_{d\in\mathbb{N}}\right) \to \left(B, \{\mathcal{G}^d\}_{d\in\mathbb{N}}\right)$ we define

$$\mathrm{gr}(f) : \mathrm{gr}(A) \to \mathrm{gr}(B) \qquad \left(x_d + \mathcal{F}^{d-1}\right)_{d\in\mathbb{N}} \mapsto \left(f(x_d) + \mathcal{G}^{d-1}\right)_{d\in\mathbb{N}}$$

One can check that it is indeed well defined and preserves composition.

Remark 2.2. Given an $\mathbb{N}$-graded Poisson algebra $R = \oplus_{d\in\mathbb{N}} R_d$, one has a natural way to associate a filtered Poisson algebra to it. Namely, let $\mathcal{F}^d := \oplus_{k\leq d} R_k$. In this case, the associated graded Poisson algebra gr$R$ of $\left(R, \{\mathcal{F}^d\}_{d\in\mathbb{N}}\right)$ is isomorphic to R.

2.3. **The Kirillov-Kostant-Souriau bracket.** A classical example of a Poisson algebra is given by the Kirillov-Kostant-Souriau (KKS) bracket on $\mathcal{O}(\mathfrak{g}^*)$, the coordinate ring of the dual of a finite-dimensional (real or complex) Lie algebra $\big(\mathfrak{g}, [.,.]\big)$ (see [ChP] Example 1.1.3, or [W] Section 3).

It is defined as follows: a function $f \in \mathcal{O}(\mathfrak{g}^*)$ at a point $v \in \mathfrak{g}^*$ has a differential $\mathrm{d}f_v \in T_v^*\mathfrak{g}^*$ where we can canonically identify the spaces $T_v^*\mathfrak{g}^* \cong T_0^*\mathfrak{g}^* \cong \mathfrak{g}^{**} \cong \mathfrak{g}$. Hence, we may define the Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$ as

$$\{f, g\}(v) := [\mathrm{d}f_v, \mathrm{d}g_v](v)$$

for all $f, g \in \mathcal{O}(\mathfrak{g}^*)$ and $v \in \mathfrak{g}^*$. It is clear that it is a Lie-bracket but it can be checked that the Leibniz-identity is also satisfied. For $\mathfrak{g} = \mathfrak{gl}_n$, it gives a Poisson bracket on $\mathcal{O}(M_n)$.

Alternatively, one can define this Poisson structure via semiclassical limits.

2.4. **Semiclassical limits.** Let $A = \cup_{d\in\mathbb{Z}}\mathcal{A}^d$ be a $\mathbb{Z}$-filtered algebra such that its associated graded algebra $\mathrm{gr}(A) := \oplus_{d\in\mathbb{Z}}\mathcal{A}^d/\mathcal{A}^{d-1}$ is commutative. The Rees ring of A is defined as

$$\mathrm{Rees}(A) := \bigoplus_{d\in\mathbb{Z}} \mathcal{A}^d h^d \subseteq A[h, h^{-1}]$$

Using the obvious multiplication, it is a $\mathbb{Z}$-graded algebra. The semiclassical limit of A is the Poisson algebra $\mathrm{Rees}(A)/h\mathrm{Rees}(A)$ together with the bracket

$$\{a + h\mathcal{A}^m, b + h\mathcal{A}^n\} := \frac{1}{h}[a, b] + \mathcal{A}^{n+m-2} \in \mathcal{A}^{n+m-1}/\mathcal{A}^{n+m-2}$$

for all homogeneous elements $a + h\mathcal{A}^m \in \mathcal{A}^{m+1}/h\mathcal{A}^m$, $b + h\mathcal{A}^n \in \mathcal{A}^{n+1}/h\mathcal{A}^n$. The definition is valid as the underlying algebra of $\mathrm{Rees}(A)/h\mathrm{Rees}(A)$ is $\mathrm{gr}(A)$ that is assumed to be commutative, hence $[a, b] \in h\mathcal{A}^{m+n-1}$.

The Poisson algebra $\mathcal{O}(\mathfrak{g}^*)$ with the KKS bracket can be obtained as the semiclassical limit of $U\mathfrak{g}$, see [G], Example 2.6.

2.5. **Quantized coordinate rings.** Assume that $n \in \mathbb{N}^+$ and define $\mathcal{O}_t(M_n)$ as the unital $\mathbb{C}$-algebra generated by the n^2 generators $x_{i,j}$ for $1 \le i, j \le n$ over $\mathbb{C}[t, t^{-1}]$ that are subject to the following relations:

$$x_{i,j}x_{k,l} = \begin{cases} x_{k,l}x_{i,j} + (t - t^{-1})x_{i,l}x_{k,j} & \text{if } i < k \text{ and } j < l \\ t x_{k,l}x_{i,j} & \text{if } (i = k \text{ and } j < l) \text{ or } (j = l \text{ and } i < k) \\ x_{k,l}x_{i,j} & \text{if } (i > k \text{ and } j < l) \text{ or } (j > l \text{ and } i < k) \end{cases}$$

for all $1 \le i, j, k, l \le n$. It turns out to be a finitely generated $\mathbb{C}[t, t^{-1}]$-algebra that is a Noetherian domain. (For a detailed exposition, see [BG].) Furthermore, it can be endowed with a coalgebra structure by setting $\varepsilon(x_{i,j}) = \delta_{i,j}$ and $\Delta(x_{i,j}) = \sum_{k=1}^n x_{i,k} \otimes x_{k,j}$. It turns $\mathcal{O}_t(M_n)$ into a bialgebra.

