LINEAR SUBSPACE ARRANGEMENTS ASSOCIATED WITH NORMAL SURFACE SINGULARITIES

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Abstract. Let us fix a normal surface singularity with rational homology sphere link and one of its good resolutions. It is known that each coefficient of the analytic Poincaré series associated with the multivariable divisorial filtration is the topological Euler characteristic of the complement of a certain linear subspace arrangement (determined by the divisorial filtration). In this note we construct the topological analogue valid for the multivariable topological series (zeta function) associated with the resolution graph. In this way the motivic version of this topological series can also be considered.

To the memory of Egbert Brieskorn

1. Introduction

Let us fix a good resolution ϕ of a normal surface singularity (X, o) with irreducible exceptional divisors $\{E_v\}_{v\in\mathcal{V}}$. (For details and notations see 2.1.) We assume that the link is a rational homology sphere. It is known that the coefficients of the Poincaré series $P(\mathbf{t})$ associated with the multivariable $\{E_v\}_v$–divisorial filtration can be identified with the topological Euler characteristic of certain projectivized linear subspace arrangements [CDGZ04, CDGZ08, N07, N08, N12], see section 2 and Corollary 3.1.3 here. More precisely, if $P(\mathbf{t}) = \sum_{l'} \mathfrak{p}(l')\mathbf{t}^{l'}$ (where the sum is over the dual lattice L' of ϕ), then for each l' there exists a finite dimensional vector space $A(l')$ and for each $v \in \mathcal{V}$ a linear subspace $A_v(l') \subset A(l')$ such that $\mathfrak{p}(l') = \chi_{top}(\mathbb{P}(A(l') \setminus \cup_v A_v(l')))$. Let us denote this linear subspace arrangement $\{A_v(l')\}_{v\in\mathcal{V}}$ of $A(l')$ by $\mathcal{A}_{\rm an}(l')$.

In this note we prove the existence of the topological analogue of this fact, valid for the multivariable topological series (zeta function) $Z(\mathbf{t}) = \sum_{l'} \mathfrak{z}(l')\mathbf{t}^{l'}$. (For its definition see 4.1.) Namely, for each l' we construct a finite dimensional vector space $T(l')$ and linear subspace arrangement $\{T_v(l')\}_{v\in\mathcal{V}}$ of $T(l')$, such that the following facts hold:

1. the arrangement $\mathcal{A}_{\rm top}(l') := \{T_v(l')\}_{v\in\mathcal{V}}$ of $T(l')$ depends only on the resolution graph;
2. $\mathfrak{z}(l') = \chi_{top}(\mathbb{P}(T(l') \setminus \cup_v T_v(l')))$;
3. for each l' the vector space $A(l')$ embeds linearly into $T(l')$ such that $A_v(l') = A(l') \cap T_v(l')$ for every $v \in \mathcal{V}$.

These facts can be interpreted as follows. Each topological type (of normal surface singularities with rational homology sphere link) with fixed dual resolution graph (or fixed lattice L and dual lattice L') determines for any $l' \in L'$ a 'topological linear subspace arrangement' $\mathcal{A}_{\rm top}(l')$. Furthermore, any analytic structure supported on this topological type determines canonically for any $l' \in L'$ an 'analytic linear subspace arrangement' $\mathcal{A}_{\rm an}(l')$ inside $T(l')$ and cut out from $\mathcal{A}_{\rm top}(l')$ by a subspace $A(l') \subset T(l')$. The analytic linear subspace arrangement might depend essentially on the analytic

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structure (since we know topologically equivalent pairs of analytical types, for which one type satisfies $Z(\mathbf{t}) = P(\mathbf{t})$ while the other not, see e.g. [N08]).

In this way, $(A(l'), \{A_v(l')\}_v) \subset (T(l'), \{T_v(l')\}_v)$ looks a natural pairing. It immediately induces (by taking the Euler characteristic of the corresponding spaces) the pairing of the two series $Z(\mathbf{t})$ and $P(\mathbf{t})$. Though these two series looked apparently artificially paired in the earlier articles, now, after the present setup, this fact is totally motivated and justified. Furthermore, taking the periodic constants of the series Z and P, we get the pairing of the Seiberg–Witten invariants of the link, respectively the equivariant geometric genera of (X, o), cf. [N12], hence the pairing predicted by the Seiberg–Witten Invariant Conjecture is indeed very natural and totally justified.

In particular, these steps provide a totally conceptual explanation for the appearance of the Seiberg–Witten invariant in the theory of complex surface singularities.

Examples show that usually the identity $Z(\mathbf{t}) = P(\mathbf{t})$ can happen even if $\mathcal{A}_{\rm top}(l') \neq \mathcal{A}_{\rm an}(l')$.

For each l' having a quasiprojective space $T(l') \setminus \cup_v T_v(l')$, instead of its topological Euler characteristic, has big advantages. Indeed, this provides a new source of invariants: one can replace $\chi_{top}(T(l') \setminus \cup_v T_v(l'))$ by several stronger invariants of this space, e.g. cohomology ring, the mixed Hodge structures, or the class in the Grothendieck ring of varieties. For the details of this last version see subsection 4.4.

2. Preliminaries. Divisorial filtration and its multivariable series

2.1. **Notations regarding a resolution.** Let (X, o) be the germ of a complex analytic normal surface singularity, and let us fix a good resolution $\phi : \widetilde{X} \to X$ of (X, o). We denote the exceptional curve $\phi^{-1}(o)$ by E, and let $\cup_{v \in \mathcal{V}} E_v$ be its irreducible components. Set also $E_I := \sum_{v \in I} E_v$ for any subset $I \subset \mathcal{V}$. For more details see [N07, N12, N99b].

Let Γ be the dual resolution graph associated with ϕ; it is a connected graph. Then $M := \partial \widetilde{X}$ can be identified with the link of (X, o), it is also an oriented plumbed 3–manifold associated with Γ. It is known that (X, o) locally is homeomorphic with the real cone over M, and M contains the same information as Γ. We will assume that M is a rational homology sphere, or, equivalently, Γ is a tree and all genus decorations of Γ are zero. We use the same notation $\mathcal{V}$ for the set of vertices, and δ_v for the valency of a vertex v.

$L := H_2(\widetilde{X}, \mathbb{Z})$, endowed with its negative definite intersection form $I = (\,,\,)$, is a lattice. It is freely generated by the classes of 2–spheres $\{E_v\}_{v \in \mathcal{V}}$. The dual lattice $L' := H^2(\widetilde{X}, \mathbb{Z})$ is generated by the (anti)dual classes $\{E^*_v\}_{v \in \mathcal{V}}$ defined by $(E^*_v, E_w) = -\delta_{vw}$ (where δ_{vw} stays for the Kronecker symbol). The intersection form embeds L into L'. Then $H_1(M, \mathbb{Z}) \simeq L'/L$, abridged by H. Usually one also identifies L' with those rational cycles $l' \in L \otimes \mathbb{Q}$ for which $(l', L) \in \mathbb{Z}$, or, $L' = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$.

There is a natural (partial) ordering of L' and L: we write $l'_1 \geq l'_2$ if $l'_1 - l'_2 = \sum_v r_v E_v$ with all $r_v \geq 0$. We set $L_{\geq 0} = \{l \in L : l \geq 0\}$ and $L_{>0} = L_{\geq 0} \setminus \{0\}$.

