FIRST COUNTABLE AND ALMOST DISCRETELY LINDELÖF T_3 SPACES HAVE CARDINALITY AT MOST CONTINUUM

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Abstract. A topological space X is called *almost discretely Lindelöf* if every discrete set $D \subset X$ is included in a Lindelöf subspace of X. We say that the space X is *μ-sequential* if for every non-closed set $A \subset X$ there is a sequence of length $\leq \mu$ in A that converges to a point which is not in A. With the help of a technical theorem that involves elementary submodels, we establish the following two results concerning such spaces.

(1) For every almost discretely Lindelöf T_3 space X we have $|X| \leq 2^{\chi(X)}$.
(2) If X is a μ-sequential T_2 space of pseudocharacter $\psi(X) \leq 2^\mu$ and for every free set $D \subset X$ we have $L(\overline{D}) \leq \mu$, then $|X| \leq 2^\mu$.

The case $\chi(X) = \omega$ of (1) provides a solution to Problem 4.5 of [5], while the case $\mu = \omega$ of (2) is a partial improvement on the main result of [2].

Our main aim in this note is to prove what is stated in the title and thus give a solution to problem 4.5 of [5].

All spaces in here are assumed to be T_1. Consequently, if X is any space and A is any subset of X then the pseudocharacter $\psi(A, X)$ of A in X, i.e. the smallest size of a family of open sets whose intersection is A, is well-defined.

We recall that a transfinite sequence $\{x_\alpha : \alpha < \eta\} \subset X$ is called a free sequence in X if for every $\beta < \eta$ we have

$$\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \beta \leq \alpha < \eta\}} = \emptyset.$$

We say that a subset $D \subset X$ is free if it has a well-ordering that turns it into a free sequence. Clearly, every free set is discrete and

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every countable discrete set is free in X. We shall use $\mathcal{F}(X)$ to denote the family of all free subsets of X. The freeness number $F(X)$ of X is defined by $F(X) = \sup\{|D| : D \in \mathcal{F}(X)\}$, while its hat version $\widehat{F}(X)$ is the smallest cardinal κ such that X has no free subset of size κ.

Given a space X and an infinite cardinal κ, we are going to consider elementary submodels M of $H(\lambda)$ for a large enough regular cardinal λ such that $X \in M$ and M is $< \kappa$-closed, i.e. $M^\varrho \subset M$ for all $\varrho < \kappa$. (Note that by $X \in M$ we really mean $\langle X, \tau\rangle \in M$ where τ is the topology on X.) We may also assume, without any loss of generality, that $\mu + 1 \subset M$ where $\mu = |M|$. The following proposition is an easy consequence of standard cardinal arithmetic.

Proposition 1. *The minimum cardinality of a $< \kappa$-closed elementary submodel M (of some $H(\lambda)$) is $2^{<\kappa}$ if κ is regular and 2^κ if κ is singular.*

We now present a (somewhat technical) result that, in addition to a space X and a cardinal κ, involves such an elementary submodel. This result plays a crucial role in the proof of our main result and we suspect that it will have numerous other interesting consequences as well.

Theorem 2. *Fix a space X and a cardinal κ, moreover let M be a $< \kappa$-closed elementary submodel (of some $H(\lambda)$) such that $X \in M$ and $\mu + 1 \subset M$ where $\mu = |M|$. If for every $D \in \mathcal{F}(X) \cap [X \cap M]^{<\kappa}$ we have $\psi(\overline{D}, X) \le \mu$, then either*

$$X = \bigcup\{\overline{D} : D \in \mathcal{F}(X) \cap [X \cap M]^{<\kappa}\}$$

or $\widehat{F}(X) > \kappa$, i.e. there is a free set of size κ in X.

Proof. Let us put $Y = \bigcup\{\overline{D} : D \in \mathcal{F}(X) \cap [X \cap M]^{<\kappa}\}$ and assume that $X \ne Y$. We may then fix a point $p \in X \setminus Y$. Note that, as M is $< \kappa$-closed, we have $\mathcal{F}(X) \cap [X \cap M]^{<\kappa} \subset M$, hence for every $D \in \mathcal{F}(X) \cap [X \cap M]^{<\kappa}$ we have $\overline{D} \in M$ as well. Consequently, by elementarity, for every such D there is a family of open sets $\mathcal{V}_D \in M$ such that $\bigcap \mathcal{V}_D = \overline{D}$ and $|\mathcal{V}_D| \le \mu$. Note that then we have $\mathcal{V}_D \subset M$ as well.

