A General Bin Packing Game: Interest Taken into Account*

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Abstract

In this paper we study a general bin packing game with an interest matrix, which is a generalization of all the currently known bin packing games. In this game, there are some items with positive sizes and identical bins with unit capacity as in the classical bin packing problem; additionally we are given an interest matrix with rational entries, whose element a_{ij} stands for how much item i likes item j. The payoff of item i is the sum of a_{ij} over all items j in the same bin with item i, and each item wants to stay in a bin where it can fit and its payoff is maximized. We find that if the matrix is symmetric, a Nash Equilibrium (NE) always exists. However the PoA (Price of Anarchy) may be very large, therefore we consider several special cases and give bounds for PoA. We present some results for the asymmetric case, too. Moreover we introduce a new metric, called the Price of Harmony (PoH for short), which we think is more accurate to describe the quality of an NE in the new model.

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1 Introduction

In the classical bin packing problem, there are n items with positive rational sizes $s_1, s_2, ..., s_n$, where each item has size at most 1, and infinitely many bins with unit capacity are available. The items have to be packed into bins in a feasible way, what means that the sum of sizes of the items being packed in any bin B_k (called the level of the bin) does not exceed the capacity of the bin; i.e., the quantity $s(B_k) = \sum_{i \in B_k} s_i$ is at most 1. The goal is to minimize the number of bins. There are many papers on this topic; we refer to [8, 5, 6, 2] for details.

The first bin packing game was introduced by Bilò [1]. Later another version was proposed by Ma et al. [9], and recently a general version was developed by Dósa and Epstein [3]. We call these bin packing games (or models) as BPG1, BPG2, and BPG3, respectively.

In case of the BPG1 model, the items in the same bin pay unit cost in total for being in that bin. The items share the cost proportionally to their sizes: a bigger item pays more, a smaller item pays less, i.e. an item with size s_i pays $s_i/s(B_k)$ for being in bin B_k.

In case of the BPG2 model, the cost of any bin is again 1, but the items of any bin pay the same price for being in this bin, i.e. any item pays $1/t$ if there are t items in the bin.

In case of model BPG3, each item i has two parameters s_i and u_i, where s_i is the size of the item (as usually), and a nonnegative weight u_i is also specified for item i. Then, for being in a bin B_k, the items in B_k pay proportionally to their weights rather than to their sizes or their cardinality; i.e., item i pays u_i/U_k cost for being in B_k, where $U_k = \sum_{i \in B_k} u_i$. This is a common generalization of the two previous models BPG1 and BPG2, since if $u_i = s_i$ for any item, we get model BPG1, or if $u_i = 1$ for any item, we get model BPG2.

Generalized Bin Packing Game. We introduce a new type of bin packing games. This new game is even more general than BPG3, thus we call it Generalized Bin Packing Game and abbreviate it as GBPG for short.

The motivation of this new model is to express that people make their decisions not only considering money or cost, but they often also take into account how much they like a certain situation. Let us consider the next simple example: There is a party where the people sit down at tables (tables = bins). Then a person is interested not only in the cost of sitting at some table (and paying for the food and drinks he will have), but would also like to enjoy the party, and therefore chooses a table where he/she finds the people appealing, i.e., if his/her profit can be improved, then he/she would like to move to another situation.

Shortly, we think that in many situations the main organizing power is not (only) the money or cost of something, but also the interest or "sympathy" what people follow.

Formally, an instance $\mathcal{I}$ of GBPG is given as follows. There are n items with sizes s_i for $1 \leq i \leq n$, where $0 < s_i \leq 1$, and an $n \times n$ rational matrix $A = [a_{ij}]$, called the interest matrix, is also given. The *payoff* of item i is $p_i = \sum_{j \in B_k} a_{ij}$ if i is packed into bin B_k. And we define the total payoff $P = \sum_{i=1}^{n} p_i$. Each item wants to stay in a bin where it can fit and its payoff is maximized. All bins are assumed to be identical with unit capacity. In the discussion below we assume that $a_{ij} \geq 0$ for all i and j, although some facts remain valid for negative values, too. (Some remarks of this kind will be given.) We note that a_{ii} is also taken into account when defining the payoff p_i of item i.

A packing of the items is called a Nash Equilibrium [10], or NE for short, if no item can improve its payoff by moving to another bin in which it can fit. Moreover, if all the items are packed into the minimum number of bins, we call this packing an optimum packing, and denote its value by b^*. We denote the number of bins in a Nash Equilibrium as b_{NE}.

It should be noted that neither all equilibria are the optimal packings, nor all the optimal packing are NE, although in some cases they may coincide.

Price of Anarchy (PoA). An often used metric in case of bin packing games is the Price of Anarchy (PoA, for short); it measures how large an NE can be compared to an optimal packing, when b^* gets large. More exactly,

$$PoA = \limsup_{b^* \to \infty, \forall NE} \left\{ \frac{b_{NE}}{b^*} \right\}.$$

There are further metrics, too, such as price of stability (PoS), strong price of anarchy, and so on, cf. e.g. [3]. In this paper, however, we only deal with PoA from the mentioned ones. Moreover we introduce a new metric for which we coin the name 'Price of Harmony' (PoH), which is more accurate to describe the quality of an NE and leads to a more sensitive analysis of the game; we define this metric a bit later.

$a_{ij} =$	general	$\max\{s_i, s_j\}$	$\min\{s_i, s_j\}$	$\max\{u_i, u_j\}$	$u_i + u_j$
PoA	∞	[1.6416, 1.6428]	∞	[1.6966, 1.723]	[1.6966, 2]
PoH	-	[1.55, 1.6428]	∞	[1, 1.723]	[1, 2]

Table 1: Summary of bounds on PoA and PoH, where $[a, b]$ means that the lower bound is a and the upper bound is b.

Previous results. For the bin packing game BPG1, Bilò [1] proved that i) this game admits an NE, ii) there is a worst NE with a cost at least $1.6b^*$, and iii) the cost of any NE is at most $1.6667b^* + 2$. Later Epstein and Kleiman [7] obtained stronger estimates, proving that there is an NE which uses $1.6416b^*$ bins, but on the other hand the cost of any NE is bounded above by $1.6428b^* + 2$, i.e. the PoA is in $[1.6416, 1.6428]$.

For the BPG2 model it was proven in [9] that its PoA is at most 1.7; moreover, from any initial feasible packing, making an arbitrarily chosen feasible move as long as at least one exists, the process always reaches an NE in at most $O(n^2)$ selfish steps by the items. This result got further improved in [4].

In case of model BPG3 [3], it was shown that many kinds of Nash equlibria (NE, Strong NE, Strongly Pareto Optimal NE and Weakly Pareto Optimal NE) exist. For the case of unit weights (which is equivalent to model BPG2), the PoA is in [1.6966, 1.6994]; and for unrestricted weights, both of the lower and upper bounds are 1.7. For other results, we refer to [4, 12].

Our Contribution. When the interest matrix is symmetric, we prove that there exists an NE for any instance. Generally, the value of PoA can be arbitrarily large, therefore we consider several specific types of the interest matrix $[a_{ij}]$. The results are listed in Table 1, where s_i is the size of item i and u_i is the weight of item i. The bounds for $a_{i,j} = \max\{s_i, s_j\}$ are quoted from [7], since we can prove the bound by using exactly the same instance as in [7]. The lower bound in the last two columns is from [3], since if we set $u_i = 1$ for all i, the lower bound 1.6966 follows directly.