Set $\mathfrak{C} := \{\sum l'_v E_v \in L',\ 0 \leq l'_v < 1\}$. For any $l' \in L'$ write its class in H by $[l']$, and or any $h \in H$ let $r_h \in L'$ be its unique representative in $\mathfrak{C}$.

All the E_v–coordinates of any E^*_u are strict positive. We define the Lipman cone as $\mathcal{S}' := \{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\}$. It is generated over $\mathbb{Z}_{\geq 0}$ by $\{E^*_v\}_v$.

Finally, denote by $\theta : H \to \widehat{H}$ the isomorphism $[l'] \mapsto e^{2\pi i (l', \cdot)}$ of H with its Pontrjagin dual $\widehat{H}$.

2.1.1. The module $\mathbb{Z}[[L']]$. We denote by $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, \ldots, t_s]$, respectively by $\mathbb{Z}[[\mathbf{t}]] := \mathbb{Z}[[t_1, \ldots, t_s]]$, the ring of polynomials, respectively the ring of formal power series, in variables $\{t_v\}_{v=1}^s$, where $s = |\mathcal{V}|$. Set also the ring of Laurent polynomials $\mathbb{Z}[\mathbf{t}][t_1^{-1}, \ldots, t_s^{-1}]$ too.

Then the formal Laurent series additive group $\mathbb{Z}[[\mathbf{t}^{\pm 1}]] := \mathbb{Z}[[t_1^{\pm 1}, \dots, t_s^{\pm 1}]]$ is a $\mathbb{Z}[\mathbf{t}][t_1^{-1}, \dots, t_s^{-1}]$–module. It is contained in the larger module $\mathbb{Z}[[\mathbf{t}^{\pm 1/d}]] = \mathbb{Z}[[t_1^{\pm 1/d}, \dots, t_s^{\pm 1/d}]]$, the module of formal Laurent series in variables $t_v^{\pm 1/d}$, where $d := |H|$. $\mathbb{Z}[[L']]$ embeds into $\mathbb{Z}[[\mathbf{t}^{\pm 1/d}]]$ as a submodule: it consists of the Laurent series with monomials of type

$$\mathbf{t}^{l'} = t_1^{l'_1} \cdots t_s^{l'_s}, \quad \text{where } \ l' = \textstyle\sum_v l'_v E_v \in L'.$$

In this way $\mathbb{Z}[[L]]$ identifies with $\mathbb{Z}[[\mathbf{t}^{\pm 1}]]$. $\mathbb{Z}[[L']]$ also admits several $\mathbb{Z}$–submodules corresponding to different cones of L'; e.g. $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$, consisting of series with monomials of type $\mathbf{t}^{l'}$ with $l' \in L'_{\geq 0}$, or $l' \in \mathcal{S}'$ respectively. Both $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$ have natural ring structure as well.

$\mathbb{Z}[[\mathcal{S}']]$ is a usual formal power series ring in variables $\mathbf{t}^{E_v^*}$: its elements are

$$\Phi(f)(\mathbf{t}) := f(\mathbf{t}^{E_1^*}, \dots, \mathbf{t}^{E_s^*}), \quad \text{where } \ f(x_1, \dots, x_s) \in \mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, \dots, x_s]]. \tag{2.1.2}$$

Definition 2.1.3. Any series $S(\mathbf{t}) = \sum_{l'} a_{l'} \mathbf{t}^{l'} \in \mathbb{Z}[[L']]$ decomposes in a unique way as

$$S = \sum_{h \in H} S_h, \quad \text{where } \ S_h = \sum_{[l'] = h} a_{l'} \mathbf{t}^{l'}. \tag{2.1.4}$$

S_h is called the h-component of S. In fact, if $F(\mathbf{t}) := \Phi(f)(\mathbf{t})$ for some $f \in \mathbb{Z}[[\mathbf{x}]]$ then

$$F_h(\mathbf{t}) = \frac{1}{|H|} \cdot \sum_{\rho \in \widehat{H}} \rho(h)^{-1} \cdot f(\rho([E_1^*]) \mathbf{t}^{E_1^*}, \dots, \rho([E_s^*]) \mathbf{t}^{E_s^*}). \tag{2.1.5}$$

Indeed, if $l' := \sum n_v E_v^*$ and $\prod x_v^{n_v}$ is a monomial of f, then $\Phi(\prod x_v^{n_v})(\mathbf{t}) = \mathbf{t}^{l'}$ and the Fourier transform $(1/d) \sum_\rho \rho(h)^{-1} \rho([l']) \mathbf{t}^{l'}$ is $\mathbf{t}^{l'}$ if $[l'] = h$ and it is zero otherwise.

2.2. **Natural line bundles.** Some line bundles on $\widetilde{X}$ are distinguished. They are provided by the splitting of the cohomological exponential exact sequence (see e.g. [N07, 4.2]):

$$0 \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} L' \to 0.$$

The first Chern class c_1 has a natural section on the subgroup L, namely $l \mapsto \mathcal{O}_{\widetilde{X}}(l)$. One shows that this section has a unique extension $\mathcal{O}(\cdot)$ to L'. We call a line bundle *natural* if it is in the image of this section. Hence, by definition, a line bundle is natural if and only if one of its powers has the form $\mathcal{O}_{\widetilde{X}}(l)$ for some $l \in L$.

The natural line bundle associated with $l' \in L'$ will be denoted by $\mathcal{O}_{\widetilde{X}}(l')$.

2.3. **The universal abelian covering.** Let $c : (X_a, o) \to (X, o)$ be the universal abelian covering of (X, o), the unique normal singular germ corresponding to the regular covering of $X \setminus \{o\}$ associated with $\pi_1(X \setminus \{o\}) \to H_1(X \setminus \{o\}, \mathbb{Z}) = H$. It has a natural $H = L'/L$–action. Since $\widetilde{X} \setminus E \approx X \setminus \{o\}$, $\pi_1(\widetilde{X} \setminus E) = \pi_1(X \setminus \{o\}) \to H$ defines a regular Galois covering of $\widetilde{X} \setminus E$ as well. This has a unique extension $\widetilde{c} : Z \to \widetilde{X}$ with Z normal and $\widetilde{c}$ finite. In other words, $\widetilde{c} : Z \to \widetilde{X}$ is the normalized pullback of c via ϕ. The (reduced) branch locus of $\widetilde{c}$ is included in E, and the Galois action of H extends to Z as well. Since E is a normal crossing divisor, the only singularities what Z might have are cyclic quotient singularities. Let $r : \widetilde{Z} \to Z$ be a resolution of these singular points such that $(\widetilde{c} \circ r)^{-1}(E)$ is a normal crossing divisor. We have the following diagram:

$$\begin{array}{ccccc} \widetilde{Z} & \xrightarrow{r} & Z & \xrightarrow{\psi_a} & (X_a, o) \\ & & \big\downarrow{\scriptstyle \widetilde{c}} & & \big\downarrow{\scriptstyle c} \\ & & (\widetilde{X}, o) & \xrightarrow{\phi} & (X, o) \end{array} \tag{2.3.1}$$

Set $\phi_a = \psi_a \circ r$ and $p = \widetilde{c} \circ r$. One verifies (see [N07, Lemma 4.2.3]) $p^*(l')$ is an integral cycle for any $l' \in L'$.

