

Bounds on the 2-domination number

^{1,2}Csilla Bujtás and ¹Szilárd Jaskó

¹Faculty of Information Technology
University of Pannonia, Veszprém, Hungary
Email: bujtas@dcs.uni-pannon.hu, jasko.szilard@uni-pen.hu

²Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences, Budapest, Hungary

Abstract

In a graph G , a set $D \subseteq V(G)$ is called 2-dominating set if each vertex not in D has at least two neighbors in D . The 2-domination number $\gamma_2(G)$ is the minimum cardinality of such a set D . We give a method for the construction of 2-dominating sets, which also yields upper bounds on the 2-domination number in terms of the number of vertices, if the minimum degree $\delta(G)$ is fixed. These improve the best earlier bounds for any $6 \leq \delta(G) \leq 21$. In particular, we prove that $\gamma_2(G)$ is strictly smaller than $n/2$, if $\delta(G) \geq 6$. Our proof technique uses a weight-assignment to the vertices where the weights are changed during the procedure.

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1 Introduction

We study the graph invariant $\gamma_2(G)$, called 2-domination number, which is in close connection with the fault-tolerance of networks. Our main contributions are upper bounds on $\gamma_2(G)$ in terms of the number of vertices, when the minimum degree $\delta(G)$ is fixed. The earlier upper bounds of this type are tight for $\delta(G) \leq 4$, here we establish improvements for the range of $6 \leq \delta(G) \leq 21$. Our approach is based on a weight-assignment to the vertices, where the weights are changed according to some rules during a 2-domination procedure.

1.1 Basic terminology

Given a simple undirected graph G , we denote by $V(G)$ and $E(G)$ the set of its vertices and edges, respectively. The *open neighborhood* of a vertex $v \in V(G)$ is defined as $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, while the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Then, the

degree $d(v)$ is equal to $|N(v)|$ and the *minimum degree* of G is the smallest vertex degree $\delta(G) = \min\{d(v) \mid v \in V(G)\}$. We say that a vertex v *dominates* itself and its neighbors, that is exactly the vertices contained in $N[v]$. A set $D \subseteq V(G)$ is a *dominating set* if each vertex of G is dominated or equivalently, if the closed neighborhood of D , defined as $N[D] = \bigcup_{v \in D} N[v]$, equals $V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of such a set D . Domination theory has a rich literature, for results and references see the monograph [13].

There are two different natural ways to generalize the notion of (1-)domination to multiple domination. As defined in [10], a *k-dominating set* is a set $D \subseteq V(G)$ such that every vertex not in D has at least k neighbors in D . Moreover, D is a *k-tuple dominating set* if the same condition $|N[v] \cap D| \geq k$ holds not only for all $v \in V(G) \setminus D$ but for all $v \in V(G)$. The minimum cardinalities of such sets are the *k-domination number* $\gamma_k(G)$ and the *k-tuple domination number* of G , respectively.

1.2 2-domination and applications

A sensor network can be modeled as a graph such that the vertices represent the sensors and two vertices are adjacent if and only if the corresponding devices can communicate with each other. Then, a dominating set D of this graph G can be interpreted as a collection of cluster-heads, as each sensor which does not belong to D has at least one head within communication distance.

A k -dominating set D may represent a dominating set which is $(k-1)$ -fault tolerant. That is, in case of the failure of at most $(k-1)$ cluster-heads, each remaining vertex is either a head or keeps in connection with at least one head. The price of this k -fault tolerance might be very high. In the extremal case, when k is greater than the maximum degree in the network, the only k -dominating set is the entire vertex set. But for the usual cases arising in practice, 2-domination might be enough and it does not require extremely many heads.

Note that k -tuple domination might need much more vertices (cluster-heads) than k -domination. As proved in [11], for each real number $\alpha > 1$ and each natural number n large enough, there exists a graph G on n vertices such that its k -tuple domination number is at least $\frac{k}{\alpha}$ times larger than its k -domination number. There surely exist some practical problems where k -tuple domination is needed, but for many problems arising k -domination seems to be sufficient. Indeed, if a cluster-head fails and is deleted from the network, we may not need further heads to supervise it. This motivates our work on the 2-domination number γ_2 .

Another potential application of our results in sensor networks concerns the data collection problem. Here, each sensor has two capabilities: either measures and reports, or receives and collects data. Only one position from those two can be active at the same time. After deploying, the organization process determines exactly which sensors supply the measuring and the collector function in the given network. Since it is a natural condition that every measurement should be saved in at least two different devices, the set of

collector sensors should form a 2-dominating set in the network.

We mention shortly that many further kinds of application exist. For example a facility location problem may require that each region is either served by its own facility or has at least two neighboring regions with such a service [17]. In this context, facility location may also mean allocation of a camera system, or that of ambulance service centers.