Moreover, we introduce and study the behavior of a new metric for our general game, which we call the Price of Harmony, or PoH for short, which is defined later. It measures the quality of an NE in the new model, simultaneously taking two criteria of the number bins used and the total payoff into account.

Lastly, we investigate the case of asymmetric matrices and find that an

NE does not exist for some instances. We give a sufficient condition to recognize this situation of non-existing NE. On the other hand we also give a sufficient condition which guarantees that the model with an asymmetric interest matrix can be converted to a model with a symmetric one.

2 Preliminaries

In this section, we first recall the definition of a classical algorithm for the bin packing problem, recall several important results for bin packing, and then prove that the game we study is a generalization of all the known bin packing games.

First Fit for Bin Packing. For an input I of bin packing, let $ALG(I)$ be the number of bins used by algorithm ALG to pack this input, and let b^* denote the number of bins by an optimal algorithm. Algorithm First Fit (FF) is a classical algorithm, which packs each item into the first bin where it fits. (If the item does not fit into any opened bin, it is packed into a new bin.) First Ullmann [11] proved that $FF(I) \leq 1.7\,b^* + 3$. Then, after several attempts to decrease the additive constant, finally Dósa and Sgall [5] proved that $FF(I) \leq 1.7\,b^*$, what means that the absolute approximation ratio of FF is at most 1.7. In another work, Dósa and Sgall [6] give a matching lower bound, thus the bound 1.7 is tight.

There are "better" algorithms, as well, for both the asymptotic and absolute approximation ratios. (The asymptotic approximation ratio of an algorithm ALG is the smallest R for which $ALG(I) \leq R \cdot b^* + C$ holds for all inputs I, where C is a suitably chosen constant, independent of the inputs.) For further details we recommend the survey [2].

2.1 Relation to the Earlier Models

The players in the games introduced earlier wish to minimize their costs, while in our game the players wish to maximize their payoff. In spite of this, the former bin packing game models can be considered as special cases of Model GBPG, in the following way:

- BPG1 model: let $a_{ij} = s_i \cdot s_j$ for all i; or $a_{ij} = s_j$ for all i.
- BPG2 model: let $a_{ij} = 1$ for all i; or $a_{ij} = s_i$ for all i.

- BPG3 model: let $a_{ij} = u_i \cdot u_j$ for all i; or $a_{ij} = u_j$ for all i.

We only prove this claim for the BPG1 model; the assertions for the other two models can be shown in a similar way.

Lemma 1 *For j, if $a_{ij} = s_i \cdot s_j$ for all i, or if $a_{ij} = s_j$ for all i, then our game is equivalent to the BPG1 model in the sense that an NE for GBPG is also an NE for BPG1, and vice versa.*

Proof. Suppose that $a_{ij} = s_i \cdot s_j$, and an item i is packed into bin B_k. Then the payoff of this item is $p_i = \sum_{j \in B_k}(s_i \cdot s_j) = s_i \cdot s(B_k)$. This item intends to go into another bin B_l if its payoff will be bigger there (and item i fits there). This new payoff is $p'_i = \sum_{j \in B_l}(s_i \cdot s_j) + s_i^2 = s_i \cdot (s(B_l) + s_i)$. Thus the movement is possible if and only if $p'_i > p_i$, i.e. $s(B_l) + s_i > s(B_k)$. This is exactly the same case (when the movement is possible) like the one in model BPG1. ■

2.2 New Definition: the Price of Harmony

After this pre-treatment, we find general properties of the new model, and then we investigate several special versions of interest. Before that, we introduce a new metric: the PoH, to measure the quality of a packing, which appears to be useful for this new general packing game. We find several properties and quantitative bounds for it.

Let b be the number of bins in an NE, and let b^* be the number of optimal bins, i.e. the minimum number of bins in a feasible packing, disregarding the interest matrix. Recall that $P = \sum_{i=1}^{n} p_i$, where p_i is the payoff of item i. Let P^* be the largest P.

Example 2 *Let n be even, and consider n very small items $1, \ldots, n$, each of size $2/n$. Define the interest $a_{i,i+n/2} = 1$ for all $1 \leq i \leq n/2$, and let $a_{i,j} = 0$ for all the other pairs (i, j) of items. This instance admits two extreme Nash Equilibria of quite different nature.*

(i) Sparse packing ($P = P^*$, $b \gg b^*$). *We use $n/2$ bins, each bin B_i contains the items i and $i + n/2$. Then no item intends to move because its payoff would decrease.*

(ii) Dense packing ($b = b^*$, $P \ll P^*$). *We use just two bins, pack the items $1, \ldots, n/2$ into B_1 and the items $n/2+1, \ldots, n$ into B_2. Then no item can move because both bins are full, moreover the opening of a new bin would not increase the payoff.*

Thus, both packings are NE, the sparse one with optimal $P = n/2$ but with very large $b = n/2$, while the dense one with optimal $b = 2$ but extremely poor $P = 0$.

Thus it makes sense to consider both parameters together. We do this by the next definition.

Definition 3 (Price of Harmony) *Given a packing, define $P = \sum_{i=1}^{n} p_i$, where p_i is the payoff of item i. Let P^* be the largest P. Price of Harmony(*PoH*) is defined as below:*

$$PoH = \limsup_{b^* \to \infty, \forall NE} \left\{ \alpha \mid \frac{b_{NE}}{b^*} \geq \alpha, \frac{P^*}{P_{NE}} \geq \alpha, \right\},$$

where NE stands for a Nash Equilibrium and P_{NE} is the total payoff in an NE.

Some concrete examples are given in Propositions 10 and 12.

Corollary 4 *For any game, $PoH \leq PoA$.*

3 General bounds for the symmetric case

We prove that for any symmetric matrix A, there always exists an NE for our model GBPG. We give the proof by using a potential function.

Definition 5 (Potential function of a packing.) *We consider the total payoff of items — defined as $P = \sum_i p_i$ — to be the potential of the packing.*

This approach can be used not only to prove the existence of an NE, but also to estimate the number of steps to converge to an NE from any unstable state.

Theorem 6 *If matrix A is symmetric, then GBPG always has an NE. Moreover, if all entries a_{ij} of A are rational, and the least common multiple of their denominators is q, then an NE is reached from any feasible initial packing after making at most $q\,n^2 \max_{i,j}\{a_{ij}/2\}$ feasible steps, which can be chosen arbitrarily as long as the packing is not an NE.*

Proof. The high-level idea is to associate each feasible packing with a potential in such a way that the potential function is upper-bounded by a value computable from the input, and to prove that once an item moves from one bin to another, the total potential strictly increases. If this property holds, then no previous state can occur again, thus an NE surely exists. Since there is an upper bound on the total payoff and each greedy step will increase the total payoff by some fix value $\Delta > 0$, after some finite number of steps we would come to a configuration that allows no more greedy step, and this configuration is at an NE.

Recall that if item i is packed into bin B_k, then its payoff is $p_i = \sum_{j \in B_k} a_{ij}$. We define a potential function as

$$P = \sum_{i=1}^{n} p_i \leq n^2 \max_{i,j}\{a_{ij}\}.$$

Next we prove that if item i moves from bin B_k to bin B_h, then the value of P increases. Before moving from bin B_k to bin B_h, let p_j be the payoff of item j and $P = \sum_{i \in I} p_i$. After the movement, let p'_j be the payoff of item j and $P' = \sum_{i \in I} p'_i$. Without danger of confusion, we also let B_k and B_h denote the set of items in the bins B_k and B_h before the move, and B'_k and B'_h be those after the move.