One can recover the natural line bundles via the universal abelian covering as follows.

$$p_*\mathcal{O}_{\widetilde{Z}}=\widetilde{c}_*\mathcal{O}_Z=\bigoplus_{l'\in\mathfrak{C}}\mathcal{O}_{\widetilde{X}}(-l')\quad (\mathcal{O}_{\widetilde{X}}(-l')\text{ being the }\theta([l'])\text{-eigenspace of }\widetilde{c}_*\mathcal{O}_Z). \tag{2.3.2}$$

2.4. **The divisorial filtration. The series $H(\mathbf{t})$ and $P(\mathbf{t})$.** We will define an L–filtration of the local ring of (X,o) and a compatible H–equivariant L'–filtration of the local ring of (X_a,o). Fore more see [N12].

Definition 2.4.1. The L'–filtration on $\mathcal{O}_{X_a,o}$ is defined as follows. For any $l'\in L'$, we set

$$\mathcal{F}(l'):=\{f\in\mathcal{O}_{X_a,o}\mid \mathrm{div}(f\circ\phi_a)\geq p^*(l')\}. \tag{2.4.2}$$

Notice that the natural action of H on (X_a,o) induces an action on $\mathcal{O}_{X_a,o}$, which keeps $\mathcal{F}(l')$ invariant. Therefore, H acts on $\mathcal{O}_{X_a,o}/\mathcal{F}(l')$ as well. For any $l'\in L'$, let $\mathfrak{h}(l')$ be the dimension of the $\theta([l'])$-eigenspace $(\mathcal{O}_{X_a,o}/\mathcal{F}(l'))_{\theta([l'])}$. Then one defines the Hilbert series $H(\mathbf{t})$ by

$$H(\mathbf{t}):=\sum_{l'\in L'}\mathfrak{h}(l')\cdot\mathbf{t}^{l'}\in\mathbb{Z}[[L']]. \tag{2.4.3}$$

By [N07, Prop. 4.3.3], for any $l'\in L'$ there exists a unique minimal $s(l')\in\mathcal{S}'$ such that $l'\leq s(l')$ and $[l']=[s(l')]$. Since for any $f\in\mathcal{O}_{X_a,o}$, that part of $div(f\circ\phi_a)$, which is supported by the exceptional divisor of ϕ_a, is in the Lipman cone of $\widetilde{Z}$, we get

$$\mathcal{F}(l')=\mathcal{F}(s(l')). \tag{2.4.4}$$

For a fixed l' we write $[l']=h$. If $l'>0$ one has the exact sequence

$$0\to\mathcal{O}_{\widetilde{Z}}(-p^*(l'))\to\mathcal{O}_{\widetilde{Z}}\to\mathcal{O}_{p^*(l')}\to 0. \tag{2.4.5}$$

The $\theta(h)$-eigenspace constitutes the exact sequence

$$0\to\mathcal{O}_{\widetilde{X}}(-l')\to\mathcal{O}_{\widetilde{X}}(-r_h)\to\mathcal{O}_{l'-r_h}(-r_h)\to 0. \tag{2.4.6}$$

In particular, for $l'>0$,

$$\mathfrak{h}(l')=\dim\Big(\frac{H^0(\widetilde{Z},\mathcal{O}_{\widetilde{Z}})}{H^0(\widetilde{Z},\mathcal{O}_{\widetilde{Z}}(-p^*(l')))}\Big)_{\theta(h)}=\dim\,\frac{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-r_h))}{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-l'))}. \tag{2.4.7}$$

Example 2.4.8. In (2.4.7) if $l'\in L$ then $r_h=0$. Hence the 0–component of $H(\mathbf{t})$ is

$$H_0(\mathbf{t})=\sum_{l\in L}\dim\frac{\mathcal{O}_{X,o}}{\{f\in\mathcal{O}_{X,o}:\mathrm{div}_E(f\circ\phi)\geq l\}}\ \mathbf{t}^l.$$

This is the Hilbert series of $\mathcal{O}_{X,o}$ associated with the divisorial filtration $L\ni l\mapsto\mathcal{F}_0(l)=\{f\in\mathcal{O}_{X,o}:\mathrm{div}_E(f\circ\phi)\geq l\}$ of all irreducible exceptional divisors of ϕ.

2.4.9. Next, we define the Poincaré series $P(\mathbf{t})=\sum_{l'\in L'}\mathfrak{p}(l')\mathbf{t}^{l'}$ associated with the filtration $\{\mathcal{F}(l')\}_{l'}$ as

$$P(\mathbf{t})=-H(\mathbf{t})\cdot\prod_v(1-t_v^{-1}). \tag{2.4.10}$$

Using (2.4.7) one verifies that for any $l'\in\mathcal{S}'$ one has

$$\mathfrak{p}(l')=\sum_{I\subset\mathcal{V}}(-1)^{|I|+1}\dim\,\frac{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-l'))}{H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-l'-E_I))}. \tag{2.4.11}$$

The series $P(\mathbf{t})$ is supported in $\mathcal{S}'$, and the following 'invertion identities' hold [N12, Prop. 3.2.4]:

$$\mathfrak{h}(l')=\sum_{l\in L,\,l\ngeq 0}\mathfrak{p}(l'+l). \tag{2.4.12}$$

2.4.13. A reformulation of $P(\mathbf{t})$. For $l'-r_h\in L_{>0}$ from (2.4.6) follows that

$$\mathfrak{h}(l')=\chi(\mathcal{O}_{l'-r_h}(-r_h))-h^1(\mathcal{O}_{\widetilde{X}}(-l'))+h^1(\mathcal{O}_{\widetilde{X}}(-r_h)).$$

Since $\chi(\mathcal{O}_{l'-r_h}(-r_h))=\chi(l')-\chi(r_h)$, this reads as

$$\mathfrak{h}(l')=\chi(l')-h^1(\mathcal{O}_{\widetilde{X}}(-l'))-\chi(r_h)+h^1(\mathcal{O}_{\widetilde{X}}(-r_h)).$$

Hence, using the definition of P, we get

$$P(\mathbf{t})=\sum_{l'\in\mathcal{S}'}\sum_{I\subset\mathcal{V}}(-1)^{|I|+1}\Big(\chi(l'+E_I)-h^1(\mathcal{O}_{\widetilde{X}}(-l'-E_I))\Big)\,\mathbf{t}^{l'}. \tag{2.4.14}$$

2.4.15. The definition of P by Campillo, Delgado and Gusein-Zade [CDGZ04, CDGZ08]**.**

The infinite dimensional arrangement $\{\mathcal{F}(l'+E_I)\}_I$ in the infinite dimensional linear space $\mathcal{F}(l')$ can be reduced to a finite dimensional arrangement as follows. Since all these subspaces contain $\mathcal{F}(l'+E)$, and $\mathcal{F}(l')/\mathcal{F}(l'+E)$ is finite dimensional, it is natural to set the series

$$L(\mathbf{t}):=\sum_{l'\in L'}\dim\,(\mathcal{F}(l')/\mathcal{F}(l'+E))_{\theta([l'])}\cdot\mathbf{t}^{l'}\in\mathbb{Z}[[L']]. \tag{2.4.16}$$

Since $\mathcal{F}(l')=\mathcal{F}(l'+E_v)$ if $(l',E_v^*)>0$, one obtains that $L(\mathbf{t})\prod_v(t_v-1)$ is an element of $\mathbb{Z}[[L'_{\geq 0}]]$. Hence the next infinite power series in $\mathbb{Z}[[L'_{\geq 0}]]$ is well-defined:

$$P(\mathbf{t}):=-\frac{L(\mathbf{t})\prod_v(t_v-1)}{1-\mathbf{t}^E}=-L(\mathbf{t})\prod_v(t_v-1)\cdot\sum_{k\geq 0}\mathbf{t}^{kE}. \tag{2.4.17}$$

Since $\mathfrak{h}_h(l')=0$ for $l'\leq 0$ one has $L(\mathbf{t})=-H(\mathbf{t})(1-\mathbf{t}^{-E})$ and $P(\mathbf{t})=-H(\mathbf{t})\cdot\prod_v(1-t_v^{-1})$, cf. (2.4.10).