1.3 Upper bounds on the 2-domination number

Although this subject attracts much attention (see the recent survey [8] for results and references) and it seems very natural to give upper bounds for γ_2 in terms of the minimum degree, there are not too many results of this type. The following general upper bounds are known. (As usual, n denotes the order of the graph, that is the number of its vertices.)

- If the minimum degree $\delta(G)$ is 0 or 1, then $\gamma_2(G)$ can be equal to n .
- If $\delta(G) = 2$ then $\gamma_2(G) \leq \frac{2}{3}n$. This statement follows from a general upper bound on $\gamma_k(G)$ proved in [9]. The bound is tight for graphs each component of which is a K_3 .
- If $\delta(G) \geq 3$ then $\gamma_2(G) \leq \frac{1}{2}n$. The general theorem, from which the bound follows, was established in [7]. Note that a 2-dominating set of cardinality at most $n/2$ can be constructed by a simple algorithm. We divide the vertex set into two parts and then in each step, a vertex which has more neighbors in its own part than in the other one, is moved into the other part. If the minimum degree is at least 3, this procedure results in two disjoint 2-dominating sets. Note that for $\delta(G) = 3$ and 4 the bound is tight. For example, it is easy to check that $\gamma_2(K_4) = 2$ and $\gamma_2(K_4 \square K_2) = 4$.¹
- For every graph G of minimum degree $\delta \geq 0$,

$$\gamma_2(G) \leq \frac{2 \ln(\delta + 1) + 1}{\delta + 1} n.$$

This upper bound was obtained in [12] using probabilistic method and it is a strong result when δ is really high. On the other hand, it gives an upper bound better than $0.5n$ only if $\delta(G) \geq 11$.

In this paper we present a method which can be used to improve the existing upper bounds when the minimum degree is in the “middle” range. Particularly, we show that if $\delta(G) \geq 6$ then $\gamma_2(G)$ is strictly smaller than $n/2$; $\delta(G) = 7$ implies $\gamma_2(G) < 0.467n$; $\delta(G) = 8$ implies $\gamma_2(G) < 0.441n$; and $\gamma_2(G) < 0.418n$ holds for every graph whose minimum degree is at least 9.

The paper is organized as follows. In Section 2, we state our main theorem and its corollaries which are the new upper bounds for specified minimum degrees. In Section 3 our main theorem is proved. Finally, we make some remarks on the algorithmic aspects of our results.

¹The Cartesian product $K_4 \square K_2$ is the graph of order 8 which consists of two copies of K_4 with a matching between them. Note that $\gamma_3(K_4 \square K_2)$ also equals 4.

2 Our results

To avoid the repetition of the analogous argumentations for different minimum degrees, we will state our theorem in a general form which is quite technical. Then, the upper bounds will follow as easy consequences. First, we introduce a set of conditions which will be referred to in our main theorem. We assume that $d \geq 4$ holds.

$$s > a \geq y_{d+1} \geq y_d \geq \cdots \geq y_0 \geq b_0 = 0 \quad (1)$$

$$0 \leq b_{d+1} - b_d \leq b_d - b_{d-1} \leq \cdots \leq b_2 - b_1 \leq b_1 \quad (2)$$

$$0 \leq y_{d+1} - b_{d+1} \leq y_d - b_d \leq \cdots \leq y_1 - b_1 \leq y_0 \quad (3)$$

$$y_{d+1} \leq a - \frac{s-a}{d+2} \quad (4)$$

$$y_d \leq a - \frac{s-a}{d+1} \quad (5)$$

$$b_{d+1} \leq a - \frac{s-a}{d+3} - \frac{s-a}{d+1} \quad (6)$$

$$b_d \leq a - \frac{s-a}{d+2} - \frac{s-a}{d+1} \quad (7)$$

$$b_{d-1} \leq a - 2 \cdot \frac{s-a}{d+1} \quad (8)$$

$$a + d(a - y_{d-1}) \geq s \quad (9)$$

$$a + d(y_{d+1} - b_d) \geq s \quad (10)$$

$$y_{d+1} + (d+1)(a - y_{d-2}) \geq s \quad (11)$$

$$y_{d+1} + (d+1)(y_{d+1} - b_d) \geq s \quad (12)$$

$$a + (d-1)(a - y_{d-2}) \geq s \quad (13)$$

$$a + (d-1)(y_d - b_{d-1}) \geq s \quad (14)$$

$$y_d + d(a - y_{d-3}) \geq s \quad (15)$$

$$y_d + d(y_d - b_{d-1}) \geq s \quad (16)$$

$$b_{d+1} + (d+1)(a - y_{d-2}) \geq s \quad (17)$$

$$b_{d+1} + (d+1)(y_{d-1} - b_{d-1}) \geq s \quad (18)$$

$$a + (d-1)(b_3 - b_2) + (y_2 - b_1) + (d-3)(b_3 - b_2) \geq s \quad (19)$$

$$y_2 + (d-3)(b_3 - b_2) + 2(y_2 - b_1 + (d-3)(b_3 - b_2)) \geq s \quad (20)$$

$$a + 0.5b_3 + (d-1.5)(b_3 - b_2) + (a - y_1) \geq s \quad (21)$$

$$a + 0.5b_3 + (d-1.5)(b_3 - b_2) + (y_1 - b_1 + (d-3)(b_3 - b_2)) \geq s \quad (22)$$