Observe that for item j which is not packed in bin B_k or B_h, its payoff does not change. Then we have

$$
\begin{aligned}
P' - P &= \sum_{j \in B'_k} p'_j + \sum_{j \in B'_h} p'_j - \sum_{j \in B_k} p_j - \sum_{j \in B_h} p_j \\
&= (\sum_{j \in B'_h} p'_j - \sum_{j \in B_h} p_j) + (\sum_{j \in B'_k} p'_j - \sum_{j \in B_k} p_j) \\
&= (p'_i + \sum_{j \in B_h} a_{ji}) - (p_i + \sum_{j \in B'_k} a_{ji}) \\
&= (p'_i - p_i) + (\sum_{j \in B_h} a_{ij} - \sum_{j \in B'_k} a_{ij}) \quad \text{by } a_{ij} = a_{ji} \\
&= (p'_i - p_i) + (a_{ii} + \sum_{j \in B_h} a_{ij} - a_{ii} - \sum_{j \in B'_k} a_{ij}) \\
&= (p'_i - p_i) + (p'_i - p_i) = 2(p'_i - p_i) > 0.
\end{aligned}
$$

For the number of convergence steps, suppose that A is a rational matrix. Let $\Delta > 0$ be the smallest integer such that Δa_{ij} is integer for all entries of A. Then $\Delta(p'_i - p_i) \geq 1$, and the potential function will increase by at least $2/\Delta$ after each selfish movement. After at most $\Delta n^2 \max_{i,j}\{a_{ij}/2\}$ steps, we will have an NE. ■

Remark 1 *One can observe that the above proof works even when matrix A has zero or negative entries. A natural interpretation of this extension is that a_{ij} is positive if person i likes person j, and is negative if i dislikes j.*

Remark 2 *Example 2(i) shows that, for arbitrarily large k, there exists an interest matrix A, for which the PoA is bigger than k, even if $a_{ij} \in \{0, 1\}$ is required for all $1 \leq i, j \leq n$.*

Recall that if all entries in the interest matrix A have the same value $a_{ij} = 1$ (i.e. we consider the BPG2 model), the PoA is upper-bounded by 1.6994 [3]. However, we find that the PoA can be very large even if almost all entries are $a_{ij} = 1$ and all the other elements satisfy $a_{ij} = 1 - \delta$ where $\delta > 0$ could be arbitrarily small. The exact conditions for "almost all entries" are given in the proposition below. Beside giving the proof considering the PoA, we show the stronger result that even the PoH can be very large.

Proposition 7 *Given $0 < \delta \leq 1/2$, let k be an integer for which $k\delta > 1$. Then there is a symmetric matrix A with size $n \times n$, where $n = k^4$, in which $a_{ij} = 1$ for at least $(1 - 1/k) \cdot n^2$ different pairs (i, j), moreover $a_{ij} = 1 - \delta$ for all the other entries, and the PoH is at least $k^2 - 1$.*

Proof. First we give a short and simple proof for a special version, which gives an insight into the proof in the general version. Let $p = 1 - \delta = 0.5$ for the sake of simplicity. Suppose that there are many bins, and each bin contains three items of size $1/n$. Let $a_{ii} = 1$ for all $1 \leq i \leq n$, $a_{ij} = 1$ if the items share a bin, and $a_{ij} = 0.5$ otherwise. Then the payoff of any item is $p_i = 3$, and this payoff decreases to 2.5 if an item moves to any other bin. On the other hand, all items can fit into one bin, and this latter packing has total payoff $\frac{1}{2}(n^2 + 3n)$, moreover it decreases the number of bins from $n/3$ to 1. Thus, the PoH can be arbitrarily large, by choosing the original packing. Note that in that packing we have only three items per bin, and consequently the number of $a_{ij} = 1$ entries is not large enough compared to the number of entries with $a_{ij} = 0.5$. To ensure that the non-unit entries are sparse, we need a more careful construction, where there are many items in each bin. Below we give this complete proof, in details.

Recall that we have $k\delta > 1$ by assumption. We construct a packing, with $n = k^4$ items and k^2 bins. In each bin there are k^2 very small items, of size $1/k^4$ each, thus all items fit into one bin.

Since the interest matrix contains only nonnegative entries, the biggest value of the total payoff is given if all items are in one bin.

We will have either $a_{ij} = 1$ or $a_{ij} = p$, where $1-\delta = p < 1$. For simplicity, let us say that $a_{ij} = 1$ if i and j know each other; otherwise if $a_{ij} = p$, we say they do not know each other. Let each item know all items in the same bin (including itself); and suppose that any item knows exactly $k^2 - k$ items in any other bin.

This construction is possible to be made in many different ways. Here we describe one: Partition the contents of each bin B_i into k parts, say $B_{i,1}, B_{i,2}, ..., B_{i,k}$ where each part $B_{i,\ell}$ contains exactly k items. We require that, for any choice of subscripts i, i', ℓ, ℓ' with $1 \leq i \neq i' \leq k^2$ and $1 \leq \ell, \ell' \leq k$ an item from $B_{i,\ell}$ knows an item from $B_{i',\ell'}$ if and only if $\ell \neq \ell'$.

We claim that no item has the intention to move. Consider any item i. The payoff for this item is exactly $p_i = k^2$. If this item moves into another bin, its payoff will be there $(k^2 - k) + 1 + k \cdot (1 - \delta) = k^2 + 1 - k\delta < k^2$, since $k\delta > 1$. Thus the claim holds, i.e. the packing is an NE.

Now let us count the number of entries with $a_{ij} = p$ in the matrix; we need to show that this number is at most n^2/k. Indeed, between any two bins there are only k^3 pairs where $a_{ij} = p$ holds, and no such pair occurs inside any bin. Moreover, an ordered pair of bins can be chosen in $k^2 \cdot (k^2 - 1)$ different ways. Thus, the number of entries with $a_{ij} = p$ satisfies

$$k^5 \cdot (k^2 - 1) < k^7 = n^2/k,$$

i.e., the total number of all the pairs with $a_{ij} = 1$ is at least $(1 - 1/k) \cdot n^2$.

Thus it follows that the conditions of the lemma hold. On the other hand, since all contents of the current k^2 bins fit into one single bin, we obtain that $PoA \geq k^2$.

Now we show that even the PoH is very big. For this, we must show that the current packing is also "weak" regarding the value of the total payoff of the items.

We have seen that the payoff of any item is $p_i = k^2$. Thus the total payoff given for all items (which is the value of the potential function is $P = k^4 \cdot k^2 = k^6$. If all items were packed into one bin (which is possible), the payoff of any item would be $p'_i = k^4 - (k-1)k \cdot \delta$, since any item knows almost all items, with exception only $(k-1)$ times k items, thus we need to decrease k^4 with $(k-1)k \cdot \delta$. For this latter payoff, we get $p'_i = k^4 - (k-1)k \cdot \delta > k^4 - k^2 \cdot \delta > k^4 - k^2$, since $\delta < 1/2$. Thus $P^* > k^4 \cdot (k^4 - k^2) = k^8 - k^6$, and thus $P^*/P > k^2 - 1$. ■

Remark 3 *In the previous proposition we have seen that although in model BPG2 the PoA is bounded, changing the entries of the interest matrix only in a small percentage and only slightly, this damages the PoA, and even the PoH to be unbounded.*

What is the case with model BPG1, if we make a similar small change? This question will be investigated in the next proposition. We show that again there is an NE for which the PoH is arbitrarily large. The proof is similar, namely the construction is almost the same.