Example 2.4.18. Consider the cyclic quotient singularity whose minimal resolution ϕ has only one irreducible component E with self-intersection $-n$ ($n\geq 2$). Then $H=\mathbb{Z}_n$,$\mathcal{O}_{X_a,o}=\mathbb{C}\{z_1,z_2\}$. Moreover, $E^*=E/n$. The action of H is given by $h*z_i=\theta(E^*)(h)z_i$; hence, $z_1^iz_2^j$ is in the $\theta(E^*)^{i+j}$–eigenspace. Below, for a character χ of this action on $\mathbb{C}\{\mathbf{z}\}$ we denote the corresponding eigenspace by $\mathbb{C}\{\mathbf{z}\}^{\chi}$.

Therefore, the Poincaré series of the H–eigenspaces (with $\deg(z_i)=\frac{1}{n}\in\frac{1}{n}\mathbb{Z}$) is

$$P((\mathbb{C}\{\mathbf{z}\}^{\theta([qE^*])},t)=\sum_{k\geq 0}(1+q+nk)t^{k+\frac{q}{n}},\ \text{and}\ P(\mathbb{C}\{\mathbf{z}\},t)=\sum_{\ell\geq 0}(1+\ell)t^{\ell/n}.$$

$$\sum_{\rho\in\widehat{H}}P(\mathbb{C}\{\mathbf{z}\}^{\rho},t)\cdot\rho=\frac{1}{(1-\theta(E^*)\cdot t^{E^*})^2}\in\mathbb{Z}[[t]][\widehat{H}]; \tag{2.4.19}$$

$\widetilde{Z}=Z$ is just the blow up ϕ_a of X_a at 0 with exceptional divisor $\widetilde{E}$ a (-1)-curve. Since $\widetilde{c}^*(E)=n\widetilde{E}$, we get that $\mathcal{F}(k'E)$ contains all the monomials of degree $\geq nk'$. We claim that this inclusion is, in fact, an isomorphism. Indeed, if $f=\sum_{i+j=nk'}c_iz_1^iz_2^j\in\mathbb{C}\{\mathbf{z}\}$, such that f is not identically zero, then $\phi_a^*(f)$ will have vanishing order exactly nk' (and never higher) along $\widetilde{E}$, independently of the choice of the coefficients $c_i\in\mathbb{C}$. Therefore, for $k'=k+q/n$ ($k\in\mathbb{Z}$), $(\mathcal{F}(k'E)/\mathcal{F}(k'E+E))_{\theta([qE^*])}$ can be identified with the vector space of monomials of degree $nk+q$ ($0\leq q<p$), and $P(\mathbf{t})=1/(1-t^{E^*})^2$. Its h–components are

$$\sum_{h\in H}P_h(t)\cdot h=\frac{1}{(1-[E^*]\cdot t^{E^*})^2}\in\mathbb{Z}[[t]][H]. \tag{2.4.20}$$

Note that $(1-[E^*]\cdot t^{E^*})^{-2}$ agrees exactly with the H–decomposition $\sum_{h\in H}Z_h(t)\cdot h$ of the topological series $Z(t)$, which will be considered in 4.1.

3. Linear subspace arrangements associated with the filtration

3.1. Fix a normal surface singularity as in 2.3, one of its resolutions and the filtration $\{\mathcal{F}(l')\}_{l'\in L'}$, $\mathcal{F}(l')\subset \mathcal{O}_{X_a,o}$, from 2.4.1. For any $l'\in L'$, the linear space

$$(\mathcal{F}(l')/\mathcal{F}(l'+E))_{\theta([l'])}=H^0(\mathcal{O}_{\widetilde{X}}(-l'))/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$$

naturally embeds into

$$T(l'):=H^0(\mathcal{O}_E(-l')).$$

Let its image be denoted by $A(l')$. Furthermore, for every $v\in\mathcal{V}$, consider the linear subspace $T_v(l')$ of $T(l')$ given by

$$T_v(l'):=H^0(\mathcal{O}_{E-E_v}(-l'-E_v))=\ker\big(\,H^0(\mathcal{O}_E(-l'))\to H^0(\mathcal{O}_{E_v}(-l'))\,\big)\subset T(l').$$

Then the image $A_v(l')$ of $H^0(\mathcal{O}_{\widetilde{X}}(-l'-E_v)/H^0(\mathcal{O}_{\widetilde{X}}(-l'-E))$ in $T(l')$ satisfies $A_v(l')=A(l')\cap T_v(l')$. This fact follows from the following diagram:

$$\begin{array}{ccccccc}
 & & H^0(\mathcal{O}_{\widetilde{X}}(-l'-E)) & = & H^0(\mathcal{O}_{\widetilde{X}}(-l'-E)) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{O}_{\widetilde{X}}(-l'-E_v)) & \to & H^0(\mathcal{O}_{\widetilde{X}}(-l')) & \to & H^0(\mathcal{O}_{E_v}(-l')) \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & H^0(\mathcal{O}_{E-E_v}(-l'-E_v)) & \to & H^0(\mathcal{O}_E(-l')) & \to & H^0(\mathcal{O}_{E_v}(-l')) \\
 & & \| & & \| & & \\
 & & T_v(l') & \hookrightarrow & T(l') & &
\end{array}$$

Definition 3.1.1. The (finite dimensional) arrangement of linear subspaces $\mathcal{A}_{\rm top}(l')=\{T_v(l')\}_v$ in $T(l')$ is called the 'topological arrangement' at $l'\in L'$. The arrangement of linear subspaces $\mathcal{A}_{\rm an}(l')=\{A_v(l')=T_v(l')\cap A(l')\}_v$ in $A(l')$ is called the 'analytic arrangement' at $l'\in L'$. The corresponding projectivized arrangement complements will be denoted by $\mathbb{P}(T(l')\setminus\cup_vT_v(l'))$ and $\mathbb{P}(A(l')\setminus\cup_vA_v(l'))$ respectively.

If $l'\notin\mathcal{S}'$ then there exists v such that $(E_v,l')>0$, that is $h^0(\mathcal{O}_{E_v}(-l'))=0$, proving that $T_v(l')=T(l')$. Hence $A_v(l')=A(l')$ too. In particular, both arrangement complements are empty.

The connection with the series P is provided by the following topological Euler characteristic formula.

Lemma 3.1.2. *Assume that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a finite family of linear subspaces of a finite dimensional linear space V. For $I\subset\Lambda$ set $V_I:=\cap_{\alpha\in I}V_\alpha$ (where $V_\emptyset=V$). Then*

$$\chi_{\rm top}(\mathbb{P}(V\setminus\cup_\alpha V_\alpha))=\sum_{I\subset\Lambda}(-1)^{|I|}\dim V_I.$$

If $\Lambda\neq\emptyset$, then this also equals $\sum_I(-1)^{|I|+1}{\rm codim}(V_I\subset V)$.