For $q \in \mathbb{C}^\times$, the quantized coordinate ring of $n \times n$ matrices with parameter q is defined as the $\mathbb{C}$-algebra

$$\mathcal{O}_q(M_n) := \mathcal{O}_t(M_n)/(t - q)$$

In this article, we only deal with the case when q is not a root of unity, then the algebra is called the generic quantized coordinate ring of M_n.

Similarly, one can define the non-commutative deformations of the coordinate rings of GL_n and SL_n using the quantum determinant

$$\det_q := \sum_{s\in S_n} (-q)^{\ell(s)} x_{1,s(1)} x_{2,s(2)} \dots x_{n,s(n)}$$

where $\ell(\sigma)$ stands for the length of σ in the Coxeter group S_n. Then – analogously to the classical case – one defines

$$\mathcal{O}_q(SL_n) := \mathcal{O}_q(M_n)/(\det_q - 1) \qquad \mathcal{O}_q(GL_n) := \mathcal{O}_q(M_n)\big[\det_q^{-1}\big]$$

by localizing at the central element $\det_q$.

2.6. **Semiclassical limits of quantized coordinate rings.** The semiclassical limits of $\mathcal{O}_q(SL_n)$ can be obtained via the slight modification of process of Section 2.4 (see [G], Example 2.2). The algebra $R := \mathcal{O}_t(M_n)$ can be endowed with a $\mathbb{Z}$-filtration by defining $\mathcal{F}^n$ to be the span of monomials that are the product of at most n variables. However, instead of defining a Poisson structure on $\mathrm{Rees}(R)/h\mathrm{Rees}(R)$ with respect to this filtration, consider the algebra $R/(t-1)R$ that is isomorphic to $\mathcal{O}(M_n)$ as an algebra. The semiclassical limit Poisson bracket is defined as

$$\{\bar{a},\bar{b}\} := \frac{1}{t-1}(ab-ba) + (t-1)R \in R/(t-1)R$$

for any two representing elements $a, b \in R$ for $\bar{a},\bar{b} \in R/(t-1)R$. One can check that it is a well-defined Poisson bracket.

This Poisson structure of $\mathcal{O}(M_n)$ can be given explicitly by the following relations:

$$\{x_{i,j}, x_{k,l}\} = \begin{cases} 2x_{i,l}x_{k,j} & \text{if } i<k \text{ and } j<l \\ x_{i,j}x_{k,l} & \text{if } (i=k \text{ and } j<l) \text{ or } (j=l \text{ and } i<k) \\ 0 & \text{otherwise} \end{cases}$$

extended according to the Leibniz-rule (see [G]). It is a quadratic Poisson structure in the sense of [V], Definition II.2.6. The semiclassical limit for GL_n and SL_n is defined analogously using $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ or by localization (resp. by taking quotient) at the Poisson central element det (resp. $\det - 1$) in $\mathcal{O}(M_n)$.

2.7. **Coefficients of the characteristic polynomial.** Consider the characteristic polynomial function $M_n \to \mathbb{C}[x]$, $A \mapsto \det(A - xI)$. Let us define the elements $c_0, c_1, \dots, c_n \in \mathcal{O}(M_n)$ as

$$\det(A - xI) = \sum_{i=0}^{n} (-1)^i c_i x^{n-i}$$

In particular, $c_0 = 1$, $c_1 = \mathrm{tr}$ and $c_n = \det$. Their images in $\mathcal{O}(SL_n) \cong \mathcal{O}(M_n)/(\det - 1)$ are denoted by $\overline{c}_1, \dots, \overline{c}_{n-1}$. If ambiguity may arise, we will write $c_i(A)$ for the element corresponding to c_i for an algebra A with a fixed isomorphism $A \cong \mathcal{O}(M_k)$ for some k.

The coefficient functions $c_1, \dots, c_n$ can also be expressed via matrix minors as follows: For $I, J \subseteq \{1, \dots, n\}$, $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ define

$$[I \mid J] := \sum_{s\in S_k} \mathrm{sgn}(s) x_{i_1, j_{s(1)}} \dots x_{i_k, j_{s(k)}}$$

i.e. it is the determinant of the subalgebra generated by $\{x_{i,j}\}_{i\in I, j\in J}$ that can be identified with $\mathcal{O}(M_k)$. Then

$$c_i = \sum_{|I|=i} [I \mid I] \in \mathcal{O}(M_n)$$

for all $1 \leq i \leq n$. It is well.known that $c_1, \dots, c_n$ generate the same subalgebra of $\mathcal{O}(M_n)$ as the trace functions $A \mapsto \mathrm{Tr}(A^k)$, namely, the subalgebra $\mathcal{O}(M_n)^{GL_n}$ of GL_n-invariants with respect to the adjoint action.