We are now going to define points $x_\alpha \in X \cap M$ and open sets $V_\alpha \in M$ by transfinite recursion on $\alpha < \kappa$ so that the following three conditions hold true for every $\alpha < \kappa$:

(i) $p \notin V_\alpha$;
(ii) $\overline{\{x_\beta : \beta < \alpha\}} \subset V_\alpha$;
(iii) $\{x_\beta : \alpha \le \beta < \kappa\} \cap V_\alpha = \emptyset$.

Clearly then $\{x_\alpha : \alpha < \kappa\}$ will be a free sequence of length κ.

So, assume that $\alpha < \kappa$ and we have defined $D_\alpha = \{x_\beta : \beta < \alpha\} \subset X \cap M$ and $\{V_\beta : \beta < \alpha\} \subset \tau \cap M$ such that for every $\gamma < \alpha$ we have $p \notin V_\gamma$, $\overline{D_\gamma} \subset V_\gamma$, and $\{x_\beta : \gamma \le \beta < \alpha\} \cap V_\gamma = \emptyset$.

Then we have $D_\alpha \in \mathcal{F}(X) \cap [X \cap M]^{<\kappa}$, and so there is $V_\alpha \in \mathcal{V}_{D_\alpha}$ such that $p \notin V_\alpha$. As M is $< \kappa$-closed, we then have $\{V_\beta : \beta \le \alpha\} \in M$ as well and so $p \notin \bigcup\{V_\beta : \beta \le \alpha\}$ implies $X \cap M \setminus \bigcup\{V_\beta : \beta \le \alpha\} \neq \emptyset$ by elementarity. We then pick x_α as any element of $X \cap M \setminus \bigcup\{V_\beta : \beta \le \alpha\}$. It is clear that this recursive procedure carries though for all $\alpha < \kappa$, moreover $\{x_\alpha : \alpha < \kappa\}$ and $\{V_\alpha : \alpha < \kappa\}$ satisfy the three conditions (i) – (iii). □

We now turn to applying Theorem 2 to produce some new cardinal function inequalities, in particular the statement formulated in the title. Our notation and terminology of cardinal functions follow those in [3].

We recall, see [5], that a space X is called (almost) discretely Lindelöf if for every discrete set $D \subset X$ we have that $\overline{D}$ is Lindelöf (resp. D is included in a Lindelöf subspace of X). We now present several lemmas concerning such spaces. We shall use $\mathcal{D}(X)$ to denote the family of all discrete subspaces of X. The spread $s(X)$ of X is then defined by $s(X) = \sup\{|D| : D \in \mathcal{D}(X)\}$.

Lemma 3. *For every almost discretely Lindelöf T_2 space X we have (i) $s(X) \le 2^{t(X)\cdot\psi(X)}$ and (ii) $F(X) \le t(X)$.*

Proof. For every $D \in \mathcal{D}(X)$ there is a Lindelöf $Y \subset X$ with $D \subset Y$. Consequently, we have $|D| \le |Y| \le 2^{t(X)\cdot\psi(X)}$ by Shapirovskii's theorem from [6], see also 2.27 of [3]. This proves (i).

To see (ii), assume, arguing indirectly, that $D = \{x_\alpha : \alpha < t(X)^+\}$ is a free sequence in X. Then again there is a Lindelöf $Y \subset X$ with $D \subset Y$ and so D has a complete accumulation point y in Y. But then, by the definition of $t(X)$, there is an $\alpha < t(X)^+$ such that $y \in \overline{\{x_\beta : \beta < \alpha\}}$ and at the same time $y \in \overline{\{x_\beta : \alpha \le \beta < t(X)^+\}}$, contradicting that $\{x_\alpha : \alpha < t(X)^+\}$ is a free sequence. This proves (ii). □

The cardinal function $g(X) = \sup\{|\overline{D}| : D \in \mathcal{D}(X)\}$ was introduced in [1] (and is not mentioned in [3]). Since every right separated (or equivalently: scattered) space has a dense discrete subspace, for every space X we have $h(X) \le g(X)$ because $h(X)$ is just the supremum of the sizes of all right separated subspaces of X.