Proof. Use the inclusion–exclusion principle and $\dim V_I=\chi_{\rm top}(\mathbb{P}V_I)$. □

Corollary 3.1.3. *For any $l'\in\mathcal{S}'$ one has*

$$\mathfrak{p}(l')=\chi_{\rm top}(\,\mathbb{P}(A(l')\setminus\cup_vA_v(l'))\,).$$

Proof. Use (2.4.11) and 3.1.2. □

The corresponding dimensions of the linear subspaces in $\mathcal{A}_{\rm an}(l')$ are as follows.

Lemma 3.1.4. *For any $l' \in L'$ one has:*

$$\dim\, A(l') = \mathfrak{h}(l'+E) - \mathfrak{h}(l'), \quad \dim\, \cap_{v\in I} A_v(l') = \mathfrak{h}(l'+E) - \mathfrak{h}(l'+E_I).$$

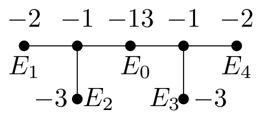
Thus, we can expect that the analytic arrangement is rather sensitive to the modification of the analytic structure, and in general, does not coincide with the topological arrangement.

Note that $\dim\, A(l')$ is the l'–coefficient of the series $L(\mathbf{t})$, cf. paragraph 2.4.15.

Since the series $P(\mathbf{t})$ and $H(\mathbf{t})$ determine each other (see (2.4.12)), once the analytic Poincaré series $P(\mathbf{t})$ is fixed all the dimensions involved in $\mathcal{A}_{\rm an}(l')$ (for all l') are determined.

Example 3.1.5. We will write $Z_{min} \in L$ for the minimal (or fundamental) cycle, which is the minimal non–zero cycle of $\mathcal{S}' \cap L$ [A62, A66]. Yau's maximal ideal cycle $Z_{max} \in L$ defines the divisorial part of the pullback of the maximal ideal $\mathfrak{m}_{X,o} \subset \mathcal{O}_{X,o}$, i.e. $\phi^*\mathfrak{m}_{X,o} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-Z_{max}) \cdot \mathcal{I}$, where $\mathcal{I}$ is an ideal sheaf with 0–dimensional support [Y80].

Consider the complete intersection singularity in $(\mathbb{C}^4, 0)$ given by $z_1^2+z_2^3-z_3^2z_4 = z_4^2+z_3^3-z_2^2z_1 = 0$. Its graph is



3.1.6. One verifies that $\mathrm{div}(z_i) = E_i^*$ for $1 \le i \le 4$; $Z_{min} = E_0^*$ is not the compact part of a divisor of an analytic function; $Z_{max} = 2E_0^* = \min\{E_2^*, E_3^*\}$. We wish to find $\mathfrak{p}(Z_{max})$. Note that $T(Z_{max}) = H^0(\mathcal{O}_E(-Z_{max})) \simeq H^0(\mathcal{O}_{E_0}(-Z_{max})) \simeq \mathbb{C}^3$. On the other hand, $A(Z_{max})$ is the image of $H^0(\mathcal{O}(-Z_{max})) = \mathfrak{m}_{X,o}$. Since z_1, z_4 and $\mathfrak{m}^2_{X,o}$ are contained in $H^0(\mathcal{O}(-Z_{max} - E)$, $A(Z_{max})$ is 2–dimensional, generated by the classes of z_2 and z_3. (To see the linear independence of their classes, check their divisors.) Hence $A(Z_{max}) \neq T(Z_{max})$.

Moreover, $\cup_v A_v(Z_{max})$ is the union of the two coordinate axes. Hence $\mathbb{P}(\mathbb{C}^2 \setminus (\mathbb{C} \cup \mathbb{C})) = \mathbb{C}^*$ and $\mathfrak{p}(Z_{max}) = 0$.

Although $A(Z_{max}) \neq T(Z_{max})$, they still can be compared. Indeed, $T(Z_{max}) = H^0(\mathcal{O}_{E_0}(2))$ and $\cup_v T_v(Z_{max})$ is a union of two 2–planes (corresponding to global sections of $\mathcal{O}_{E_0}(2)$ vanishing at the two intersection points of E_0 with the other components). Hence $T \setminus \cup_v T_v = (A_v \setminus \cup_v A_v) \times \mathbb{C}$, and $\chi_{\rm top}(\mathbb{P}(T \setminus \cup_v T_v)) = 0$ too.

Here $T_0 = 0$, which is contained in A, and A intersects all the other strata of $\{A_v\}_v$ generically. This quarantees that $\chi_{\rm top}(\mathbb{P}(A \setminus \cup_v A_v)) = \chi_{\rm top}(\mathbb{P}(T \setminus \cup_v T_v))$ holds.

Our next goal is to show that whenever the link of the singularity is a rational homology sphere the *topological arrangement* $\mathcal{A}_{\rm top}$ is indeed topological, it depends only on the combinatorics of the resolution graph.

We will need the following technical definition.

Lemma 3.1.7. *(1) For any $l' \in L'$ and subset $I \subset \mathcal{V}$ there exists a unique minimal subset $J(l', I) \subset \mathcal{V}$ which contains I, and has the following property:*

$$\text{there is no } v \in \mathcal{V} \setminus J(l', I) \text{ with } (E_v, l' + E_{J(l',I)}) > 0. \tag{3.1.8}$$

(2) $J(l', I)$ can be found by the next algorithm: one constructs a sequence $\{I_m\}_{m=0}^k$ of subsets of $\mathcal{V}$, with $I_0 = I$, $I_{m+1} = I_m \cup \{v(m)\}$, where the index $v(m)$ is determined as follows. Assume that I_m is already constructed. If I_m satisfies (3.1.8) we stop and $m = k$. Otherwise, there exists at least one v with $(E_v, l' + E_{I_m}) > 0$. Take $v(m)$ one of them and continue the algorithm with I_{m+1}. Then $I_k = J(l', I)$.

Proof. For (1) notice that if J_1 and J_2 satisfies the wished requirement (3.1.8) of $J(l',I)$ then $J_1\cap J_2$ satisfies too. Part (2) is a version of the usual Laufer type algorithm (see [La72, La77] or [N07, Prop. 4.3.3]). □

Proposition 3.1.9. *Assume that the resolution graph is a tree of rational curves. For any $l'\in L'$ and $I\subset\mathcal{V}$ write $J(I):=J(l',I)$. Then the following facts hold.*

(a) One has the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & H^0(\mathcal{O}_{E-E_{J(I)}}(-l'-E_{J(I)})) & \to & H^0(\mathcal{O}_E(-l')) & \stackrel{k}{\twoheadrightarrow} & H^0(\mathcal{O}_{E_{J(I)}}(-l')) & \to & 0\\
 & & \downarrow j & & \| & & \downarrow i & & \\
0 & \to & H^0(\mathcal{O}_{E-E_I}(-l'-E_I)) & \to & H^0(\mathcal{O}_E(-l')) & \to & H^0(\mathcal{O}_{E_I}(-l')) & & \\
 & & \| & & \| & & & & \\
 & & \cap_{v\in I}T_v(l') & \hookrightarrow & T(l') & & & &
\end{array}$$

where j is an isomorphism (hence $\cap_{v\in I}T_v(l')=\cap_{v\in J(I)}T_v(l')$), i is injective and k is surjective.