Proposition 8 *Let $0 < \delta < 1$ be an arbitrary constant, and let T be an arbitrarily large value. Then there exists an $n \times n$ symmetric matrix A, for which $a_{ij} = s_i \cdot s_j$ for at least $(1 - \delta) \cdot n^2$ different pairs (i, j), and $a_{ij} = 0$ holds for the other pairs (i, j), and the PoH is bigger than T.*

Proof. Given δ, let us chose an integer $k > 1$, for which $k\delta > 1$ holds. Let $n = k^4$. We construct a packing, with k^2 bins, and k^2 very small items with size $s_i = \varepsilon = 1/k^4 = 1/n$ in any bin. The biggest value of the potential function is given if all items are in one bin. As in the proof of Proposition 7, suppose that any item knows all items in the same bin (including itself), and any item knows exactly $k^2 - k$ items in any other bin, the construction is made in the same way as before in the proof of Proposition 7. Then we set $a_{ij} = s_i \cdot s_j$ if the items i and j know each other, and $a_{ij} = 0$ otherwise.

We claim that no item intends to move. Consider an item i in a certain bin; the payoff p_i for this item is exactly $s_i \cdot k^2\varepsilon = k^2\varepsilon^2$. If this item moves into another bin, its payoff will be there $s_i \cdot (k^2 - k + 1)\varepsilon = (k^2 - k + 1)\varepsilon^2$ which is smaller than p_i, since $k > 1$. Thus the packing is an NE.

We have seen that the number of $a_{ij} = 0$ entries of the matrix is at most n^2/k, and $PoA \geq k^2 > T$ for large enough k.

Now we show that also the PoH is very big. We have seen that the payoff of any item is $p_i = k^2\varepsilon^2$. Thus the total payoff given for all items in the actual NE is $P = k^4 \cdot k^2\varepsilon^2 = k^6\varepsilon^2$. Consider the packing where all items are packed into one bin. The payoff of any item is $p_i' = (k^4 - (k-1)k) \cdot \varepsilon^2 > (k^2-1)k^2\varepsilon^2$, since any item knows all but $(k-1)k$ exceptional ones. Therefore $P^* > k^4 \cdot (k^2 - 1)k^2\varepsilon^2 = (k^8 - k^6)\varepsilon^2$. Thus $P^*/P > k^2 - 1 > T$, for large enough k. ■

After having proved Propositions 7 and 8, let us have a comment. There are two "classical" settings: models BPG1 and BPG2. For these, we know that the PoA is below 1.7. On the other hand, there exists a choice of the matrix (a_{ij}), which is "almost the same" as in BPG1 as in BPG2, but the PoH is blown up to infinity. In the following we show that this does not mean that the PoA or PoH is always bad if we deviate from these two special versions. For this, we turn to define and investigate special cases of our new model.

4 Special symmetric models

Now we give lower and upper estimates on the PoA and PoH for several special cases of GBPG, e.g. if $a_{ij} = \max\{s_i, s_j\}$, $a_{ij} = \min\{s_i, s_j\}$, $a_{ij} = \max\{u_i, u_j\}$, or $a_{ij} = u_i + u_j$, where u_i is the weight of item i, which may be different from its size s_i. These special settings need some explanation.

Let us consider the special case $a_{ij} = \max\{u_i, u_j\}$, where u_i means how much a person or item i is important. This is a natural case, where the items correspond to persons, some of them are famous while the others are not famous. If we assume that $u_i = 1$ if a person i is famous, and $u_i = p < 1$ otherwise, we model the situation that people like to be in the presence of famous or important persons. Or, more generally, any person gets an "importance index", this is the u_i value. Then, the happiness between two persons, i and j is defined as $a_{ij} = \max\{u_i, u_j\}$.

In order to explore the relationship between a matrix A and the corresponding value of PoA, we begin with the setting $a_{ij} = \max\{s_i, s_j\}$. We prove that the PoA is at most 1.7, and find that some earlier results also remain valid for this model. Contrary to this, for the model $a_{ij} = \min\{s_i, s_j\}$ we get that it is substantially different as the PoA can be arbitrarily large.

4.1 The special case $a_{ij} = \max\{s_i, s_j\}$

In this subsection we consider the special case $a_{ij} = max\{s_i, s_j\}$ and we get estimates for both the PoA and the PoH.

Proposition 9 *If* $a_{ij} = \max\{s_i, s_j\}$, *then* PoA *is at most* 1.7.

Proof. Our key observation is as follows: For any bin in a given packing, the payoff of the smallest item is the total size of items in the bin, and the payoff of any other item in the bin is at least this value. Consider an NE with bins $B_1, B_2, ..., B_m$. Assume that the bins are sorted such that $s(B_1) \geq s(B_2) \geq ... \geq s(B_m)$. We claim that no item in B_k fits into B_h for any $h < k$, i.e. the packing can be viewed as a result of FF packing. Let i be the smallest item in B_k, and suppose for a contradiction that it fits into B_h. In B_k, the payoff of item i is $p_i = s(B_k)$, whereas the payoff of this item is at least $s(B_h) + s_i > s(B_k)$ if it moves to B_h; this contradicts the assumption that we are in an NE state. Thus the claim follows. As we know that the asymptotic approximation ratio of FF (and even the absolute approximation ratio of FF) is 1.7, we obtain that the PoA in the current model is at most 1.7. ■

Proposition 10 *If* $a_{ij} = \max\{s_i, s_j\}$ *then* $PoA \geq 19/12 \approx 1.5833$, *and the* PoH *still exceeds* $14/9 \approx 1.555$.

Proof. In order to prove our statement we give a construction. In the optimum packing (when we use as few number of bins as possible), there are $12k$ bins, each having a large item $\alpha = 1/2 + \varepsilon$, and two medium sized items, $\beta = 1/4 + \varepsilon$ and $\gamma = 1/4 - 2\varepsilon$, where k is a positive integer. Since all such bins are full, this is naturally an optimum packing. Now we construct an NE, by packing all these items. In this packing there are $3k$ bins, each having four γ items, moreover $4k$ bins, each having three β items, and finally $12k$ bins having only a large item α. It is easy to see that all items are packed in this new packing. We prove that this packing is an NE. It is easy to see that a large item does not fit into any other bin. A β item can fit only into a bin where there is only a large item, but the payoff of a β item would be decreased from $\frac{3}{4} + 3\epsilon$ to $\frac{3}{4} + 2\epsilon$ if it moves. So this kind of item is not intended in the movement. The γ item also can move only into such bin where there is only a large item, but it is neither intended to move. Thus the packing is indeed an NE. For the number of bins we get $b^* = 12k$, while $b = 19k$, and the statement follows for the PoA.

Let us count the total payoff in the two packings. For the optimum packing (regarding the number of used bins) we get: $P = 12k \cdot ((3/2 + 3\varepsilon) + (1 + 3\varepsilon) + 1) = 6k\,(7 + 12\varepsilon)$. For the NE we get: $P_{NE} = 3k \cdot 4 \cdot (1 - 8\varepsilon) + 4k \cdot 3 \cdot (3/4 + 3\varepsilon) + 12k \cdot (1/2 + \varepsilon) = 3k\,(9 - 16\varepsilon)$. Thus $PoH \geq 2\,(7 + 12\varepsilon)\,/\,(9 - 16\varepsilon) > 14/9$. ■

Remark 4 *We find that using the methods in [1], one can get* $1.6 \leq PoA \leq 1.667$*; and using the methods in [7], one can further get* $1.6416 \leq PoA \leq 1.6428$.