We are now ready to formulate and prove our main result.

Theorem 4. *For every almost discretely Lindelöf T_3 space X we have $|X| \le 2^{\chi(X)}$.*

Proof. We first show that $g(X) \le 2^{\chi(X)}$. Indeed, if $D \in \mathcal{D}(X)$ then we have $|D| \le 2^{t(X)\cdot\psi(X)} \le 2^{\chi(X)}$ by part (i) of Lemma 3. But we also have $|\overline{D}| \le |D|^{\chi(X)}$, see e.g. 2.5 of [3], hence putting these together we get $|\overline{D}| \le 2^{\chi(X)}$ as well. This, in turn, yields us $h(X) \le g(X) \le 2^{\chi(X)}$.

But as X is T_3, we have $\Psi(X) \le h(X) = hL(X)$, i.e. for every closed set $H \subset X$ we have $\psi(H, X) \le h(X)$. Indeed, every point $x \in X \setminus H$ admits a neighborhood U_x such that $\overline{U_x} \cap H = \emptyset$ and then there is a set $A \subset X \setminus H$ with $|A| \le h(X)$ such that $X \setminus H = \bigcup\{U_x : x \in A\} = \bigcup\{\overline{U_x} : x \in A\}$. This is the point where the T_3 property, and not just T_2, is essentially used.

And now we apply Theorem 2 to X with the choice $\kappa = \chi(X)^+$, by choosing an appropriate elementary submodel M of cardinality $2^{\chi(X)}$ that is $\chi(X)$-closed (i.e. $< \chi(X)^+$-closed), moreover $X \in M$ and $2^{\chi(X)} + 1 \subset M$. Note that we have established above the inequality $\Psi(X) \le 2^{\chi(X)}$ that is much more than what is needed to ensure the applicability of Theorem 2.

But by part (ii) of Lemma 3 we have $F(X) \le t(X) \le \chi(X)$, i.e. there is no free set in X of size $\kappa = \chi(X)^+$, consequently

$$X = \bigcup\{\overline{D} : D \in \mathcal{F}(X) \cap M\}$$

must be satisfied.

Finally, using again $g(X) \le 2^{\chi(X)}$ we can conclude from the above equality and from $|M| = 2^{\chi(X)}$ that $|X| \le 2^{\chi(X)}$. □

We do not know if the T_3 property can be relaxed to the T_2 property in Theorem 4, even in the countable case $\chi(X) = \omega$. However, it should be mentioned concerning this that Spadaro has given a consistent affirmative answer to this question in [7]. He proved in fact that if $\mathfrak{c} = 2^{<\mathfrak{c}}$ then every sequential T_2 space of pseudocharacter $\le \mathfrak{c}$ which is almost discretely Lindelöf has cardinality $\le \mathfrak{c}$.

Our next application of Theorem 2 also concerns sequential T_2 spaces, in fact a generalization of sequentiality will be used. Instead of the almost discretely Lindelöf property, however, a slightly weakened version of the discrete Lindelöf property will be used. On the other hand, we shall obtain a ZFC result.

Definition 5. Let μ be an infinite cardinal number. A space X is called *μ-sequential* if for every non-closed set $A \subset X$ there is a (transfinite) sequence of length $\le \mu$ of points of A that converges to a point which is not in A. Thus, ω-sequential $\equiv$ sequential.

The following proposition is well-known in the case $\mu = \omega$ and hence its proof, being a natural adaptation of the countable case, is left to the reader.

Proposition 6. *If X is a μ-sequential T_2 space then (i) $t(X) \le \mu$ and (ii) for every set $A \subset X$ we have $|\overline{A}| \le |A|^\mu$.*

Theorem 7. *Let X be a μ-sequential T_2 space such that for every $D \in \mathcal{F}(X)$ we have $L(\overline{D}) \le \mu$, moreover $\psi(X) \le 2^\mu$. Then actually $|X| \le 2^\mu$.*

Proof. Let us start by noting that, using $t(X) \le \mu$, the same argument as in the proof of part (ii) of Lemma 3 yields $F(X) \le \mu$. Thus, by part (ii) of Proposition 6, for every free set $D \in \mathcal{F}(X)$ we have $|\overline{D}| \le \mu^\mu = 2^\mu$.