(b) $\dim\cap_{v\in J(I)}T_v(l')=\chi(\mathcal{O}_{E-E_{J(I)}}(-l'-E_{J(I)}))=\chi(l'+E)-\chi(l'+E_{J(I)})$.

(c) In particular, if $J(I_1)=J(I_2)$ then $\cap_{v\in I_1}T_v(l')=\cap_{v\in I_2}T_v(l')$, and if $J(I_1)\subsetneq J(I_2)$ then $\cap_{v\in I_1}T_v(l')\supsetneq\cap_{v\in I_2}T_v(l')$. Therefore, $J(I)$ is the unique maximal subset $I_{max}\subset\mathcal{V}$, such that $I\subset I_{max}$, and $\cap_{v\in I}T_v(l')=\cap_{v\in I_{max}}T_v(l')$.

(d) Part (b) for $I=\emptyset$ reads as follows: $\dim T(l')=\dim\cap_{v\in J(\emptyset)}T_v(l')=\chi(l'+E)-\chi(l'+E_{J(\emptyset)})$. *Hence, if $l'\in\mathcal{S}'$ then* $\dim T(l')=-(l',E)+1$.

(e) $\mathrm{codim}\,(\,\cap_{v\in I}T_v(l')\hookrightarrow T(l')\,)=\chi(l'+E_{J(I)})-\chi(l'+E_{J(\emptyset)})$.
In particular, the arrangement complement is non–empty if and only if $J(\emptyset)=\emptyset$ (if and only if $l'\in\mathcal{S}'$).

Proof. First we prove the following fact: let $F\leq E$ be an effective non–zero cycle and we assume that for any $E_w\leq F$ one has $(E_w,l')\leq 0$. Then $h^1(\mathcal{O}_F(-l'))=0$. The proof runs over induction: choose E_w from the support of F such that $(F-E_w,E_w)\leq 1$, then use the cohomological long exact sequence of $\mathcal{O}_F(-l')\to\mathcal{O}_{F-E_w}(-l')$.

From the definition, $\cap_{v\in I}T_v(l')=H^0(\mathcal{O}_{E-E_I}(-l'-E_I))$. To prove (a) note that this group is stable along the steps of the algorithm 3.1.7(2). Hence j is an isomorphism. Similarly, along these steps i is injective. Since $h^1(\mathcal{O}_{E-E_J}(-l'-E_J))=0$ by the above fact , k is onto and (b) follows too. For (c) use (b) and the fact that $\chi(l'+J(I_2))>\chi(l'+J(I_1))$ whenever $J(I_1)\subsetneq J(I_2)$. □

Corollary 3.1.10. *The arrangement $\mathcal{A}_{\rm top}(l')$ depends only on the combinatorial data of the graph.*

At topological Euler characteristic level one has:

Corollary 3.1.11. *If the graph is a tree of rational curves and $l'\in\mathcal{S}'$ then*

$$\chi_{\rm top}(\,\mathbb{P}(T(l')\setminus\cup_vT_v(l'))\,)=\sum_{I\subset\mathcal{V}}(-1)^{|I|+1}\chi(l'+E_{J(l',I)}).$$

Proof. Use Lemma 3.1.2 and Proposition 3.1.9(b). □

Example 3.1.12. Consider the situation of the Example 3.1.5, and set $l'=Z_{min}$. Then $T(Z_{min})=\mathbb{C}^2$ and $\cup_vT_v(Z_{min})$ consists of the union of two different lines. Therefore, at topological Euler characterisitc level, $\chi_{\rm top}(\,\mathbb{P}(T(Z_{min}\setminus\cup_vT_v(Z_{min}))\,)=0$. At Z_{min} the complement of the analytic arrangement is empty.

Example 3.1.13. Using special vanishing theorems and computation sequences of rational and elliptic singularities (cf. [N99, N99b]) one can prove the following results as well (the details will be published elsewhere, see also [N]).

(I) Assume the following situations:

(a) either (X, o) is rational, ϕ is arbitrary resolution, and $l' \in \mathcal{S}'$ is arbitrary,

(b) or (X, o) is minimally elliptic singularity with $H^1(\widetilde{X}, \mathbb{Z}) = 0$, ϕ is a resolution whose elliptic cycle equals E, and we also assume that for the fixed $l' \in \mathcal{S}'$ there exists a computation sequence $\{x_i\}_i$ for Z_{min}, which contains E as one of its terms, and it jumps (that is, $(x_i, E_1) = 2$) at some E_1 with $(E_1, l') < 0$.

Then the topological and analytic arrangements at l' agree, $\mathcal{A}_{\rm top}(l') = \mathcal{A}_{\rm an}(l')$.

(II) For minimally elliptic singularities it can happen that $\mathcal{A}_{\rm top}(l') \neq \mathcal{A}_{\rm an}(l')$, even for the minimal resolution. E.g., in the case of the minimal good resolution of $\{x^2 + y^3 + z^7 = 0\}$, or in the case of minimal resolution of $\{x^2 + y^3 + z^{11} = 0\}$ (which is good), for $l = Z_{min}$ one has $\dim(T(Z_{min})) = 2$ and $\dim(A(Z_{min})) = 1$.

Remark 3.1.14. For any $l' \in \mathcal{S}'$ one has the exact sequence

$$0 \to A(l') \to T(l') \to H^1(\mathcal{O}_{\widetilde{X}}(-l' - E)) \to H^1(\mathcal{O}_{\widetilde{X}}(-l'))$$

Hence, $\mathcal{A}_{\rm an}(l') = \mathcal{A}_{\rm top}(l')$ whenever $H^1(\mathcal{O}_{\widetilde{X}}(-l' - E)) = 0$. This fact happens e.g. if $l' = \sum_v a_v E_v^*$ with $a_v \gg 0$, in which case $H^1(\mathcal{O}_{\widetilde{X}}(-l' - E)) = 0$ by the Grauert–Riemenschneider Vanishing Theorem.

4. The topological series $Z(\mathbf{t})$.

4.1. Using the notations of Subsection 2.1 (and under the above assumption $H^1(\widetilde{X}, \mathbb{Z}) = 0$) we define the following combinatorial/topological 'candidate' for $P(\mathbf{t})$. Sometimes we do not make distinction between a rational function and their Taylor expansion at the origin.

Definition 4.1.1. The series $Z(\mathbf{t}) \in \mathbb{Z}[[\mathcal{S}']]$ is defined as the Taylor expansion at the origin of the rational function in variables $x_v = \mathbf{t}^{E_v^*}$ (cf. 2.1.2)

$$Z(\mathbf{t}) := \text{ Taylor expansion at 0 of } \Phi(z)(\mathbf{t}), \quad \text{where } \ z(\mathbf{x}) := \prod_{v \in \mathcal{V}} (1 - x_v)^{\delta_v - 2}. \tag{4.1.2}$$

That is,

$$Z(\mathbf{t}) = \text{ Taylor expansion at 0 of } \prod_v (1 - \mathbf{t}^{E_v^*})^{\delta_v - 2}. \tag{4.1.3}$$

We call this form the **first appearance of** $Z(\mathbf{t})$.