4.2 The special case $a_{ij} = 1$ or $a_{ij} = p$ for some $0 \leq p < 1$

Another special (and natural) choice is the next model, where the items correspond to persons, some of them are famous while the other are not famous. Here, by setting $a_{ij} = 1$ if and only if at least one of i and j is famous, we model the situation that people like to be in the presence of famous persons. For another application, suppose there are families (adults and children) in groups. Being in a group (like in case of traveling, e.g. with the same train but being distributed into several passenger cars) it is safer for a child if there is at least one adult in the group. Then $a_{ij} = 1$ if any of i and j is adult, otherwise $a_{ij} = p < 1$ (possibly $p = 0$) if both i and j are children.

Proposition 11 *In the considered model, the PoA is at most* 2.

Proof. Now we give a simple proof to show that the PoA is at most 2. We will prove later that even in a more general setting, the PoA is at most 31/18. Consider an NE, assume there are two bins, say B_1 and B_2, such that the total level is at most 1. Let the cardinality of items in the two bins in consideration be, say k_1 and k_2, respectively. Suppose without loss of generality that $k_1 \geq k_2$. If there is no famous people in any of the bins, any item in the second bin is intended to move to the first bin (and it fits), we got a contradiction. Suppose there is famous people, say i, in bin B_2. Then the payoff for item i is just k_2 in the actual packing, and it will be improved to $k_1 + 1 > k_2$ if i moves to the other bin, a contradiction. Finally suppose there is a famous person in B_1, and there is not in B_2. Then any item in B_2 wants to move to B_1, a contradiction again. ■

Remark 5 *Regarding the lower bound, we can find the next: Since the considered special case is still the generalization of model BPG2, any lower bound for that also holds for this. There,* 1.6966 *is a lower bound, thus this value is also lower bound for now for the PoA.*

Next we show that the PoH is not too small either.

Proposition 12 *The PoH is at least* 1.5 *in the considered model.*

Proof. We make the next construction: Let ε be a suitable chosen small real value. Let the next items $(1/2 + \varepsilon,\ 1/3 + \varepsilon,\ 1/7 + \varepsilon)$ be in any optimal bin. Each item is chosen to be famous, except the items of the smallest size. Thus $a_{ij} = 1$ for all (i, j) pairs, except if both i and j have size of $1/7 + \varepsilon$. We will choose the value of p later. Let b^* be the number of optimal bins, let it be chosen as $b^* = 6k$ with some integer k.

Now let us pack the items by the Harmonic algorithm (for short H). We show that this is an NE with suitable chosen value of p. There are k bins with 6 items, $3k$ bins with 2 items in each and finally $6k$ bins with a single item. No item can move into an earlier bin, because of the size constraint. No item is intended to move into a later bin, except possibly the items of the smallest size. Here, we choose $p = 1/2$. So regarding an item of the smallest size, its payoff will be at most 3 if it moves to another bin with some famous item. But currently the payoff of this item is just equal to 3, thus the item does not move. Thus the claim holds.

For the number of created bins by H we get $H = (1+3+6)k = 10k$. Thus $H/b^* = 10/6 = 5/3$. Thus we need to show that P^*/P_H is not below 1.5. In case of the optimum packing, there are 3 items in each bin, and $p_i = 3$ for any item. There are $3 \cdot 6k = 18k$ items in total, so the total payoff is $P = 54k$ in case of the optimal packing. It means that $P^* \geq 54k$. Let us see the value of total payoff in the H packing. Here we get $P_H = k \cdot 6 \cdot 3 + 3k \cdot 2^2 + 6k \cdot 1^2 = 36k$, thus $P^*/P_H \geq 3/2$, which proves our statement regarding the PoH. ■

4.3 A common generalization of the two special cases

In this subsection we consider a model, which is a common generalization of the models of the two previous subsections. Assume that each item i is associated with two parameters, the size s_i and the weight $u_i > 0$, and let $a_{ij} = \max\{u_i, u_j\}$. If $u_i = s_i$ for any item, we get the model of Subsection 4.1. On the other hand, if $u_i = 1$ if item i is famous and $u_i = p$ otherwise, we get the model of Subsection 4.2. Below we analyze the upper bound of the PoA.

Theorem 13 *Assume that each item i is associated with two parameters, the size s_i and the weight $u_i > 0$. If $a_{ij} = \max\{u_i, u_j\}$, then the PoA is at most $\frac{31}{18} < 1.723$.*

Proof. Let us consider an NE with bins $B_1, B_2, ..., B_m$. Given a bin B_j, let the total weight and total size of the items in B_j be $u(B_j)$ and $s(B_j)$, respectively; and let the number of items in B_j be $|B_j|$. Suppose that there are two bins, say B_1 and B_2, such that $s(B_1) + s(B_2) \leq 1$. Assume without loss of generality that $u(B_1) \geq u(B_2)$. Let i be the item with the smallest weight in B_2. The payoff of i is exactly $u(B_2)$, and it becomes at least $u(B_1) + u_i > u(B_2)$ if the item moves to B_1, contradicting the assumption that the packing is an NE. Consequently, $s(B_1) + s(B_2) > 1$ holds for any two bins. Moreover, we have the following properties.

1. If item i is packed into B_j, its payoff satisfies $p_i \geq u(B_j)$ and equality holds only if i has the minimum weight in B_j.

2. If item k has the smallest or second smallest weight in B_j, then $p_k < u(B_j) + u_k$.

Now we divide the bins into four groups:

$$G_1 = \{B_j \mid 0 < s(B_j) \le 1/2\}, \qquad G_2 = \{B_j \mid 1/2 < s(B_j) \le 2/3\},$$
$$G_3 = \{B_j \mid 2/3 < s(B_j) \le 3/4\}, \qquad G_4 = \{B_j \mid s(B_j) > 3/4\}.$$

Claim 1. $|G_1| \le 1$.

Proof: This follows from the fact that, as we have shown the beginning of the argument, any two bins satisfy $s(B_1) + s(B_2) > 1$ in any NE. □

Define
$$G_2^1 = \{B_j \mid B_j \in G_2, |B_j| = 1\}, \quad G_3^1 = \{B_j \mid B_j \in G_3, |B_j| = 1\},$$
$$G_2^{2+} = \{B_j \mid B_j \in G_2, |B_j| \ge 2\}, \quad G_3^{2+} = \{B_j \mid B_j \in G_3, |B_j| \ge 2\}.$$

Claim 2. The sole item in each bin of $G_2^1 \cup G_3^1$ has a size larger than 1/2 (by definition).

Claim 3. $|G_2^{2+}| \le 1$.

Proof: Suppose for a contradiction that at least two bins, say B_1 and B_2 belong to G_2^{2+}; assume without loss of generality that $u(B_1) \ge u(B_2)$. From the definition of G_2^{2+}, we see that the item with the smallest weight or the second smallest weight has a size at most $\frac{1}{3}$. Let item k be the item. If item k moves to bin B_1, then its payoff is at least $u(B_1) + u_k \ge u(B_2) + u_k > p_k$, where p_k is the payoff of item k in bin B_2. So it is not difficult to see that the item in B_2 has an incentive to move to bin B_1. Hence the assumption is false and $|G_2^{2+}| \le 1$. □

Claim 4. With the exception of at most one bin, in each bin of G_3^{2+}, both the item with the minimum weight and the second minimum weight have size larger than 1/4.

