THE ROLE OF TWISTED WREATH PRODUCTS IN THE FINITE CONGRUENCE LATTICE PROBLEM

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Abstract

The problem whether every finite lattice is representable as the congruence lattice of a finite algebra has been reduced to a group theoretic question: whether every finite lattice occurs as an interval in the subgroup lattice of a finite group. Based on works of R. Baddeley, A. Lucchini, F. Börner, J. Shareshian, and M. Aschbacher the problem can be further reduced to two particular cases: intervals in subgroup lattices of finite groups where the group is either almost simple or a twisted wreath product of a restricted type. So the group theoretic construction of twisted wreath products introduced by B. H. Neumann in 1963 seems to play a crucial role in dealing with the finite congruence lattice problem.

1 The Finite Congruence Lattice Problem

A famous unsolved problem in universal algebra asks whether every finite lattice is isomorphic to the congruence lattice of a finite algebra. Since finite lattices are obviously algebraic, it follows from the fundamental Grätzer–Schmidt Theorem [7] that every finite lattice is the congruence lattice of some algebra. However, all known proofs of the Grätzer–Schmidt Theorem construct infinite algebras in almost all cases. P. Pudlák and the author [13] have shown that the finiteness problem is equivalent to a group theoretic one:

Problem 1 *Is every finite lattice isomorphic to an interval in the subgroup lattice of a finite group?*

For a group G and a subgroup $H < G$ we write

$$\mathrm{Int}(H,G) = \{\, X \mid H \leq X \leq G \,\}$$

for the lattice of intermediate subgroups (in other words: overgroups of H), and call it the **interval** between H and G in the subgroup lattice.

One direction of the equivalence is obvious. Let G act on the set of right cosets of the subgroup H, and consider each permutation in G as an operation with one variable. Then the congruences are exactly the partitions into cosets of subgroups belonging to the interval $\mathrm{Int}(H, G)$, hence the congruence lattice of this multi-unary algebra is isomorphic to this interval. Concerning the reverse implication, it should be emphasized that we do not claim that the congruence lattices of finite algebras are (up to isomorphism) the same as the intervals in subgroup lattices of finite groups. What we proved is that if *all* finite lattices can be represented as congruence lattices of finite algebras then *all* finite lattices can be represented as intervals in subgroup lattices of finite groups. In fact, we embed any finite lattice into a finite lattice with some useful properties, and then we show that the smallest algebra with a congruence lattice having these properties is a transitive permutation group considered as a multi-unary algebra.

It was shown by Jiří Tůma [17] that every algebraic lattice is isomorphic to an interval in the subgroup lattice of an infinite group. So it is the finiteness of the group what seems to constitute a severe restriction. Therefore, it is generally believed that the answer to the finite congruence lattice problem is negative.

2 Twisted Wreath Products

The notion of twisted wreath product was introduced by B. H. Neumann [10] in 1963. At first glance his definition looks quite complicated. M. Suzuki [16, Chapter 2, §10] presented a more elegant treatment of this construction. In [12] we gave a natural explanation for the occurrence of twisted wreath products. Although originally Neumann used twisted wreath products for constructing infinite groups with peculiar properties, here in the present paper we will stick to finite groups.

Twisted wreath products occur in the O'Nan–Scott–Aschbacher Theorem on the classification of primitive finite permutation groups. They were erroneously omitted from the first version [14] of the theorem, and were only added later to the list in the paper of Michael Aschbacher and Leonard Scott [2], and independently by László Kovács [8]. (See also [9].)

The fundamental role of twisted wreath products in the problem of representing finite lattices as intervals in subgroup lattices of finite groups was explicitly or implicitly observed in the papers of Robert Baddeley and Andrea Lucchini [4], Baddeley [3], Ferdinand Börner [5], John Shareshian [15], and Michael Aschbacher [1].

The ingredients of the twisted wreath product are the following: (finite) groups D (the domain) and T (the target), a subgroup $D_0 \leq D$ and a homomorphism $\varphi : D_0 \to \mathrm{Aut}(T)$ into the automorphism group of T. Let us decompose $D = D_0x_1 \cup D_0x_2 \cup \cdots \cup D_0x_m$ into a disjoint union of right cosets. Now let

$$\mathrm{Sdp}(D_0, \varphi) = \{f : D \to T \mid f(ax_i) = \varphi_a(t_i),\, a \in D_0,\, t_i \in T\, (i = 1, \ldots, m)\}.$$

It is easy to check that $\mathrm{Sdp}(D_0, \varphi)$ is a D-invariant subdirect product in T^D, where D acts on T^D via the natural action $f^d(x) = f(xd^{-1})$ for $f \in T^D$, $d, x \in D$. The **twisted wreath product** of T and D with respect to the subgroup $D_0 \leq D$ and the homomorphism $\varphi : D_0 \to \mathrm{Aut}(T)$ is defined as the semidirect product

$$\mathrm{Twr}(T, D, D_0, \varphi) = \mathrm{Sdp}(D_0, \varphi) \rtimes D.$$

3 The Reduction Theorem

Slighly improving Börner's result [5, Theorem 6.1] — by using a different lattice embedding lemma — we gave a proof [12] of the following reduction theorem.