By (2.1.5), its h-component $Z_h(\mathbf{t})$ is the expansion of

$$\frac{1}{|H|} \cdot \sum_{\rho \in \widehat{H}} \rho(h)^{-1} \cdot \prod_{v \in \mathcal{V}} \left(1 - \rho([E_v^*])\mathbf{t}^{E_v^*}\right)^{\delta_v - 2}. \tag{4.1.4}$$

4.2. We start the list of its properties by the following observation. If Σ is a topological space, let $S^a\Sigma$ $(a \geq 0)$ denote its symmetric product $\Sigma^a / \mathfrak{S}_a$. For $a = 0$, by convention, $S^0\Sigma$ is a point. Then, by Macdonald formula [Macd62],

$$\sum_{a \geq 0} \chi_{\rm top}(S^a\Sigma)\, x^a = (1 - x)^{-\chi(\Sigma)}. \tag{4.2.1}$$

Let E_v° denote the regular part of $E_v \simeq \mathbb{P}^1$. Then $\chi_{\rm top}(E_v^\circ) = 2 - \delta_v$.

Corollary 4.2.2. The second appearance of $Z(\mathbf{t})$, *cf.* [CDGZ04, CDGZ08]. *With the notation* $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}\cdots x_s^{a_s}$,

$$z(\mathbf{x}) = \prod_v \sum_{a_v\geq 0} \chi_{\mathrm{top}}(S^{a_v}E_v^\circ)\, x_v^{a_v} = \sum_{\mathbf{a}\geq 0}\prod_v \chi_{\mathrm{top}}(S^{a_v}E_v^\circ)\mathbf{x}^{\mathbf{a}}. \tag{4.2.3}$$

4.3. In the next paragraphs we provide another interpretation of $Z(\mathbf{t})$. For the definition of the cycle $J(l',I)$ associated with $l'\in L'$ and $I\subset\mathcal{V}$ see 3.1.7.

Theorem 4.3.1. The third appearance of $Z(\mathbf{t})$**.**

$$Z(\mathbf{t}) = \sum_{l'\in\mathcal{S}'}\ \sum_{I\subset\mathcal{V}} (-1)^{|I|+1}\chi(l'+E_{J(l',I)})\mathbf{t}^{l'}. \tag{4.3.2}$$

This formula can be compared with (2.4.14) valid for $P(\mathbf{t})$.

Proof. With the notation $l'=\sum_v a_vE_v^*$ set

$$y(\mathbf{x}) := \sum_{\mathbf{a}\geq 0}\sum_I (-1)^{|I|+1}\chi(l'+E_{J(l',I)})\mathbf{x}^{\mathbf{a}}. \tag{4.3.3}$$

We wish to show that $y(\mathbf{x})=z(\mathbf{x})$. In the proof we use induction over $|\mathcal{V}|$. The verification of the $|\mathcal{V}|=1$ case is left to the reader. Hence, we assume $|\mathcal{V}|>1$. Fix a vertex $w\in\mathcal{V}$ so that $\delta_w=1$. Let $\Gamma_0:=\Gamma\setminus\{w\}$, and let u be that vertex of Γ_0 which is adjacent to w in Γ. Let $\mathbf{x}_0$ be the $\mathbf{x}$–vector associated with $\mathcal{V}(\Gamma_0)$. Clearly, one has

$$z_\Gamma(\mathbf{x}) = z_{\Gamma_0}(\mathbf{x}_0)\cdot(1-x_u)/(1-x_w).$$

We will establish similar identity for y_Γ. For this we write $l_0':=R(l')=\sum_{v\neq w}a_vE_v^{*,\Gamma_0}$ for any $l'=\sum_v a_vE_v^*$ (here E_v^{*,Γ_0} is the anti-dual of E_v in Γ_0). This is the restriction, the dual operator $L(\Gamma)'\to L(\Gamma_0)'$ of the inclusion $L((\Gamma_0)\to L(\Gamma)$. Hence, for $Z\in L(\Gamma_0)$

$$(l',Z)=(l_0',Z) \ \text{ and } \ (-E_u^{*,\Gamma_0},Z)=(E_w,Z). \tag{4.3.4}$$

First, we fix some $l'\in\mathcal{S}'$ and a subset $I\subset\mathcal{V}$ with $w\notin I$. If $w\in J(l',I)$, we may assume (cf. the notations of 3.1.7) that $I_{k-1}=J(l',I)\setminus w$. Since $(l',E_w)\leq 0$, $\delta_w=1$ and $(l'+E_{I_{k-1}},E_w)>0$, we get that, in fact, $(l'+E_{I_{k-1}},E_w)=1$. Hence

$$\chi(l'+E_{J(l',I)})=\chi(l'+E_{J(l',I)\setminus w}).$$

Comparing the two algorithms on Γ and Γ_0 we get that $J(l',I)\setminus w = J^{\Gamma_0}(l_0',I)$, and

$$\chi(l'+E_{J(l',I)\setminus w})-\chi(l'+E_I)=\chi(l_0'+E_{J^{\Gamma_0}(l_0',I)})-\chi(l_0'+E_I).$$

By (4.3.4) one also has $\chi(l'+E_I)-\chi(l')=\chi(l_0'+E_I)-\chi(l_0')$. All these implies the next identity, where in the right hand side all invariants are considered in Γ_0:

$$\chi(l'+E_{J(l',I)})-\chi(l')=\chi(l_0'+E_{J^{\Gamma_0}(l_0',I)})-\chi(l_0'). \tag{4.3.5}$$

The same is true if $w\notin J(l',I)$. Next, fix again $l'\in\mathcal{S}'$ and take $I\subset\mathcal{V}$ with $w\in I$.

We need to distinguish two cases. In the first case we assume that $(l',E_u)=0$. This happens exactly when $l_0'-E_u^{*,\Gamma_0}\notin\mathcal{S}'(\Gamma_0)$. In this case, for any $K\subset\mathcal{V}\setminus\{u,w\}$ one has $J(l',K\cup w\})=J(l',K\cup\{w,u\})$. Indeed, $(l'+E_{K\cup w},E_u)\geq(E_w,E_u)=1$, hence in the very first step of the algorithm of $J(l',K\cup w)$ we can add E_u. Thus,

$$\sum_{M\not\ni w}(-1)^{|M|}\chi(l'+E_{J(l',M\cup w)})=0. \tag{4.3.6}$$

Next, assume that $l'_0 - E_u^{*,\Gamma_0} \in \mathcal{S}'(\Gamma_0)$. Then, compared the two algorithms we get

$$J(l', I) = J^{\Gamma_0}(l'_0 - E_u^{*,\Gamma_0}, I \setminus w) \cup w,$$

$$\chi(l' + E_{J(l',I)}) - \chi(l' + E_I) = \chi(l'_0 - E_u^{*,\Gamma_0} + E_{J^{\Gamma_0}(l'_0 - E_u^{*,\Gamma_0}, I\setminus w)}) - \chi(l'_0 - E_u^{*,\Gamma_0} + E_{I\setminus w}).$$

Finally, (4.3.4) implies

$$\chi(l' + E_w + E_{I\setminus w}) - \chi(l' + E_w) = \chi(l'_0 - E_u^{*,\Gamma_0} + E_{I\setminus w}) - \chi(l'_0 - E_u^{*,\Gamma_0}).$$

Hence

$$\chi(l' + E_{J(l',I)}) - \chi(l' + E_w) = \chi(l'_0 - E_u^{*,\Gamma_0} + E_{J^{\Gamma_0}(l'_0 - E_u^{*,\Gamma_0}, I\setminus w)}) - \chi(l'_0 - E_u^{*,\Gamma_0}). \tag{4.3.7}$$