$$1.5a + 0.5b_3 - 0.5y_0 + (d-2)(b_2 - b_1) \geq s \quad (23)$$

$$a + 0.5b_3 + 0.5y_1 - 0.5b_1 + (d-2)(b_2 - b_1) + 0.5(d-3)(b_3 - b_2) \geq s \quad (24)$$

$$1.5a + b_3 + 0.5(d-3)(b_3 - b_1) + 0.5(d-2)(b_3 - b_2) \geq s \quad (25)$$

$$a + b_3 + 0.5y_1 - 0.5b_1 + 0.5(d-3)(b_3 - b_1) + 0.5(d-4)(b_3 - b_2) \geq s \quad (26)$$

$$b_3 + 3(a - y_0) \geq s \quad (27)$$

$$b_3 + 3(y_1 - b_1 + (d - 3)(b_3 - b_2)) \geq s \quad (28)$$

$$2y_1 + 2(d - 2)(b_2 - b_1) \geq s \quad (29)$$

$$a + 0,5b_2 + (d - 1,5)(b_2 - b_1) + 0,5(a - y_1) \geq s \quad (30)$$

$$a + 0,5b_2 + (d - 1,5)(b_2 - b_1) + 0,5(y_1 - b_1 + (d - 3)(b_2 - b_1)) \geq s \quad (31)$$

$$a + 0,5(d - 1)b_2 \geq s \quad (32)$$

$$b_2 + 2y_0 + 2(d - 2)(b_2 - b_1) \geq s \quad (33)$$

$$b_2 + y_0 + (d - 2)(b_2 - b_1) + (a - y_0) \geq s \quad (34)$$

$$y_0 + (d - 1)b_1 \geq s \quad (35)$$

For every $2 \leq i \leq d - 2$:

$$a + (d - i)(b_{i+2} - b_{i+1}) + i(a - y_{i-1}) \geq s \quad (36)$$

$$a + (d - i)(b_{i+2} - b_{i+1}) + i(y_{i+1} - b_i + (d - i - 2)(b_{i+2} - b_{i+1})) \geq s \quad (37)$$

$$y_{i+1} + (d - i - 2)(b_{i+2} - b_{i+1}) + (i + 1)(a - y_{i-2}) \geq s \quad (38)$$

$$y_{i+1} + (d - i - 2)(b_{i+2} - b_{i+1}) + (i + 1)(y_{i+1} - b_i + (d - i - 2)(b_{i+2} - b_{i+1})) \geq s \quad (39)$$

$$b_{i+2} + (i + 2)(a - y_{i-1}) \geq s \quad (40)$$

$$b_{i+2} + (i + 2)(y_i - b_i + (d - i - 2)(b_{i+2} - b_{i+1})) \geq s \quad (41)$$

Now we are in a position to state our main theorem. Its proof will be given in Section 3.

Theorem 1. Assume that G is a graph of order n and with minimum degree $\delta(G) = d \geq 6$. If $a, y_0, \dots, y_{d+1}, b_0, \dots, b_{d+1}$ are nonnegative numbers and s is a positive number such that conditions (1)–(35), and also for every $2 \leq i \leq d - 2$ the inequalities (36)–(41) are satisfied, then

$$\gamma_2(G) \leq \frac{a}{s} n.$$

If we fix an integer d , set $s = 1$, and want to minimize a under the conditions given in Theorem 1, we have a linear programming problem. The solution a^* of this LP-problem gives an upper bound on $\frac{\gamma_2(G)}{n}$ which holds for every graph with $\delta(G) \geq d$. In Table 1, we summarize these upper bounds for several values of d .

The following consequences for $d = 6, 7, 8, 9$ can be directly obtained by using the integer values given for the variables $s, a, y_0, \dots, y_{d+1}, b_0, \dots, b_{d+1}$ in Table 2. Substituting them into the conditions (1)–(41) of Theorem 1, one can check that all inequalities are satisfied. This yields the following upper bounds on the 2-domination number.

Corollary 1. Let G be a graph of order n .

(i) If $\delta(G) = 6$ then $\gamma_2(G) \leq \frac{456883}{918298} n < 0.498n$.

(ii) If $\delta(G) = 7$ then $\gamma_2(G) \leq \frac{140835095}{301690439} n < 0.467n$.