Proof: Consider any two bins of G_3^{2+}, say B_1 and B_2. Assume without loss of generality that $u(B_1) \ge u(B_2)$. In B_2, if any item i with the smallest or the second smallest weight has a size at most $\frac{1}{4}$, then its payoff will get improved from at most $u(B_2) + u_i$ to at least $u(B_1) + u_i$, by Properties 1 and 2. Therefore each of them has a size larger than 1/4. □

Now we can proceed further with the proof of the theorem. We have an upper bound

$$\begin{aligned} b_{NE} &= |G_1| + |G_2^1| + |G_3^1| + |G_2^{2+}| + |G_3^{2+}| + |G_4| \\ &\le 2 + |G_2^1| + |G_3^1| + |G_3^{2+}| + |G_4|, \end{aligned}$$

and a lower bound

$$b^* \geq \frac{|G_2^1| + |G_3^1|}{2} + \frac{2|G_3^{2+}|}{3} + \frac{3|G_4|}{4}. \tag{1}$$

The size of each item in the bins of $G_2^1 \cup G_3^1$ is larger than $1/2$; let us call these items big items. Hence, there are $|G_2^1| + |G_3^1|$ big items. In each bin of G_3^{2+}, except at most one bin, there are at least two items with a size larger than $1/4$ each; let us call these items medium-sized items. Hence, there are at least $2(|G_3^{2+}| - 1)$ medium-sized items. Note that no two big items can be packed into the same bin, therefore

$$b^* \geq |G_2^1| + |G_3^1|. \tag{2}$$

Case 1. $|G_2^1| + |G_3^1| \geq 2(|G_3^{2+}| - 1)$. Then we also have

$$b^* \geq 2|G_3^{2+}| - 2. \tag{3}$$

We multiply the inequalities (1), (2), (3) by $\frac{24}{18}$, $\frac{6}{18}$, and $\frac{1}{18}$, respectively. Adding them we obtain

$$\frac{31}{18} b^* \geq |G_2^1| + |G_3^1| + |G_3^{2+}| + |G_4| - 1/9,$$

thus

$$b_{NE} \leq |G_2^1| + |G_3^1| + |G_3^{2+}| + |G_4| + 2 \leq \frac{31}{18} b^* + \frac{19}{9} < 1.723 \cdot b^* + 2.12.$$

Case 2. $|G_2^1| + |G_3^1| < 2(|G_3^{2+}| - 1)$. Now, in any feasible packing, at most one medium-sized item can be packed with a big item into the same bin, and the remaining medium-sized items need at least $\frac{2(|G_3^{2+}|-1)-|G_2^1|-|G_3^1|}{3}$ bins. Therefore,

$$\begin{aligned} b^* &\geq |G_2^1| + |G_3^1| + \frac{2(|G_3^{2+}| - 1) - |G_2^1| - |G_3^1|}{3} \\ &= \frac{2(|G_2^1| + |G_3^1|)}{3}| + \frac{2|G_3^{2+}| - 2}{3}. \end{aligned} \tag{4}$$

Now we multiply the inequalities (1), (2), and (4) by $\frac{24}{18}$, $\frac{4}{18}$, and $\frac{3}{18}$, respectively. Adding them we obtain

$$\frac{31}{18} b^* \geq |G_2^1| + |G_3^1| + |G_3^{2+}| + |G_4| - 1/9,$$

therefore we also have $b_{NE} \leq \frac{31}{18} b^* + \frac{19}{9} < 1.723 b^* + 2.12$. ■

Proposition 14 *Assume that $s_i \geq s_j$ implies $u_i \geq u_j$ for any two items i and j. If $a_{ij} = \max\{u_i, u_j\}$, then the PoA is at most* 1.7.

Proof. The proof can be done in the same argument as the proof for the model of Subsection 4.1, replacing the s_i sizes by the u_i values. ■

4.4 The special case $a_{ij} = \min\{s_i, s_j\}$

Now we turn to consider the special case $a_{ij} = \min\{s_i, s_j\}$. In this special case, instead of "how i and j like each other", the situation is as "how i and j dislike each other". We again find estimates for both the PoA and the PoH.

Proposition 15 *If $a_{ij} = \min\{s_i, s_j\}$, then the PoA, and even the PoH can be arbitrarily large.*

Proof. We give the proof by constructing a class of bad instances. Consider a packing with c bins.

Let each bin B_k ($1 \leq k \leq c$) contain $k+1$ items whose sizes are $\alpha_k = \frac{2\varepsilon}{k(k+1)}$ (so $\alpha_1 = \varepsilon$, $\alpha_2 = \varepsilon/3$, $\alpha_3 = \varepsilon/6$, and so on), where ε is a small positive constant. It is easy to see that the sequence $(\alpha_k)_{k\geq 1}$ is strictly decreasing. The heart of the matter is that the actual packing is an NE, whereas all items can be packed into one bin if ϵ is sufficiently small. Let $1 \leq k \leq c$ be an arbitrary index, and consider item i in bin B_k. We get $p_i = (k+1) \cdot \alpha_k$. Suppose that item i moves to another bin, say with index l. Here $l < k$ is not possible, because then the payoff of i would change to $p_i' = (l+2) \cdot \alpha_k \leq (k+1) \cdot \alpha_k$, hence the item does not intend to move there. Otherwise, if $l > k$, then the payoff changes to

$$\begin{aligned} p_i' &= (l+1) \cdot \alpha_l + \alpha_k = (l+1) \cdot \frac{2\epsilon}{l(l+1)} + \alpha_k = \frac{2\epsilon}{l} + \alpha_k \\ &< \frac{2\epsilon}{(k+1)} + \alpha_k = k \cdot \frac{2\epsilon}{k(k+1)} + \alpha_k = k \cdot \alpha_k + \alpha_k \\ &= (k+1) \cdot \alpha_k = p_i. \end{aligned}$$

We claim that this is at most $p_i = k \cdot \alpha_k + \alpha_k$, it suffices to show that $(l+1) \cdot \alpha_l \leq k \cdot \alpha_k$. After dividing by 2ε, this is the same as $\frac{l+1}{l(l+1)} \leq \frac{1}{k+1}$, or $k+1 \leq l$, which trivially holds. Consequently, item i does not intend to move in this latter case either. Since all items fit into one bin, the PoA is not smaller than the number of bins in our actual NE, that is c. Thus, we obtain that PoA in the current model can be any large, as c can be chosen arbitrarily large.