We claim that then $\psi(\overline{D}, X) \le 2^\mu$ also holds for every free set $D \in \mathcal{F}(X)$. To see this, let us fix for every point $x \in \overline{D}$ a family $\mathcal{U}_x$ of open neighborhoods of x with $|\mathcal{U}_x| \le 2^\mu$ and $\bigcap \mathcal{U}_x = \{x\}$ and then put $\mathcal{U} = \bigcup\{\mathcal{U}_x : x \in \overline{D}\}$. Clearly, we have $|\mathcal{U}| \le 2^\mu$ as well. Now, if we fix any point $p \in X \setminus \overline{D}$ then for every $x \in \overline{D}$ there is $U_x \in \mathcal{U}_x$ such that $p \notin U_x$. But then $L(\overline{D}) \le \mu$ implies that for some $A \subset \overline{D}$ with $|A| \le \mu$ we have $\overline{D} \subset \bigcup\{U_x : x \in A\}$. This shows that the family

$$\mathcal{W} = \{\cup\mathcal{V} : \mathcal{V} \in [\mathcal{U}]^{\le\mu} \text{ and } \overline{D} \subset \cup\mathcal{V}\}$$

of open sets satisfies $\cap\mathcal{W} = \overline{D}$ and, as clearly $|\mathcal{W}| \le 2^\mu$, the family $\mathcal{W}$ witnesses $\psi(\overline{D}, X) \le 2^\mu$.

Consequently, if M is an appropriate μ-closed elementary submodel of cardinality 2^μ such that $\{X\} \cup 2^\mu + 1 \subset M$ then the assumptions of Theorem 2 are satisfied with $\kappa = \mu^+$. Thus, from $F(X) \le \mu$ we conclude that

$$X = \bigcup\{\overline{D} : D \in \mathcal{F}(X) \cap M\}.$$

But then $|M| = 2^\mu$ and $|\overline{D}| \le 2^\mu$ for all $D \in \mathcal{F}(X)$ clearly imply $|X| \le 2^\mu$. □

Arhangel'skii and Buzyakova has shown in [2] that the cardinality of a sequential linearly Lindelöf Tikhonov space X does not exceed 2^ω if the pseudocharacter of X does not exceed 2^ω. Clearly, the case $\mu = \omega$ of Theorem 7 is a partial improvement on their result.

It is obvious that radial spaces of tightness $\le \mu$ are μ-sequential but it is also clear that the converse of this statement fails, as is demonstrated by the existence of sequential spaces that are not Frèchet.

On the other hand, every μ-sequential space is pseudoradial and has tightness $\le \mu$. Our next example shows that, at least consistently, the converse of this statement also fails, even for compact spaces.

Example 8. *The one-point compactification $X = K \cup \{p\}$ of the so called Kunen line K, constructed from CH in* [4], *is pseudoradial and hereditarily separable, hence countably tight, but not sequential.*

Proof. X is hereditarily separable because K is. K is a non-closed subset of X that is sequentially closed in X because K is countably compact, hence X is not sequential.

Finally, assume that $A \subset X$ is not closed in X. If there is a point $x \in K$ with $x \in \overline{A} \setminus A$ then an ω-sequence in A converges to x because K is first countable. Otherwise, $A \subset K$ is closed in K and $\overline{A} = A \cup \{p\}$, hence A is not compact. But every countable closed subset of K is compact, again by the countable compactness of K, hence we have $|A| = \omega_1$. Every neighborhood of p in X is clearly co-countable, hence we have $\psi(p, A \cup \{p\}) = \chi(p, A \cup \{p\}) = \omega_1$, and so there is an ω_1-sequence in A that converges to p. □

The following intriguing problem, however remains open.

Problem 1. *Is there a ZFC example of a countably tight pseudoradial space that is not sequential?*

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