3. Equivalence of the statements

Consider $\mathcal{O}(M_n)$ endowed with the semiclassical limits Poisson bracket. As it is discussed in the Introduction, Theorem 1.1 follows directly from Theorem 1.2 and Proposition 5.1.

The following proposition shows that it is enough to prove Theorem 1.2 for the case of $\mathcal{O}(M_n)$.

Proposition 3.1. *For any $n \in \mathbb{N}^+$ the following are equivalent:*

1. *The Poisson-centralizer of $c_1 \in \mathcal{O}(M_n)$ is generated by $c_1, \dots, c_n$.*
2. *The Poisson-centralizer of $c_1 \in \mathcal{O}(GL_n)$ is generated by $c_1, \dots, c_n, c_n^{-1}$.*
3. *The Poisson-centralizer of $\overline{c}_1 \in \mathcal{O}(SL_n)$ is generated by $\overline{c}_1, \dots, \overline{c}_{n-1}$.*

Proof. The first and second statements are equivalent as det is a Poisson-central element, so we have $\{c_1, h \cdot \det^k\} = \{c_1, h\} \cdot \det^k$ for any $h \in \mathcal{O}(GL_n)$ and $k \in \mathbb{Z}$. Hence,

$$\mathcal{O}(GL_n) \supseteq C(c_1) = \big(\mathcal{O}(M_n) \cap C(c_1)\big)[\det^{-1}]$$

proving 1) $\iff$ 2).

1) $\iff$ 3): First, assume 1) and let $\overline{h} \in \mathcal{O}(SL_n)$ such that $\{\overline{c}_1, \overline{h}\} = 0$. Since $\mathcal{O}(SL_n)$ is $\mathbb{Z}/n\mathbb{Z}$-graded (inherited from the $\mathbb{N}$-grading of $\mathcal{O}(M_n)$) and $\overline{c}_1$ is homogeneous with respect to this grading, its Poisson-centralizer is generated by $\mathbb{Z}/n\mathbb{Z}$-homogeneous elements, so we may assume that $\overline{h}$ is $\mathbb{Z}/n\mathbb{Z}$-homogeneous.

Let $k = \deg(\overline{h}) \in \mathbb{Z}/n\mathbb{Z}$. Let $h \in \mathcal{O}(M_n)$ be a lift of $\overline{h} \in \mathcal{O}(SL_n)$ and consider the $\mathbb{N}$-homogeneous decomposition $h = \sum_{j=0}^{d} h_{jn+k}$ of h, where h_{jn+k} is homogeneous of degree $jn + k$ for all $j \in \mathbb{N}$. Define

$$h' := \sum_{j=0}^{d} h_{jn+k} \det^{d-j} \in \mathcal{O}(M_n)_{dn+k}$$

that is a homogeneous element of degree $dn + k$ representing $\overline{h} \in \mathcal{O}(SL_n)$ in $\mathcal{O}(M_n)$. Then $\{c_1, h'\} \in (\det -1) \cap \mathcal{O}(M_n)_{dn+k+1}$ since $\{\overline{c}_1, \overline{h'}\} = \{\overline{c}_1, \overline{h}\} = 0$, c_1 is homogeneous of degree 1 and the Poisson-structure is graded. Clearly, $(\det -1) \cap \mathcal{O}(M_n)_{dn+k+1} = 0$ hence $\{c_1, h'\} = 0$. Applying 1) gives $h' \in \mathbb{C}[c_1, \dots, c_n]$ so $\overline{h} \in \mathbb{C}[\overline{c}_1, \dots, \overline{c}_{n-1}]$ as we claimed.

Conversely, assume 3) and let $h \in \mathcal{O}(M_n)$ such that $\{c_1, h\} = 0$. Since c_1 is $\mathbb{N}$-homogeneous, we may assume that h is also $\mathbb{N}$-homogeneous and so the image $\overline{h} \in \mathcal{O}(SL_n)$ of h is $\mathbb{Z}/n\mathbb{Z}$-homogeneous. By the assumption, $\overline{h} = p(\overline{c}_1, \dots, \overline{c}_{n-1})$ for some $p \in \mathbb{C}[t_1, \dots, t_{n-1}]$. Endow $\mathbb{C}[t_1, \dots, t_n]$ with the $\mathbb{N}$-grading $\deg(t_i) = i$. As $\overline{h}$ is $\mathbb{Z}/n\mathbb{Z}$-homogeneous, we may choose $p \in \mathbb{C}[t_1, \dots, t_{n-1}]$ so that its homogeneous components are all of degree $dn + \deg(\overline{h}) \in \mathbb{N}$ with respect to the above grading for some $d \in \mathbb{N}$.