Since for any constant c, one has $\sum_{I: I \not\ni w} (-1)^{|I|+1} c = \sum_{I: I \ni w} (-1)^{|I|+1} c = 0$, the identities (4.3.5), (4.3.6) and (4.3.7) read as

$$\sum_{\mathbf{a}\geq 0} \sum_{I \not\ni w} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \mathbf{x}^{\mathbf{a}} = y_{\Gamma_0}(\mathbf{x}_0) \cdot \sum_{n_w \geq 0} x_w^{a_w};$$

$$\sum_{\mathbf{a}\geq 0} \sum_{I \ni w} (-1)^{|I|+1} \chi(l' + E_{J(l',I)}) \mathbf{x}^{\mathbf{a}} = -y_{\Gamma_0}(\mathbf{x}_0) x_u \cdot \sum_{n_w \geq 0} x_w^{a_w}.$$

Hence $y_\Gamma(\mathbf{x}) = y_{\Gamma_0}(\mathbf{x}_0)(1 - x_u)/(1 - x_w)$. □

Corollary 4.3.8. The forth appearance of $Z(\mathbf{t})$.

$$Z(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \chi_{\mathrm{top}}(\,\mathbb{P}(T(l') \setminus \cup_v T_v(l'))\,) \cdot \mathbf{t}^{l'}.$$

Proof. Combine Corollary 3.1.11 and Proposition 4.3.1. □

This formula can be compared with the statement of Corollary 3.1.3 valid for $P(\mathbf{t})$.

Example 4.3.9. Using Example 3.1.13 and Corollaries 3.1.3 and 4.3.8 we obtain that $P(\mathbf{t}) = Z(\mathbf{t})$ in the following cases (see also [N08]):

(a) (X, o) is rational, and ϕ is arbitrary resolution,

(b) or (X, o) is minimally elliptic singularity, and it satisfies the assumptions of Theorem 3.1.13.

More generally, the identity $Z(\mathbf{t}) = P(\mathbf{t})$ is true for any splice quotient singularity [N12]. Note that $Z(\mathbf{t}) = P(\mathbf{t})$ can happen even if $\mathcal{A}_{\mathrm{an}}(l') \neq \mathcal{A}_{\mathrm{top}}(l')$; see Example 3.1.5, which is a splice quotient singularity.

Remark 4.3.10. There is another incarnation of $Z(\mathbf{t})$, which uses **weighted cubes**. This realizes a connection with the lattice complex of the lattice cohomology, for details see e.g. [N11].

The set of q–cubes (where $q \in \mathbb{Z}_{\geq 0}$) consists of pairs $(l', I) \in L' \times \mathcal{P}(\mathcal{V})$, $|I| = q$, where $\mathcal{P}(\mathcal{V})$ denotes the power set of $\mathcal{V}$. $\square_q = (l', I)$ can be identified with the 'vertices' $\{l' + \sum_{j\in I'} E_j\}_{I'}$, where I' runs over all subsets of I, of a q–cube in $L' \otimes \mathbb{R}$. One defines the weight function

$$w : L' \to \mathbb{Q}, \qquad w(k) := \chi(l') = -(l', l' + K)/2. \tag{4.3.11}$$

This extends to a weight–function defined on the set of all q–cubes

$$w(\square_q) = w((l', I)) = \max_{I' \subset I} \{\, w(l' + \sum_{j \in I'} E_j)\,\}.$$

Then the **fifth appearance of** $Z(\mathbf{t})$ is

$$Z(\mathbf{t}) = \sum_{l' \in L'} \Big(\sum_{I \in \mathcal{P}(\mathcal{V})} (-1)^{|I|+1}\, w((l', I)) \Big)\, \mathbf{t}^{l'}. \tag{4.3.12}$$

4.4. The extension of $Z(\mathbf{t})$ in the Grothendieck ring. The information contained in $Z(\mathbf{t})$ can be improved if we replace in the 'forth appearance'

$$Z(\mathbf{t}) = \sum_{l'\in\mathcal{S}'} \chi_{\rm top}(\mathbb{P}(T(l')\setminus\cup_v T_v(l'))\cdot \mathbf{t}^{l'}$$

the topological Euler characteristic of $\mathbb{P}(T(l')\setminus\cup_v T_v(l'))$ with the class of this space in the Grothendieck group of complex quasi–projective varieties. (In the analytic case the extension of $P(\mathbf{t})$ to the series $\sum_{l'\in\mathcal{S}'}[\mathbb{P}(A(l')\setminus\cup_v A_v(l'))]\cdot\mathbf{t}^{l'}$ with coefficients in the Grothendieck ring was already considered e.g. in [CDGZ07].)

Let $\mathbb{L}$ be the class of the 1–dimensional affine space. Then, by inclusion–exclusion principle (as the analogue of 3.1.2) one has the following. If $\{V_\alpha\}_{\alpha\in\Lambda}$ is a finite family of linear subspaces of a finite dimensional linear space V, and for $I\subset\Lambda$ one writes $V_I:=\cap_{\alpha\in I}V_\alpha$, then

$$[V\setminus\cup_\alpha V_\alpha]=\sum_I(-1)^{|I|}\mathbb{L}^{\dim(V_I)},\ \ [\mathbb{P}(V\setminus\cup_\alpha V_\alpha)]=(\sum_I(-1)^{|I|}\mathbb{L}^{\dim(V_I)})/(\mathbb{L}-1).$$

According to this, one defines

$$Z(\mathbb{L},\mathbf{t})=\sum_{l'\in\mathcal{S}'}[\mathbb{P}(T(l')\setminus\cup_v T_v(l'))]\cdot\mathbf{t}^{l'}, \tag{4.4.1}$$

which, using 3.1.9 reads as

$$\begin{aligned} Z(\mathbb{L},\mathbf{t}) &= \frac{1}{\mathbb{L}-1}\cdot\sum_{l'\in\mathcal{S}'}\sum_{I\subset\mathcal{V}}(-1)^{|I|}\,\mathbb{L}^{\chi(l'+E)-\chi(l'+E_{J(l',I)})}\,\mathbf{t}^{l'} \\ &= \sum_{l'\in\mathcal{S}'}\sum_{I\subset\mathcal{V}}(-1)^{|I|}\ \cdot\frac{\mathbb{L}^{\chi(l'+E)-\chi(l'+E_{J(l',I)})}-1}{\mathbb{L}-1}\,\mathbf{t}^{l'}. \end{aligned}\tag{4.4.2}$$

Note that $\lim_{\mathbb{L}\to 1}Z(\mathbb{L},\mathbf{t})=Z(\mathbf{t})$. The analogue of the topological/combinatorial identity (4.1.3) is the following.

Theorem 4.4.3.

$$Z(\mathbb{L},\mathbf{t})=\frac{\prod_{(u,v)\in\mathcal{E}}\ (1-\mathbf{t}^{E^*_u}-\mathbf{t}^{E^*_v}+\mathbb{L}\mathbf{t}^{E^*_u+E^*_v})}{\prod_{v\in\mathcal{V}}\ (1-\mathbf{t}^{E^*_v})(1-\mathbb{L}\mathbf{t}^{E^*_v})}. \tag{4.4.4}$$

Proof. Follow the steps and all the identities of the proof of 4.3.1. □

This formula was reproved in the Master Thesis of János Nagy as well [Nagy16]. In this Thesis also several cohomological properties of the linear subspace arrangement complements are studied.

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