δ	6	7	8	9	10	11
Our result	0.49754	0.46682	0.44016	0.41702	0.39679	0.37957
Earlier best bound	0.5	0.5	0.5	0.5	0.5	0.49749
δ	12	13	14	15	16	17
Our result	0.36459	0.35117	0.33914	0.33385	0.33052	0.32762
Earlier best bound	0.47154	0.44844	0.42775	0.40908	0.39215	0.37671
δ	18	19	20	21	22	23
Our result	0.32505	0.32277	0.32074	0.31891	0.31726	0.31574
Earlier best bound	0.36258	0.34958	0.33758	0.32646	0.31613	0.30651
δ	24	25	26	27	30	40
Our result	0.31436	0.31309	0.31192	0.31084	0.30803	0.30178
Earlier best bound	0.29752	0.28909	0.28118	0.27373	0.25381	0.20555
δ	50	60	70	80	90	100
Our result	0.29806	0.29560	0.29385	0.29254	0.29152	0.29071
Earlier best bound	0.17380	0.15118	0.13416	0.12086	0.11013	0.10129

Table 1: Comparison of our results and earlier best upper bounds on $\frac{\gamma_2(G)}{n}$, if the minimum degree δ is fixed.

(iii) If $\delta(G) = 8$ then $\gamma_2(G) \leq \frac{292954593}{665571713} n < 0.441n$.

(iv) If $\delta(G) \geq 9$ then $\gamma_2(G) \leq \frac{60805963517}{145812382205} n < 0.418n$.

3 Proof of Theorem 1

To prove Theorem 1 we apply an algorithmic approach, where weights are assigned to the vertices and these weights change according to some rules during the greedy 2-domination procedure. A similar proof technique was introduced in [2], later it was used in [3, 4, 18] for obtaining upper bounds on the game domination number (see [1] for the definition) and in [15, 16] for proving bounds on the game total domination number [14]. Based on this approach we also obtained improvements for the upper bounds on the domination number [6], and in the conference paper [5] we presented a preliminary version of this algorithm to estimate the 2-domination number of graphs of minimum degree 8.

3.1 Selection procedure with changing weights

Throughout, we assume that a graph G is given with $\delta(G) \geq d \geq 6$. We will consider an algorithm in which the vertices of the 2-dominating set are selected one-by-one. A step in the algorithm means that one vertex is selected (or chosen) and put into the set D which was empty at the beginning of the process. Hence, after any step of the procedure, D denotes the set of vertices chosen up to this point. We make difference between the following four main types of vertices:

	$\delta = 6$	$\delta = 7$	$\delta = 8$	$\delta = 9$
a	502562162340	9858456650	215321625855	93641183816180
s	1010109434040	21118330730	489195209055	224551068595700
y_{10}	—	—	—	78747157548500
y_9	—	—	180637395519	78747157548500
y_8	—	8265018290	180637395519	78747157548500
y_7	422846061750	8265018290	180637395519	77277448218740
y_6	422846061750	8093880725	176196828255	75612599739380
y_5	409645123200	7981810970	170236790715	73000318746740
y_4	401052708000	7754608778	164408232975	69343125357044
y_3	387969820875	7321226150	153359038875	64634747985500
y_2	357968691360	6598921770	138571857655	57524154844772
y_1	296456709780	5196793700	105895928425	43483590947181
y_0	254021681340	4492799990	87943795415	33987088151324
b_{10}	—	—	—	64166766443780
b_9	—	—	146353194015	64166766443780
b_8	—	6656464850	146353194015	61811868322820
b_7	338254849800	6656464850	139847385195	59456970201860
b_6	338254849800	6286147490	133341576375	57102072080900
b_5	313665896880	5915830130	126835767555	54125243789540
b_4	289076943960	5545512770	118110911835	50365444145324
b_3	264487991040	5021360750	107061717735	45588781601132
b_2	226888474680	4278173340	89997559750	37861061453138
b_1	151217550540	2770921790	57321630520	23820497555547

Table 2: Weights assigned to the vertices for graphs of minimum degree $\delta = 6, 7, 8$ and 9 .

- A vertex v is *white*, if v is not dominated, that is if $|N[v] \cap D| = 0$.
- A vertex v is *yellow*, if $|N(v) \cap D| = 1$ and $v \notin D$.
- A vertex v is *blue*, if $|N(v) \cap D| \geq 2$ and $v \notin D$.
- A vertex v is *red*, if $v \in D$.

The sets of the white, yellow, blue and red vertices are denoted by W , Y , B and R , respectively. After any step of the algorithm, we consider the graph G together with the set D . Hence, the current colors of the vertices, that is the partition $V(G) = W \cup Y \cup B \cup R$, are also determined. The graph G together with a $D \subseteq V(G)$ will be called *colored graph* and denoted by G^D . We define the *WY-degree* of a vertex v in G^D to be $\deg_{WY}(v) = |N(v) \cap (W \cup Y)|$. The sets W , Y and B are partitioned according to the WY-degrees of the vertices. For every integer $i \geq 0$ and for $X = W, Y, B$, let $X_i = \{v \in X \mid \deg_{WY}(v) = i\}$. Since $R = D$, we may assume that red vertices are not selected in any steps of the procedure.