By $h - p(c_1, \dots, c_{n-1}) \in (\det -1)$ and the assumptions on degrees, we may choose a polynomial $q \in \mathbb{C}[t_1, \dots, t_n]$ that is homogeneous with respect to the above grading and $q(t_1,\dots,t_{n-1},1)=p$. Let $h':=h\cdot\det^r$ where $r:=\frac{1}{n}(\deg q-\deg h)\in\mathbb{Z}$ so $\deg(h')=\deg(q)\in\mathbb{N}$. Then

$$h'-q(c_1,\dots,c_n)\in(\det-1)\cap\mathcal{O}(M_n)_{\deg q}=0$$

hence $h'\in\mathbb{C}[c_1,\dots,c_n]$ and $h\in\mathbb{C}[c_1,\dots,c_n,c_n^{-1}]$. This is enough as $\mathbb{C}[c_1,\dots,c_n,c_n^{-1}]\cap\mathcal{O}(M_n)=\mathbb{C}[c_1,\dots,c_n]$ by the definitions. $\square$

4. Case of $\mathcal{O}(SL_2)$

In this section, we prove Theorem 1.2 for $\mathcal{O}(SL_2)$ that is the first step of the induction in the proof of the general case.

We denote by a,b,c,d the generators $\overline{x}_{1,1},\overline{x}_{1,2},\overline{x}_{2,1},\overline{x}_{2,2}\in\mathcal{O}(SL_2)$ and $\mathrm{tr}:=\overline{c}_1=a+d$.

Proposition 4.1. *The centralizer of* $\mathrm{tr}\in\mathcal{O}(SL_2)$ *is* $\mathbb{C}[\mathrm{tr}]$.

By $ad-bc=1$ we have a monomial basis of $\mathcal{O}(SL_2)$ consisting of

$$a^ib^kc^l,\ b^kc^ld^j,\ b^kc^l \qquad (i,j\in\mathbb{N}^+,\ k,l\in\mathbb{N})$$

The Poisson bracket on the generators is the following:

$$\{a,b\}=ab \qquad \{a,c\}=ac \qquad \{a,d\}=2bc$$
$$\{b,c\}=0 \qquad \{b,d\}=bd \qquad \{c,d\}=cd$$

The action of $\{\mathrm{tr},.\}$ on the basis elements can be written as

$$\begin{aligned}&\{(a+d),a^ib^kc^l\}=\\&=(k+l)a^{i+1}b^kc^l-2ia^{i-1}b^{k+1}c^{l+1}-(k+l)a^ib^kc^ld\\&=(k+l)a^{i+1}b^kc^l-(2i+k+l)a^{i-1}b^{k+1}c^{l+1}-(k+l)a^{i-1}b^kc^l\end{aligned}$$

By the same computation on b^kc^l and $b^kc^ld^j$ one obtains

$$\{(a+d),b^kc^l\}=(k+l)ab^kc^l-(k+l)b^kc^ld$$

$$\{(a+d),b^kc^ld^j\}=(k+l+2j)b^{k+1}c^{l+1}d^{j-1}+(k+l)b^kc^ld^{j-1}-(k+l)b^kc^ld^{j+1}$$

Hence, for a polynomial $p\in\mathbb{C}[t_1,t_2]$ and $i\geq 1$:

$$\begin{aligned}\{(a+d),a^ip(b,c)\} &= a^{i+1}\sum_m m\cdot p_m(b,c)\\ &\quad -a^{i-1}\sum_m\big((2i+m)bc+m\big)p_m(b,c)\end{aligned}\tag{4.1}$$

where p_m is the m-th homogeneous component of p. The analogous computations for $p(b,c)d^j$ $(j\geq 1)$ and $p(b,c)$ give

$$\begin{aligned}\{(a+d),p(b,c)d^j\} &= -d^{j+1}\sum_m m\cdot p_m(b,c)\\ &\quad +d^{j-1}\sum_m\big((m+2j)bc+m\big)p_m(b,c)\end{aligned}\tag{4.2}$$

$$\{(a+d),p(b,c)\}=(a-d)\sum_m m\cdot p_m(b,c)\tag{4.3}$$

Proof of Proposition 4.1. Assume that $0 \neq g \in C(\mathrm{tr})$ and write it as

$$g = \sum_{i=1}^{\alpha} a^i r_i + \sum_{j=1}^{\beta} s_j d^j + u$$

where r_i, s_j and u are elements of $\mathbb{C}[b, c]$, and α and β are the highest powers of a and d appearing in the decomposition.