Theorem 1 *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group if and only if one of the following is true:*

(1) Every finite lattice consisting of more than one element is isomorphic to an interval $\mathrm{Int}(H, G)$ *in the subgroup lattice of an* ***almost simple*** *finite group* G *with a core-free subgroup* H *(that is,* $\bigcap_{g \in G} g^{-1}Hg = 1$*).*

(2) Every finite lattice consisting of more than one element is isomorphic to an interval $\mathrm{Int}(D, G)$ *in the subgroup lattice of a* ***twisted wreath product*** $G = \mathrm{Twr}(T, D, D_0, \varphi)$ *of a non-abelian finite simple group* T *and a finite group* D *with respect to a subgroup* $D_0 < D$ *and a homomorphism* $\varphi : D_0 \to \mathrm{Aut}(T)$ *satisfying* $\varphi(D_0) \geq \mathrm{Inn}(T)$*, the group of inner automorphisms of* T*.*

With some extra work one can show also (as it was done by Börner [5]) that in case (2) we can force D_0 to be core-free in D.

We should note that the proof uses the classification of finite simple groups via one of its well-known consequences, **Schreier's Hypothesis**, claiming that the outer automorphism group $\mathrm{Out}(T) = \mathrm{Aut}(T)/\mathrm{Inn}(T)$ of every finite non-abelian simple group T is solvable.

As for many questions in finite group theory it would be desirable to reduce the problem to case (1) of **almost simple groups** (groups G with a simple normal subgroup T with $\mathbf{C}_G(T) = 1$). However, it seems inevitable to consider also certain twisted wreath products.

On the lattice theoretical side the proof does not use any deep considerations. If there is a lattice L_1 not representable with an almost simple group as in case (1) of the theorem, and another lattice L_2 that cannot be represented as an interval as in case (2), then one constructs a lattice that cannot be represented as an interval in the subgroup lattice of any finite group, see Figure 1. (Here L^d denotes the dual of the lattice L and $\hat{L}$ refers to a suitable extension of L that is generated by its coatoms and contains L as a filter.)

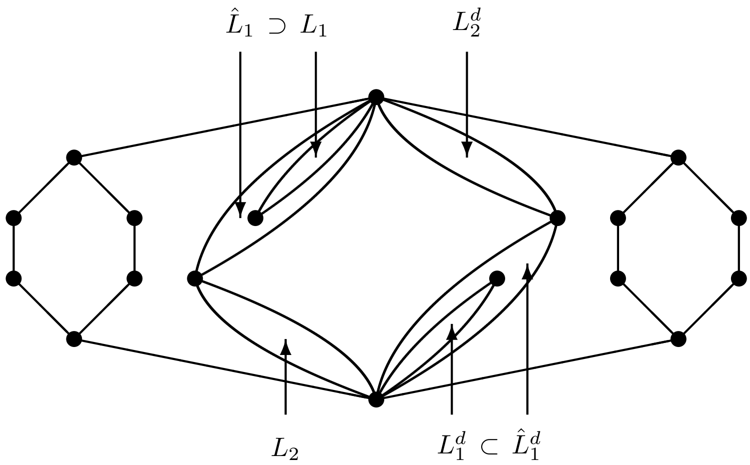

Figure 1: A possibly non-representable lattice

4 Intervals in the Subgroup Lattice of a Twisted Wreath Product

In case (2) of Theorem 1 one can describe the interval $\mathrm{Int}(D, G)$ in the following way. If $D < X \leq G = \mathrm{Twr}(T, D, D_0, \varphi)$, then $X = \mathrm{Sdp}(D_1, \varphi_1) \rtimes D$ for some subgroup $D_0 \leq D_1 \leq D$ and homomorphism $\varphi_1 : D_1 \to \mathrm{Aut}(T)$ extending φ. Moreover, $\mathrm{Sdp}(D_1, \varphi_1) \rtimes D \leq \mathrm{Sdp}(D_2, \varphi_2) \rtimes D$ iff $D_1 \geq D_2$ and $\varphi_1\big|_{D_2} = \varphi_2$. Hence we obtain:

Theorem 2 *Let $G = \mathrm{Twr}(T, D, D_0, \varphi)$ be the twisted wreath product of a non-abelian finite simple group T and a finite group D with respect to a subgroup $D_0 < D$ and a homomorphism $\varphi : D_0 \to \mathrm{Aut}(T)$ satisfying $\varphi(D_0) \geq \mathrm{Inn}(T)$. Then the interval $\mathrm{Int}(D, G)$ in the subgroup lattice of G is dually isomorphic to the lattice formed by all extensions of φ to subgroups of D together with a largest element added.*

The largest element on the top of all extensions corresponds to $D \in \mathrm{Int}(D, G)$ by the dual isomorphism.

For example, let A_5 and S_5 denote the alternating and the symmetric group of degree 5, and let $T = A_5$, $D = S_5 \times A_5$, $D_0 = \mathrm{diag}(A_5) = \{(a, a) \mid a \in A_5\} < D$, and fix an embedding $\varphi : D_0 \cong A_5 \to \mathrm{Aut}(T) \cong S_5$. It is easy to see that the subgroups of D containing D_0 are $D_0 = \mathrm{diag}(A_5)$, $A_5 \times A_5$, and $D = S_5 \times A_5$. Now φ has two extensions to $A_5 \times A_5$, corresponding to the first and the second projection. Likewise, there are two extensions to $S_5 \times A_5$. Together with the additional top element this gives a hexagon lattice, see Figure 2 (where $a, b \in A_5$, $s \in S_5$).

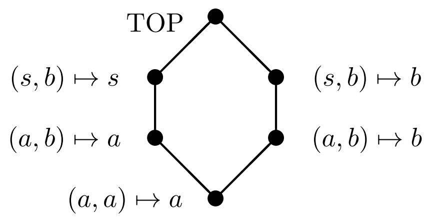

Figure 2: A representation for the hexagon lattice

Hence by Theorem 2 the interval $\mathrm{Int}(D, G)$ in the subgroup lattice of $G = \mathrm{Twr}(A_5, S_5 \times A_5, \mathrm{diag}(A_5), \varphi)$ is the hexagon lattice.

Actually, Aschbacher was motivated by a paper Yasuo Watatani [18], where it was proved that whenever a lattice can be represented as an interval in a subgroup lattice of a finite group, then it also occurs as a lattice of intermediate subfactors of a von Neumann algebra. With the exception of two lattices, Watatani was able to find intervals isomorphic to every lattice with at most six elements. One of the missing cases was the hexagon lattice. Aschbacher [1] gave a general construction whose particular cases provided examples for the hexagon and for the other six-element lattice Watatani was not able to handle. Aschbacher's example was slighly different from ours, he used $D = A_6 \times A_6$ instead of $S_5 \times A_5$ (but the same T, D_0, and φ). The hexagon also occurs in the subgroup lattice of a simple group, for example, as the interval of overgroups of a solvable subgroup of order 55 in the alternating group A_{11}, see [11, p. 477].[1]

5 Open Cases

William DeMeo [6] found representations of all lattices consisting of at most 7 elements, with two exceptions shown in Figure 3.

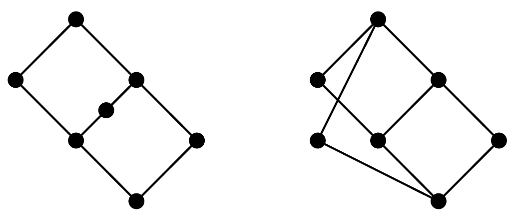

Figure 3: Open cases

So currently these are the smallest lattices for which no representation as

[1] I am very grateful to the referee for calling my attention to this example from my own old paper that I have forgotten.

an interval in the subgroup lattice of a finite group is known. (However, he showed that the lattice on the left hand side is the congruence lattice of a finite algebra.)

John Shareshian [15] suggested some candidates for lattices that may not be representable as intervals in subgroup lattices of finite groups. The smallest among these lattices is shown in Figure 4.

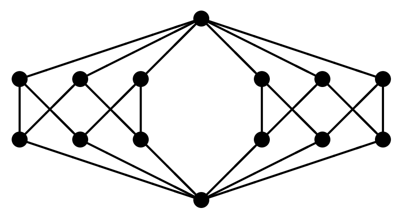

Figure 4: A lattice conjectured not to be representable

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