We distinguish between two types of colored graphs. G^D belongs to *Type 1* if $\max\{i \mid W_i \cup Y_{i+1} \neq \emptyset\} \geq d + 1$, otherwise G^D is of *Type 2*. Hence, a colored graph is of Type 2 if

and only if $\deg_{WY}(v) \leq d$ for every white vertex v and $\deg_{WY}(u) \leq d + 1$ for every yellow vertex u .

During the 2-domination algorithm, weights are assigned to the vertices. The weight $w(v)$ of vertex v is defined with respect to the current type of the colored graph and to the current color and WY-degree of v .

	$w(v)$ if G^D is of Type 1	$w(v)$ if G^D is of Type 2
$v \in W$	a	a
$v \in Y_i$	$a - \frac{s-a}{i+1}, \quad \text{if } i \geq d$ $a - \frac{s-a}{d+1}, \quad \text{if } i < d$	y_i
$v \in B_i$	$a - \frac{s-a}{i+2} - \frac{s-a}{i}, \quad \text{if } i > d$ $a - \frac{s-a}{d+2} - \frac{s-a}{d+1}, \quad \text{if } i = d$ $a - 2\frac{s-a}{d+1}, \quad \text{if } i < d$	$b_{d+1} \quad \text{if } i > d$ $b_i \quad \text{if } i \leq d$
$v \in R$	0	0

The weight of the colored graph G^D is just the sum of the weights assigned to its vertices. Formally, $w(G^D) = \sum_{v \in V(G)} w(v)$.

Assume that a vertex $v \in W \cup Y$ is selected from G^D in a step of our algorithm. Hence, v is recolored red in $G^{D \cup \{v\}}$. By definition, if a neighbor u of v belongs to W_i in G^D , then u is recolored yellow. Moreover, the WY-degree of u decreases by at least one, as its neighbor, v , was white or yellow and now it is recolored red. Similarly, if the neighbor u belongs to Y_i in G^D , then $u \in B_j$ for a $j \leq i - 1$ in $G^{D \cup \{v\}}$. In the other case, if a blue vertex v is selected, v is also recolored red. For any neighbor u of v , if $u \in W_i$ in G^D then $u \in Y_j$ with $j \leq i$ in $G^{D \cup \{v\}}$, and if $u \in Y_i$ in G^D then $u \in B_j$ with $j \leq i$ in $G^{D \cup \{v\}}$. No further vertices are recolored, but the WY-degree of vertices from $N[N(v)]$ might decrease.

Hence, assuming that the weights are nonnegative and inequalities (1)-(8) are satisfied, we can observe that the weight of the colored graph and that of any vertex does not increase in any step of the algorithm. By conditions (1), (2), (4)-(8), the weights y_i, b_i , used in a colored graph of Type 2, are not greater than the corresponding weights in a graph of Type 1. Thus, the following statement is also valid if G^D belongs to Type 1 while $G^{D \cup \{v\}}$ belongs to Type 2.

Lemma 2. *If the conditions (1)-(8) are satisfied, for any colored graph G^D and for any vertex $v \in V(G) \setminus D$, the inequality $w(G^D) \geq w(G^{D \cup \{v\}})$ holds. Moreover, no vertex u has greater weight in $G^{D \cup \{v\}}$ than in G^D .*

3.2 The s -property

For a positive number s , we will say that a colored graph G^D satisfies the s -property, if either D is a 2-dominating set of G or there exists a positive integer k and a set D^* of k vertices² such that

$$w(G^D) - w(G^{D \cup D^*}) \geq ks.$$

Assume that a 2-domination procedure is applied for a graph G which is of order n . At the beginning, we have weight a on every vertex and $w(G^\emptyset) = an$. At the end, when D is a 2-dominating set, all vertices are associated with weight 0, as they all are contained in $R \cup B_0$. Consequently, if we show that for every $D \subseteq V(G)$ the colored graph G^D satisfies the s -property, a 2-dominating set of cardinality at most an/s can be obtained, from which $\gamma_2(G) \leq \frac{a}{s} n$ follows.

Lemma 3. *Assume that G is a graph of order n and with a minimum degree of $\delta(G) = d \geq 6$. If $a, y_0, \dots, y_{d+1}, b_0, \dots, b_{d+1}$ are nonnegative numbers and s is a positive number such that conditions (1)–(35), and for every $2 \leq i \leq d-2$ the inequalities (36)–(41) are also satisfied, then for every $D \subseteq V(G)$, the colored graph G^D satisfies the s -property.*

Proof. We prove the lemma via a series of claims. Lemma 2 will be used in nearly all argumentations here (but in most of the cases we do not mention it explicitly). The only exception is Claim A, which immediately follows from the definition of s -property.