We prove that $r_\alpha \in \mathbb{C} \cdot 1$. If $\alpha = 0$ then $r_\alpha = u$ so the $a^i b^k c^l$ terms $(i > 0)$ in $\{a + d, g\}$ are the same as the $a^i b^k c^l$ terms in $\{a + d, u\}$ by Eq. 4.1, 4.2 and 4.3. However, by 4.3, these terms are nonzero if $u \notin \mathbb{C}$ and that is a contradiction. Assume that $\alpha \geq 1$ and for a fixed $k \in \mathbb{N}$ define the subspace

$$\mathcal{A}^k := \sum_{l \leq k} a^l \mathbb{C}[b, c, d] \subseteq \mathcal{O}(SL_2)$$

By $\left\{\mathrm{tr}, \mathcal{A}^{\alpha-1}\right\} \subseteq \mathcal{A}^\alpha$ we have

$$\mathcal{A}^\alpha = \{\mathrm{tr}, g\} + \mathcal{A}^\alpha = \left\{\mathrm{tr}, a^\alpha r_\alpha + \mathcal{A}^{\alpha-1}\right\} + \mathcal{A}^\alpha =$$

$$= a^\alpha \{\mathrm{tr}, r_\alpha\} + \alpha a^{\alpha-1} bc r_\alpha + \mathcal{A}^\alpha = a^\alpha \{\mathrm{tr}, r_\alpha\} + \mathcal{A}^\alpha$$

By Eq. 4.3 it is possible only if $\{\mathrm{tr}, r_\alpha\} = 0$ so $r_\alpha \in \mathbb{C}[b, c] \cap C(\mathrm{tr}) = \mathbb{C} \cdot 1$.

If $\alpha > 0$ we may simplify g by subtracting polynomials of tr from it. Indeed, by $r_\alpha \in \mathbb{C}^\times$ we have $g - r_\alpha \mathrm{tr}^\alpha \in \mathcal{A}^{\alpha-1} \cap C(\mathrm{tr})$ so we can replace g by $g - r_\alpha \mathrm{tr}^\alpha$. Hence, we may assume that $\alpha = 0$. Then, again, $r_\alpha = u \in \mathbb{C} \cdot 1 \subseteq C(\mathrm{tr})$ so we may also assume that $u = 0$.

If g is nonzero after the simplification, we get a contradiction. Indeed, let $p(b, c)d^\gamma$ be the summand of g with the smallest $\gamma \in \mathbb{N}$. By the above simplifications, $\gamma \geq 1$. Then the coefficient of $d^{\gamma-1}$ in $\{\mathrm{tr}, g\}$ is the same as the coefficient of $d^{\gamma-1}$ in

$$\{\mathrm{tr}, p(b, c)d^\gamma\} = \{\mathrm{tr}, p(b, c)\}d^\gamma + 2\gamma bcp(b, c)d^{\gamma-1}$$

so it is $2\gamma bcp(b, c)d^{\gamma-1}$ that is nonzero if $p(b, c) \neq 0$ and $\gamma \geq 1$. That is a contradiction. $\square$

5. Proof of the main result

Let $n \geq 2$ and let us denote $A_n := \mathcal{O}(M_n)$.

Proposition 5.1. *$\mathbb{C}[\sigma_1, \ldots, \sigma_n] \leq A_n$ is a Poisson-commutative subalgebra.*

Proof. Consider the principal quantum minor sums

$$\sigma_i = \sum_{|I|=i} \sum_{s \in S_i} t^{-\ell(s)} x_{i_1, i_{s(1)}} \ldots x_{i_t, i_{s(t)}} \in \mathcal{O}_t(M_n)$$

When A_n is viewed as the semiclassical limit $R/(t-1)R$ where $R = \mathcal{O}_t(M_n)$ (see Subsection 2.5), one can see that σ_i represents $c_i \in R/(t-1)R \cong \mathcal{O}(M_n)$. In [DL1], it is proved that $\sigma_i \sigma_j = \sigma_j \sigma_i$ in $\mathcal{O}_q(M_n)$ if q is not a root of unity, in particular, if q is transcendental.

Since the algebra $\mathcal{O}_q(M_n)$ is defined over $\mathbb{Z}[q, q^{-1}]$, the elements $\sigma_1, \ldots, \sigma_n$ (that are defined over $\mathbb{Z}[q, q^{-1}]$) commute in $\mathcal{O}_q\big(M_n(\mathbb{Z})\big) \leq \mathcal{O}_q\big(M_n(\mathbb{C})\big)$ as well. Hence, $\sigma_1, \ldots, \sigma_n$ also commute after extension of scalars, i.e. in the ring $\mathcal{O}_q\big(M_n(\mathbb{Z})\big) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathcal{O}_t\big(M_n(\mathbb{C})\big)$. Consequently, in $A_n \cong R/(t-1)R$ the subalgebra $\mathbb{C}[c_1, \ldots, c_n]$ is a Poisson-commutative subalgebra, by the definition of semiclassical limit. $\square$

By Proposition 5.1, $\mathbb{C}[c_1,\ldots,c_n]$ is in the Poisson-centralizer $C(c_1)$. To prove the converse, Theorem 1.2, we need further notations. Consider the Poisson ideal

$$I := (x_{1,j},\, x_{i,1} \mid 2 \leq i,j \leq n) \lhd A_n$$

We will denote its quotient Poisson algebra by $B_{2,n} := A_n/I$ and the natural surjection by $\varphi : A_n \to B_{2,n}$. Note that $B_{2,n} \cong A_{n-1}[t]$ as Poisson algebras by $x_{i,j} + I \mapsto x_{i-1,j-1}$ $(2 \leq i,j \leq n)$ and $x_{1,1} \mapsto t$ where the bracket of $A_{n-1}[t]$ is the trivial extension of the bracket of A_{n-1} by $\{t,a\} = 0$ for all $a \in A_{n-1}[t]$.