Now we show that the PoH is also big. In the present packing, the payoff of an item i, being in bin B_k, is $p_i = (k+1) \cdot \alpha_k = \frac{2\varepsilon}{k}$. Since there are $k+1$ items in this bin (with the same size), the total payoff of items in this bin is $\frac{2(k+1)}{k}\varepsilon$. Then the total payoff for the packing is

$$P = 2\varepsilon \sum_{k=1}^{c} \frac{k+1}{k} \leq 2\varepsilon \sum_{k=1}^{c} \frac{k+k}{k} = 4c\varepsilon.$$

Now let us consider the optimal packing where all items are in one bin. Let us consider an arbitrary item i, and suppose it was packed in bin B_k in the NE. Note that this item is smaller than any item which were packed into any earlier bin, and item i is bigger than any item that was packed into any later bin. Thus, when we compute the payoff of item i, $a_{ij} = s_i = \alpha_k$, if j is packed into not later bin, and $a_{ij} \geq \alpha_c \geq 0$ otherwise. There are $2+3+...+(k+1) = k\,(k+3)\,/2$ items in not later bins. Thus the payoff of item i is $p'_i \geq k\,(k+3)\,/2 \cdot \alpha_k = \frac{k+3}{k+1}\varepsilon$. Since there were $k+1$ items in bin B_k, the total payoff of all items in bin B_k is at least $(k+3)\,\varepsilon$. And thus the total payoff for this packing is at least

$$P' \geq \varepsilon \sum_{k=1}^{c} (k+3) > \varepsilon \sum_{k=1}^{c} k = \varepsilon c(c+1)/2.$$

Comparing P and P' we get

$$P'/P \geq \frac{\varepsilon c(c+1)/2}{4c\varepsilon} = (c+1)/8,$$

which can be arbitrarily big, if c is chosen big enough. ■

4.5 A model for summing the weights

Finally we consider the case $a_{ij} = u_i + u_j$, where $u_i > 0$ is the weight of item i. In this version if item i is packed together in a bin B with k other items, then its payoff is $k \cdot u_i + \sum_{j \in B} u_j$.

Below we give estimate for the PoA.

Proposition 16 *Assume that $a_{ij} = u_i + u_j$, where $u_i > 0$ is the weight of item i. Then PoA is at most 2.*

Proof. It suffices to show that the average level of bins in any NE is larger than $1/2$. Consider an NE, and assume for a contradiction that there are two bins, say B_1 and B_2, such that their total level is at most 1. Let $u(B_1)$ and $u(B_2)$ be defined as $u(B_1) = \sum_{j \in B_1} u_j$ and $u(B_2) = \sum_{j \in B_2} u_j$, respectively; we assume without loss of generality that $u(B_1) \geq u(B_2)$. Let l and k be the numbers of items in B_1 and B_2, respectively. Suppose first that $l \geq k$, and let i be an arbitrary item in B_2. We claim that i would like to move to B_1. The actual payoff for item i is

$$p_i = \sum_{j \in B_2} a_{ij} = \sum_{j \in B_2} (u_i + u_j) = k \cdot u_i + u(B_2).$$

If i moves to bin B_1, its payoff will be there

$$p_i' = \sum_{j \in B_1} a_{ij} + a_{ii} = \sum_{j \in B_1} (u_i + u_j) + 2u_i = (l + 2) \cdot u_i + u(B_1),$$

which is bigger than p_i, thus the claim is verified. This contradicts the assumption that the packing is an NE, therefore we must have $l < k$. Let i be the item in B_1 for which u_i is the biggest, and j be the item in B_2, for which u_j is the smallest. Then

$$u_i \geq u(B_1)/l \geq u(B_2)/l > u(B_2)/k \geq u_j. \tag{5}$$

If we move i from B_1 to B_2, its payoff changes by

$$\begin{aligned} c_1 &= p_i{}' - p_i = (k \cdot u_i + u(B_2) + 2u_i) - (l \cdot u_i + u(B_1)) \\ &= (k + 2 - l)u_i + u(B_2) - u(B_1); \end{aligned}$$

and if we move j from B_2 to B_1, its payoff changes by

$$\begin{aligned} c_2 &= p_j{}' - p_j = (l \cdot u_j + u(B_1) + 2u_j) - (k \cdot u_j + u(B_2)) \\ &= (l + 2 - k)u_j + u(B_1) - u(B_2). \end{aligned}$$

Moreover, from (5) we have

$$c_1 + c_2 = 2(u_i + u_j) + (k - l)(u_i - u_j) > 0.$$

This means that at least one of items i and j will improve its payoff by moving to the other bin, a contradiction. ■

5 Asymmetric case

In this section we deal with the case where the matrix A is not symmetric. First we observe by giving an example that NE may not always exist; more precisely, from a suitably chosen initial packing, NE is not reached after any sequence of selfish steps. Then we describe a sufficient condition which ensures that there exists an initial packing and an infinite sequence of feasible steps which never lead to an NE, and we find by another example that the condition is not necessary. Finally we also give a sufficient condition, where NE always exist.

Example 17 *If there are negative values in the matrix, even for the simplest instance: two bins and two items* 1 *and* 2*, each with size* 0.5*, and* $a_{i,i} = 1$*,* $a_{1,2} = 1$ *and* $a_{2,1} = -1$*, we can see there is no* NE *(*1 *pursuits* 2*,* 2 *escapes, and so on).*

So in the following, we assume all a_{ij} are non-negative.

Example 18 *The following instance admits an initial packing which never terminates with an* NE*, independent of the value of the parameter* $p < 1$*. Take three items* 1*,* 2*, and* 3*, each with size* 0.5*. Let* $a_{i,i} = 1$ *for* $i = 1, 2, 3$*,* $a_{1,2} = a_{2,3} = a_{3,1} = 1$*, and* $a_{2,1} = a_{3,2} = a_{1,3} = p$*.*

Proof. Assume that the three items are packed in three distinct bins. Then item 1 moves to share a bin with item 2. Then item 2 leaves item 1 alone and moves to share a bin with item 3. Then item 3 leaves item 2 alone and moves to share a bin with item 1. Then item 1 moves and again shares a bin with item 2, and the movement can be continued to infinity. Then there is no NE for the above instance. ■

In the following we give a generalization of the instance in Example 18, where NE does not exits after any sequence of selfish steps. This part is related to line graphs.

Partial Line Graph. According to standard terminology, the vertices in the line graph $L(G)$ of G represent the edges of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share a vertex. So, each edge in $L(G)$ identifies three vertices, say v_i, v_j, v_k in G, and two edges v_iv_j, v_iv_k on them, sharing one vertex v_i. Originally in G the edges

$$A = \begin{vmatrix} 1 & 1 & 1 & \epsilon \\ 1{-}\epsilon & 1 & p & \epsilon \\ p & 1 & 1 & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \end{vmatrix}$$

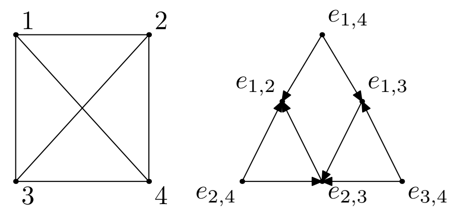

Figure 1: Matrix, Compatibility Graph and Partial Line Graph

are undirected; but now their common vertex v_i specifies the *ordered* pairs $(i, j), (i, k)$. In case if $a_{ij} = a_{ik}$, we remove the corresponding edge from $L(G)$; and if equality does not hold, then we orient the edge of $L(G)$ from the smaller to the larger a-value; see Fig. 1 for an illustration. We denote by $H = (X, F)$ the oriented graph obtained in this way. (This H, obviously, does not contain cycles of length 2.)

Compatibility Graph. Let $\mathcal{I} = (A, S)$ be an instance of GBPG. We define the *compatibility graph* to represent the pairs of items which can occur together in a bin. This undirected graph, which we denote by $G = (V, E)$, is described by the following rules:

- the vertices are $v_1, v_2, ..., v_n$, indexed according to the items;
- an unordered vertex pair $v_i v_j$ is an edge if and only if $s_i + s_j \leq 1$.