Claim A *If D is a 2-dominating set of G then G^D satisfies the s -property.*

Claim B *If G^D belongs to Type 1, it satisfies the s -property.*

Proof. Let $k = \max\{i \mid W_i \cup Y_{i+1} \neq \emptyset\}$. As G^D is of Type 1, $k \geq d+1$. We assume in the next argumentations that $G^{D \cup \{v\}}$ (or $G^{D \cup \{v'\}}$) also is of Type 1. If this is not the case, then, by conditions (1), (2), (4)–(8) and by the definition of the weight assignment, the decrease in $w(G^D)$ may be even larger than counted.

If $W_k \neq \emptyset$, select a vertex $v \in W_k$. Each white neighbor u of v is from a class W_i with $i \leq k$. After the selection of v , this neighbor u is recolored yellow and its WY-degree decreases by at least 1.³ Thus, the decrease in $w(u)$ is not smaller than

$$a - \left(a - \frac{s-a}{(k-1)+1} \right) = \frac{s-a}{k}.$$

On the other hand, each yellow neighbor u' of v is from a class $Y_{i'}$ with $i' \leq k+1$. After putting v into D , u' will be a blue vertex with a WY-degree of at most $i' - 1$. Hence, $w(u')$ is decreased by at least

$$\frac{s-a}{i'-1} \geq \frac{s-a}{k}.$$

²Note that in most of the cases we will prove that the s -property holds with $|D^*| = 1$. That is, we simply show that there exists a vertex v such that the choice of v decreases $w(G^D)$ by at least s .

³It might happen that the decrease is larger than 1. For example, if we have a complete graph K_n ($n \geq 3$) with one white vertex and $n-1$ yellow vertices, and select the white vertex.

Since v has k neighbors from $W \cup Y$ in G^D , and the selection of v results in a decrease of a in the weight of v , we have

$$w(G^D) - w(G^{D \cup \{v\}}) \geq a + k \frac{s-a}{k} = s.$$

This shows that the colored graph G^D with $W_k \neq \emptyset$ satisfies the s -property.

Now, assume that $W_k = \emptyset$. This implies $Y_{k+1} \neq \emptyset$ and we can select a vertex $v' \in Y_{k+1}$ in the next step of the procedure. As v' becomes red, its weight decreases by $a - \frac{s-a}{k+2}$. Each white neighbor u of v' has a WY-degree of at most $k-1$. Hence, when u is recolored yellow and loses at least one yellow neighbor, namely v' , $w(u)$ decreases by at least

$$\frac{s-a}{(k-2)+1} > \frac{s-a}{k}.$$

On the other hand, if u' is a yellow neighbor of v' , we have the same situation as before, when a white vertex v was put into the set D . That is, the decrease in $w(u')$ is at least $\frac{s-a}{k}$. These imply

$$w(G^D) - w(G^{D \cup \{v'\}}) \geq a - \frac{s-a}{k+2} + (k+1) \frac{s-a}{k} > s$$

and again, G^D satisfies the s -property. (\square)

From now on, we consider colored graphs of Type 2. Note that the inequalities

$$0 \leq y_{d+1} - b_d \leq y_d - b_{d-1} \leq \dots \leq y_2 - b_1 \leq y_1, \quad (*)$$

easily follow from conditions (2) and (3). Hence, if a vertex v is moved from Y_i into B_{i-1} in a step of the procedure, and $i \leq j$ is assumed, the decrease in $w(v)$ is at least $y_j - b_{j-1}$. Inequalities (1), (2) and (3) ensure similar estimations if v is moved from W into Y_i , from Y_i into B_i , or from B_i into B_{i-1} , and $i \leq j$ is assumed.

Claim C *If G^D is a colored graph with $d-1 \leq \max\{i \mid W_i \cup Y_{i+1} \neq \emptyset\} \leq d$, it satisfies the s -property.*

Proof. Our condition in Claim C implies that each white vertex has a WY-degree of at most d and each yellow vertex has a WY-degree of at most $d+1$. In particular, G^D is of Type 2. In the proof we consider four cases.

First, assume that $W_d \neq \emptyset$ and choose a vertex $v \in W_d$. When v is put into D , it is recolored red and $w(v)$ decreases by a . Any white neighbor u of v is recolored yellow and $\deg_{WY}(u)$ decreases by at least 1. Together with condition (1), this implies that $w(u)$ decreases by at least $a - y_{d-1}$. A yellow neighbor u' of v is recolored blue and $\deg_{WY}(u')$, decreases by at least 1. By (*), the weight $w(u')$ is lowered by at least $y_{d+1} - b_d$. By conditions (9) and (10), $a - y_{d-1} \geq (s-a)/d$ and $y_{d+1} - b_d \geq (s-a)/d$. Hence, we obtain

$$w(G^D) - w(G^{D \cup \{v\}}) \geq a + d \frac{s-a}{d} = s,$$

and G^D satisfies the s -property.