Furthermore, D_n will stand for $\mathbb{C}[t_1,\ldots,t_n]$ endowed with the zero Poisson bracket. Define the map $\delta : B_{2,n} \to D_n$ as $x_{i,j} + I \mapsto \delta_{i,j}t_i$ that is morphism of Poisson algebras by $\{x_{i,i}, x_{j,j}\} \in I$. Note that $(\delta \circ \varphi)(c_i) = s_i$, the elementary symmetric polynomial in $t_1,\ldots,t_n$. In particular, $\delta \circ \varphi$ restricted to $\mathbb{C}[c_1,\ldots,c_n]$ is an isomorphism onto the symmetric polynomials in $t_1,\ldots,t_n$ by the fundamental theorem of symmetric polynomials. In the proof of Theorem 1.2 we verify the same property for $C(\sigma_1)$.

Although the algebras A_n, $B_{2,n}$ and D_n are $\mathbb{N}$-graded Poisson algebras (see Section 2) using the total degree of A_n and the induced gradings on the quotients, we will instead consider them as filtered Poisson algebras where the filtration is not the one that corresponds to this grading. For each $d \in \mathbb{N}$, let us define

$$\mathcal{A}^d = \{a \in A_n \mid \deg_{x_{1,1}}(a) \leq d\}$$

This is indeed a filtration on A_n. Note, that the grading $\deg_{x_{1,1}}$ is incompatible with the bracket by $\{x_{1,1}, x_{2,2}\} = x_{1,2}x_{2,1}$. The algebras $B_{2,n}$, D_n and $C(c_1)$ inherit a filtered Poisson algebra structure as they are Poisson sub- and quotient algebras of A_n so we may take $\mathcal{B}^d := \varphi(\mathcal{A}^d)$, $\mathcal{D}^d := (\delta \circ \varphi)(\mathcal{A}^d)$ and $\mathcal{C}^d = \mathcal{A}^d \cap C(c_1)$. This way, the natural surjections φ and δ and the embedding $C(c_1) \hookrightarrow A_n$ are maps of filtered Poisson algebras.

In the proof of Theorem 1.2 we use the associated graded Poisson algebras of $B_{2,n}$, D_n and $C(c_1)$ (see Section 2). First, we describe the structure of these. The filtrations on $B_{2,n}$ and D_n are induced by the $x_{1,1}$- and t_1-degrees, hence we have $\mathrm{gr}B_{2,n} \cong B_{2,n}$ and $\mathrm{gr}D_n \cong D_n$ as graded Poisson algebras (and $\mathrm{gr}\delta = \delta$), so we identify them in the following.

The underlying graded algebra of $\mathrm{gr}A_n$ is isomorphic to A_n using the $x_{1,1}$-degree but the Poisson bracket is different: it is the same on the generators $x_{i,j}$ and $x_{k,l}$ for $(i,j) \neq (1,1) \neq (k,l)$ but

$$\begin{aligned}
\{x_{1,1}, x_{i,j}\}_{\mathrm{gr}} &= 0 && (2 \leq i,j \leq n)\\
\{x_{1,1}, x_{1,j}\}_{\mathrm{gr}} &= x_{1,1}x_{1,j} && (2 \leq j \leq n)\\
\{x_{1,1}, x_{i,1}\}_{\mathrm{gr}} &= x_{1,1}x_{i,1} && (2 \leq i \leq n)
\end{aligned}$$

where $\{.,.\}_{\mathrm{gr}}$ stands for the Poisson bracket of $\mathrm{gr}A_n$. Consequently, as maps we have $\mathrm{gr}\varphi = \varphi$, we still have $\{c_i, c_j\}_{\mathrm{gr}} = 0$ for all i,j, and the underlying algebra of $\mathrm{gr}C(c_1)$ can be identified with $C(c_1)$.

Note, that $C(c_1)$ is defined by the original Poisson structure $\{.,.\}$ of A_n and not by $\{.,.\}_{\mathrm{gr}}$, even if it will be considered as a Poisson subalgebra of $\mathrm{gr}A_n$. The reason of this slightly ambiguous notation is that we will also introduce $C^{\mathrm{gr}}(x_{1,1}) \subseteq \mathrm{gr}A_n$ as the centralizer of $x_{1,1}$ with respect to $\{.,.\}_{\mathrm{gr}}$.

Our associated graded setup can be summarized as follows:

$$C(c_1) \quad \subseteq \quad \mathrm{gr}A_n \xrightarrow{\varphi} B_{2,n} \xrightarrow{\delta} D_n$$

Proof of Theorem 1.2. We prove the statement by induction on n. The statement is verified for $\mathcal{O}(SL_2)$ in Section 4 so, by Proposition 3.1 the case $n = 2$ is proved. Assume that $n \geq 3$. We shall prove that

- $(\delta \circ \varphi)|_{C(c_1)} : C(c_1) \to D_n$ is injective, and
- the image $(\delta \circ \varphi)\big(C(c_1)\big)$ is in $D_n^{S_n}$.