Theorem 19 *Let $\mathcal{I}$ be an instance of GBPG, and let $H = (X, F)$ be the oriented partial line graph as described above. If H contains a directed (cyclically oriented) cycle, then there exists an initial packing of the items and an infinite sequence of feasible steps along which NE is never reached.*

Proof. Let $C = x_1 x_2 \dots x_\ell$ be a directed cycle in H. We assume, without loss of generality, that C is a *shortest* cycle in H. Each x_k $(1 \leq k \leq \ell)$ corresponds to some edge $e_k := v_{i_k} v_{j_k}$ of G, and we have $e_k \cap e_{k+1} \neq \emptyset$ for all $1 \leq k \leq \ell$ (subscript addition is taken modulo ℓ throughout the proof). We further observe:

Claim. For all k we have $e_k \cap e_{k+1} \neq e_{k+1} \cap e_{k+2}$.

Proof. Suppose for a contradiction that $e_k \cap e_{k+1} = e_{k+1} \cap e_{k+2} = v_k$, and assume that $e_i = v_k v_{j_i}$ for $i = k, k+1, k+2$. Then, by the construction of H, we have

$$a_{k,j_k} < a_{k,j_{k+1}} < a_{k,j_{k+2}}.$$

As a consequence, also $x_k x_{k+2}$ is an arc in H, because e_k and e_{k+2} share v_k and the corresponding s-values satisfy the required inequality. This contradicts the assumption that C is a shortest cycle in H, and hence the claim follows. □

To prove the theorem we start with the initial packing where the two items corresponding to the vertices of e_1 are in the same bin, and all the other items are in mutually distinct bins. The claim implies that moving the item of $e_1 \cap e_2$ from its bin to the bin of $e_2 \setminus e_1$ is feasible. More generally, from a bin whose contents are the two items belonging to e_k, it is feasible to move $e_k \cap e_{k+1}$ to the bin of $e_{k+1} \setminus e_k$, for any $1 \leq k \leq \ell$. Consequently, in the first $\ell - 1$ bins the first ℓ items can circulate forever, without reaching NE at any time. ■

Remark 6 *If all entries of S are positive, then the conclusion of Theorem 19 also holds with the trivial initial packing which puts each item into a distinct bin. In fact it is sufficient that at least one pair of consecutive vertices involved in the cycle of H be adjacent with a directed edge of positive weight (that is, one of their a_{ij} or a_{ji} should be positive, the other one may even be negative). Then, if each item is packed into a distinct bin, we can start the moving procedure with creating this positive edge inside one bin.*

Remark 7 *The line graph of the input can be constructed in linear time in terms of the input size, and it can also be tested in polynomial time whether the line graph contains a directed cycle.*

We note, however, that the above theorem does not solve the following question: Given the sizes and matrix A, is it true that NE exists from a given packing? Can this question be answered in polynomial time? We give a further type of instance, without a directed cycle in the line graph, and prove that NE never occurs after any sequence of selfish movements.

Proposition 20 *Even when the partial line graph of an instance contains no directed cycles, NE may not occur after any number of steps of selfish improvement.*

Proof. There are four items, each with size $1/3$ and the matrix A is exactly the same as the one in Fig. 1, where $0 < p < 1$ is fixed, and $0 < \varepsilon < (1-p)/2$ is also fixed. There is no directed cycle in Fig. 1.

We prove this by the next sequence of steps. Initially, $B_1 = \{1, 2\}$ and $B_2 = \{3, 4\}$, i.e. we have only two bins, items $1, 2$ are packed into B_1, and items $3, 4$ are packed into B_2. At this moment item 1 intends to move, as he likes item 2 and item 3 equally, but he meets also item 4 in the other bin, and a_{14} is positive. Thus item 1 moves, and we get $B_1 = \{2\}$, $B_2 = \{1, 3, 4\}$. Then item 3 intends to move, since $a_{32} = 1 > p+\varepsilon = a_{31} + a_{34}$. Thus item 3 moves, and after the movement we get the packing $B_1 = \{2, 3\}$, $B_2 = \{1, 4\}$. At this moment item 4 intends to move, as $a_{41} = \varepsilon < \varepsilon + \varepsilon = a_{43} + a_{42}$, thus item 4 moves, and we get the next packing: $B_1 = \{2, 3, 4\}$, $B_2 = \{1\}$. Finally, item 2 intends to move since $a_{21} = 1 - \varepsilon > p + \varepsilon = a_{23} + a_{24}$, thus item 2 moves, and we get the packing $B_1 = \{3, 4\}$, $B_2 = \{1, 2\}$. It means that the contents of B_1 and B_2 are swapped, and thus the sequence of the movements can be continued to infinity. ■

Next we give a sufficient condition, which ensures that the game in an asymmetric case can be converted to a symmetric game.

Proposition 21 *Assume that $a_{ij} > 0$ for any pair (i, j). If there is a univariate function $g : N \to R^+$ such that $\frac{a_{ij}}{a_{ji}} = \frac{g(i)}{g(j)}$, then the asymmetric case can be transformed to the symmetric case, and hence NE exists.*

Proof. Given a game (G_1) with an asymmetric matrix $A = [a_{ij}]$ that satisfies the condition, we construct a new game (G_2) with a matrix $A' = [a_{ij}/g(i)]$ that is symmetric. Given a state of G_1, the payoff of item i is $p_i = \sum_j a_{ij}$, where j is packed into the same bin with i. We construct a state for G_2 such that each item is packed in the same bin as in the state of G_1, and then the payoff of item i is $p'_i = \sum_j a_{ij}/g(i) = 1/g(i) \sum_j a_{ij}$. In G_1, i moves to another bin to improves its payoff by $a > 0$ if and only if in G_2, i does the same action and improve its payoff by $a/g(i) > 0$. We conclude that the equilibria of the two games are in one-to-one correspondence, and the result follows. ■

Remark 8 *If $a_{ij} = s_i$, let $g(i) = s_i$. If $a_{ij} = s_j$, let $g(i) = 1/s_i$. So, in the two cases above, NE always exists. These are known cases. We can give several further choices for a_{ij} and $g(i)$, for which it is easy to verify the sufficient condition. For instance, if $a_{ij} = s_i^2 s_j$, let $g(i) = s_i$; or if $a_{ij} = s_i(s_i+s_j)$, let $g(i) = s_i$. But currently we do not have a characterization of all possible polynomials of s_i and s_j, for which the condition holds.*

6 Conclusions and further research

In this paper we introduced a new type of bin packing game, which is a common generalization of all the previously considered bin packing games. We proved general bounds and also studied several special types of the game. There are several open questions left, let us mention some of them.

1. What are the tight bounds on PoA and PoH for the models studied?
2. Determine the algorithmic complexity of the following decision problems.

   Given $\mathcal{I}$, an instance of GBPG, together with an initial packing,

   (a) is it true that NE is reached after a suitable sequence of steps?

   (b) is it true that NE is reached after a finite number of steps, no matter which feasible step is chosen at any time?

   Is any of these problems polynomial-time solvable?

Finally we make some short further notes. We defined the payoff of an item i as the sum of the a_{ij} values for all j items which are in common bin with item i. But this is not the only option. The payoff of item i could be also defined as

2.1 $p_i = \max_{j \in B_k} a_{ij}$ if item i is packed into bin B_k.

2.2 $p_i = \min_{j \in B_k} a_{ij}$ if item i is packed into bin B_k.

The explanation can be that people try to be close (in a common group) to some people whom they like very much, or try to avoid to be close to people they dislike. In this sense not the total "payoff" is taken into account, but no matter the other people being there, they want to be in the presence of someone, or they do not want to be in the presence of someone.

Many other types of payoff functions composed of the a_{ij} may also be of interest.

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