Second, assume that $W_d = \emptyset$, but there exists a vertex $v \in Y_{d+1}$. Let us select v in the next step of the algorithm. Then, v is recolored red and $w(v)$ decreases by y_{d+1} . Each white neighbor u of v has a WY-degree of at most $d - 1$ in G^D , and the weight $w(u)$ decreases by at least $a - y_{d-2}$. Similarly, if u' is a yellow neighbor of v , the decrease in $w(u')$ is not smaller than $y_{d+1} - b_d$. These facts together with conditions (11) and (12) imply

$$w(G^D) - w(G^{D \cup \{v\}}) \geq y_{d+1} + (d+1) \frac{s - y_{d+1}}{d+1} = s,$$

which proves that G^D has the s -property.

In the third case, $W_d \cup Y_{d+1} = \emptyset$, but there exists a white vertex v with $\deg_{WY}(v) = d - 1$. Similarly to the previous cases, but referring to conditions (13)–(14), one can show that

$$w(G^D) - w(G^{D \cup \{v\}}) \geq a + (d-1) \frac{s - a}{d-1} = s.$$

In the last case, we assume that for each white vertex $\deg_{WY} \leq d - 2$, for each yellow vertex $\deg_{WY} \leq d$, and also that we may select a vertex $v \in Y_d$. By (15) and (16), we obtain

$$w(G^D) - w(G^{D \cup \{v\}}) \geq y_d + d \frac{s - y_d}{d} = s.$$

This completes the proof of Claim C. \square

Claim D *If G^D is a colored graph with $\max\{i \mid W_i \cup Y_{i-1} \neq \emptyset\} \leq d - 2$, and there exists a blue vertex v with $\deg_{WY}(v) \geq d + 1$, then G^D satisfies the s -property.*

Proof. Assume that v is selected in the next step of the 2-domination procedure. Then, v is recolored red and $w(v)$ is lowered by b_{d+1} . Each white neighbor has a WY-degree of at most $d - 2$ and becomes yellow, while each yellow neighbor of v has a WY-degree of at most $d - 1$ and becomes blue. By conditions (1) and (2), the decrease in the weight of a white or in that of a yellow neighbor is at least $a - y_{d-2}$ or $y_{d-1} - b_{d-1}$, respectively. Conditions (17) and (18) imply $w(G^D) - w(G^{D \cup \{v\}}) \geq s$. \square

In the next proofs, we will use the following facts. A white vertex does not have any red neighbors and every yellow vertex has exactly one red neighbor. Hence, under the condition $\delta(G) \geq d$, each white vertex $v \in W_x$ has at least $d - x$ blue neighbors, and each $v' \in Y_y$ has at least $d - y - 1$ blue neighbors. Moreover, when this white or yellow vertex is recolored red or blue, the WY-degrees of its $d - x$ or $d - y - 1$ blue neighbors are decreased. More precisely, if a vertex v is chosen in a step of the algorithm and v is white, the sum of the WY-degrees of vertices which are blue in G^D is decreased by at least

$$d - \deg_{WY}(v) + \sum_{w \in Y \cap N(v)} (d - 1 - \deg_{WY}(w)).$$

Similarly, if $v \in Y \cup B$, this decrease is at least

$$d - \deg_{WY}(v) - 1 + \sum_{w \in Y \cap N(v)} (d - 1 - \deg_{WY}(w))$$

if v is yellow, and at least

$$\deg_{WY}(v) + \sum_{w \in Y \cap N(v)} (d - 2 - \deg_{WY}(w))$$

if v is blue. Now, let us assume that for every blue vertex $\deg_{WY}(u) \leq j$ and for a set $B' \subseteq B$ the sum $\sum_{u \in B'} \deg_{WY}(u)$ decreases by z . Then, by (2), $\sum_{u \in B'} w(u)$ decreases by at least $z(b_j - b_{j-1})$. This remains valid, if for a vertex $u \in B'$, $\deg_{WY}(u)$ is reduced by more than 1.

Claim E *If G^D is a colored graph with $d - 2 \geq \max\{i \mid W_i \cup Y_{i+1} \cup B_{i+2} \neq \emptyset\} \geq 2$, it satisfies the s -property.*

Proof. Let $k = \max\{i \mid W_i \cup Y_{i+1} \cup B_{i+2} \neq \emptyset\}$. This implies $\deg_{WY}(v) \leq k$ for every white vertex, $\deg_{WY}(v) \leq k + 1$ for every yellow vertex, and $\deg_{WY}(v) \leq k + 2$ for every blue vertex. We consider three cases.