These imply that the restriction of $\delta \circ \varphi$ to $C(c_1)$ is an isomorphism onto $D_n^{S_n}$ since $C(c_1) \ni c_i$ for $i = 1, \ldots, n$ (see Section 2) and $\delta \circ \varphi$ restricted to $\mathbb{C}[c_1, \ldots, c_n]$ is surjective onto $D_n^{S_n}$. The statement of the theorem follows.

To prove that $\delta \circ \varphi$ is injective on $C(c_1)$ it is enough to prove that δ is injective on $C\big(\varphi(c_1)\big)$ and that φ is injective on $C(c_1)$. Indeed, as φ is a Poisson map we have $\varphi\big(C(c_1)\big) \subseteq C\big(\varphi(c_1)\big)$.

First, we prove δ is injective on $C\big(\varphi(c_1)\big)$. By $B_{2,n} \cong A_{n-1}[t]$ where t is Poisson-central, we have

$$B_{2,n} \supseteq C\big(\varphi(c_1)\big) \cong C_{A_{n-1}[t]}\big(t + c_1(A_{n-1})\big) = C_{A_{n-1}}\big(c_1(A_{n-1})\big)[t] \subseteq A_{n-1}[t]$$

By the induction hypothesis

$$C_{A_{n-1}}\big(c_1(A_{n-1})\big) = \mathbb{C}\big[c_1(A_{n-1}), \ldots, c_{n-1}(A_{n-1})\big]$$

Therefore, δ restricted to $C\big(\varphi(c_1)\big)$ is an isomorphism onto $\mathbb{C}[s_1, \ldots, s_{n-1}][t_1] \subseteq D_n$ where s_i is the symmetric polynomial in the variables $t_2, \ldots, t_n$. In particular, δ is injective on $C\big(\varphi(c_1)\big)$.

To verify the injectivity of φ on $C(c_1)$, define

$$C^{\mathrm{gr}}(x_{1,1}) := \{a \in \mathrm{gr}A_n \mid \{x_{1,1}, a\}_{\mathrm{gr}} = 0\}$$

The subalgebra $C(c_1)$ is contained in $C^{\mathrm{gr}}(x_{1,1})$ since for a homogeneous element a of degree d, we have

$$\mathcal{A}^{d+1}/\mathcal{A}^d \ni \{x_{1,1}, a\}_{\mathrm{gr}} + \mathcal{A}^d = \{x_{1,1} + \mathcal{A}^0, a + \mathcal{A}^{d-1}\} + \mathcal{A}^d = \{c_1, a\} + \mathcal{A}^d$$

hence $\{c_1, a\} = 0$ implies $\{x_{1,1}, a\}_{\mathrm{gr}} = 0 \in \mathrm{gr}A_n$. Our setup can be visualized on the following diagram:

$$\begin{tikzcd}
\mathrm{gr}(A_n) \arrow[r, two heads, "\varphi"] & B_{2,n} \arrow[r, two heads, "\delta"] & D_n \\
\cup & \cup & \\
C^{\mathrm{gr}}(x_{1,1}) & C\big(\varphi(c_1)\big) \arrow[uur, hook] & \\
\cup & & \\
C(c_1) \arrow[uur] & &
\end{tikzcd}$$

Now, it is enough to prove that φ restricted to $C^{\mathrm{gr}}(x_{1,1})$ is injective.

We can give an explicit description of $C^{\mathrm{gr}}(x_{1,1})$ in the following form:

$$C^{\mathrm{gr}}(x_{1,1}) = \mathbb{C}[x_{1,1}, x_{i,j} \mid 2 \leq i, j \leq n] \leq \mathrm{gr}A_n$$

Indeed,

$$\{x_{1,1}, x_{i,j}\}_{\mathrm{gr}} = \begin{cases} x_{1,1}x_{i,j} & \text{if } j \neq i = 1 \text{ or } i \neq j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the map $\mathrm{ad}_{\mathrm{gr}}x_{1,1} : a \mapsto \{x_{1,1}, a\}_{\mathrm{gr}}$ acts on a monomial $m \in \mathrm{gr}A_n$ as $\{x_{1,1}, m\}_{\mathrm{gr}} = c(m) \cdot x_{1,1}m$ where $c(m)$ is the sum of the exponents of the $x_{1,j}$'s and $x_{i,1}$'s ($2 \leq i, j \leq n$) in m. Hence, $\mathrm{ad}_{\mathrm{gr}} x_{1,1}$ maps the monomial basis of $\mathrm{gr} A_n$ injectively into itself. In particular,

$$C^{\mathrm{gr}}(x_{1,1}) = \mathrm{Ker}\big(\mathrm{ad}_{\mathrm{gr}} x_{1,1}\big) = \{a \in \mathrm{gr} A_n \mid c(m) = 0\} \cong A_{n-1}[t]$$

using the isomorphism $x_{1,1} \mapsto t$ and $x_{i,j} \mapsto x_{i-1,j-1}$.

The injectivity part of the theorem follows: φ is injective on $C^{\mathrm{gr}}(x_{1,1})$ (in fact it is an isomorphism onto $B_{2,n}$), and φ maps $C(c_1)$ into $C\big(\varphi(c_1)\big)$ on which δ is also injective.

To prove $(\delta \circ \varphi)\big(C(c_1)\big) \subseteq D_n^{S_n}$, first note that in the above we have proved that

$$(\delta \circ \varphi)\big(C(c_1)\big) \subseteq \delta\Big(C\big(\varphi(c_1)\big)\Big) \subseteq D_n^{S_{n-1}}$$

where S_{n-1} acts on D_n by permuting $t_2, \dots, t_n$. Consider the automorphism γ of A_n given by the reflection to the off-diagonal: $\gamma(x_{i,j}) = x_{n+1-i,n+1-j}$. It is not a Poisson map but a Poisson antimap (using the terminology of [ChP]), i.e. $\gamma(\{a, b\}) = -\{\gamma(a), \gamma(b)\}$. It maps c_1 into itself and consequently $C(c_1)$ into itself. For the analogous involution $\overline{\gamma} : D_n \to D_n$, $t_i \mapsto t_{n+1-i}$ ($i = 1, \dots, n$) we have $(\delta \circ \varphi) \circ \gamma = \overline{\gamma} \circ (\delta \circ \varphi)$. Hence,

$$(\delta \circ \varphi)\big(C(c_1)\big) = (\delta \circ \varphi \circ \gamma)\big(C(c_1)\big) = (\overline{\gamma} \circ \delta \circ \varphi)\big(C(c_1)\big) \subseteq \overline{\gamma}\big(D_n^{S_{n-1}}\big)$$

proving the symmetry of $(\delta \circ \varphi)\big(C(c_1)\big)$ in $t_1, \dots, t_{n-1}$, so it is symmetric in all the variables by $n \geq 3$. □

Remark 5.2. In contrast with Theorem 1.1, in the case of the KKS Poisson structure, every Poisson-commutative subalgebra contains the Poisson center $\mathbb{C}[c_1, \dots, c_n]$, see [W]. For a maximal commutative subalgebra with respect to the KKS bracket, see [KW].

Remark 5.3. We prove that $\mathbb{C}[\overline{c}_1, \dots \overline{c}_{n-1}]$ is not an integrable complete involutive system (see Section 2). First, observe that the rank of the semiclassical Poisson bracket of $\mathcal{O}(SL_n)$ is $n(n-1)$.

Indeed, by Section 2, the rank is the maximal dimension of the symplectic leaves in SL_n. The symplectic leaves in SL_n are classified in [HL1], Theorem A.2.1, based on the work of Lu, Weinstein and Semenov-Tian-Shansky [LW], [S]. The dimension of a symplectic leaf is determined by an associated element of $W \times W$ where $W = S_n$ is the Weyl group of SL_n. According to Proposition A.2.2, if $(w_+, w_-) \in W \times W$ then the dimension of the corresponding leaves is

$$\ell(w_+) + \ell(w_-) + \min\{m \in \mathbb{N} \mid w_+ w_-^{-1} = r_1 \cdot \dots \cdot r_m \mid r_i \text{ is a transposition for all } i\} \tag{5.1}$$

where $\ell(.)$ is the length function of the Weyl group that – in the case of SL_n – is the number of inversions in a permutation. By the definition of inversion using elementary transpositions, the above quantity is bounded by

$$\ell(w_+) + \ell(w_-) + \ell(w_+ w_-^{-1})$$

The maximum of the latter is $n(n-1)$ since $\ell(w_+) = \binom{n}{2} - \ell(w_+ t)$ where $t = (n \dots 1)$ stands for the longest element of S_n. Therefore,

$$\ell(w_+) + \ell(w_-) + \ell(w_+ w_-^{-1}) = n(n-1) - \ell(w_+ t) - \ell(w_- t) + \ell\big((w_+ t)(w_- t)^{-1}\big) \leq n(n-1)$$

because $\ell(gh) \leq \ell(g) + \ell(h) = \ell(g) + \ell(h^{-1})$ for all $g, h \in S_n$. This maximum is attained on $w_+ = w_- = t$, even for the original quantity in Equation 5.1. Hence,

$\mathrm{Rk}\{.,.\} = n(n-1)$ for SL_n and $\mathrm{Rk}\{.,.\} = n(n-1)+1$ for M_n and GL_n. However, a complete integrable system should have dimension

$$\dim SL_n - \frac{1}{2}\mathrm{Rk}\{.,.\} = n^2 - 1 - \binom{n}{2} = \binom{n+1}{2} - 1$$

So it does not equal to $\dim \mathbb{C}[\overline{c}_1, \ldots \overline{c}_{n-1}] = n-1$ if $n > 1$. Similarly, the system is non-integrable for M_n and GL_n.

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