If there exists a white vertex v of $\deg_{WY}(v) = k$, assume that v is selected in the next step. Then, $w(v)$ decreases by a . Further, since v is recolored red, the sum of the WY-degrees of its blue neighbors decreases by at least $(d - k)$. This results in a further change of at least $(d - k)(b_{k+2} - b_{k+1})$ in $w(G^D)$. If $u \in W_j$ ($j \leq k$) is a white neighbor of v , in $G^{D \cup \{v\}}$ u is recolored yellow and has a WY-degree of at most $j - 1$. Hence, the decrease in $w(u)$ is at least

$$a - y_{k-1} \geq \frac{s - a - (d - k)(b_{k+2} - b_{k+1})}{k},$$

where the last inequality follows from (36) substituting $i = k$. Consider now a yellow neighbor u' of v . After the choice of v , u' is recolored blue and $w(u')$ decreases by at least $y_{k+1} - b_k$. Taking into account the decreases in the weights of vertices from $N(u') \cap B$, the recoloring of each such u' contributes to the decrease of $w(G^D)$ with at least

$$y_{k+1} - b_k + (d - (k + 1) - 1)(b_{k+2} - b_{k+1}) \geq \frac{s - a - (d - k)(b_{k+2} - b_{k+1})}{k},$$

where the lower bound follows from (37) substituting $i = k$. Therefore, if $v \in W_k$,

$$w(G^D) - w(G^{D \cup \{v\}}) \geq a + (d - k)(b_{k+2} - b_{k+1}) + k \frac{s - a - (d - k)(b_{k+2} - b_{k+1})}{k} = s.$$

Consequently, G^D has the s -property if $W_k \neq \emptyset$.

In the following two cases, we count $w(G^D) - w(G^{D \cup \{v\}})$ in a similar way. Assume that $W_k = \emptyset$ but $Y_{k+1} \neq \emptyset$, and choose a vertex v from Y_{k+1} . Vertex v is recolored red and the WY-degrees of its blue neighbors decrease. This contributes to the difference $w(G^D) -$

$w(G^{D \cup \{v\}})$ with at least $y_{k+1} + (d - k - 2)(b_{k+2} - b_{k+1})$. Further, if u is a white neighbor of v then $\deg_{WY}(u) \leq k - 1$ in G^D . Once v is recolored red, $w(u)$ decreases by at least $a - y_{k-2}$. By condition (38), it is not smaller than $(s - y_{k+1} - (d - k - 2)(b_{k+2} - b_{k+1})) / (k + 1)$. If u' is a yellow neighbor of v , then u' will be blue in $G^{D \cup \{v\}}$ and the WY-degrees in $B \cap N(u')$ are decreased. Consequently, and also referring to (39), each yellow neighbor u' contributes to the decrease of $w(G^D)$ with at least

$$y_{k+1} - b_k + (d - k - 2)(b_{k+2} - b_{k+1}) \geq \frac{s - y_{k+1} - (d - k - 2)(b_{k+2} - b_{k+1})}{k + 1}.$$

In total, v has $k + 1$ neighbors from $W \cup Y$, and we have

$$w(G^D) - w(G^{D \cup \{v\}}) \geq y_{k+1} + (d - k - 2)(b_{k+2} - b_{k+1}) + (k + 1) \frac{s - y_{k+1} - (d - k - 2)(b_{k+2} - b_{k+1})}{k + 1} = s,$$

which proves that G^D satisfies the s -property.

In the third case, $W_k \cup Y_{k+1} = \emptyset$ and we have a blue vertex v with $\deg_{WY}(v) = k + 2$. Selecting v in the next step of the procedure, v will be recolored red and $w(v)$ becomes 0. Each white neighbor u of v is recolored yellow and has a decrease of at least $a - y_{k-1}$ in $w(u)$ (in this case, $\deg_{WY}(u)$ might be unchanged). Moreover, each yellow neighbor u' of v is recolored blue and the weights of the vertices from $N(u') \cap B$ are also decreased. Then, the recoloring of u' contributes to the decrease of $w(G^D)$ by at least

$$y_k - b_k + (d - k - 2)(b_{k+2} - b_{k+1}) \geq \frac{s - b_{k+2}}{k + 2},$$

where the inequality follows from (40). On the other hand, by (41), we have $a - y_{k-1} \geq (s - b_{k+2}) / (k + 2)$. We may conclude that

$$w(G^D) - w(G^{D \cup \{v\}}) \geq b_{k+2} + (k + 2) \frac{s - b_{k+2}}{k + 2} = s.$$

Thus, in the third case G^D also satisfies the s -property. \square

Claim F *Let G^D be a colored graph with $\max\{i \mid W_i \cup Y_{i+1} \cup B_{i+2} \neq \emptyset\} = 1$ such that there exists an edge between W and Y . Then, G^D satisfies the s -property.*

Proof. Choose a white vertex v whose only neighbor from $W \cup Y$ is a yellow vertex u in G^D . By our condition, $\deg_{WY}(u) \leq 2$. In $G^{D \cup \{v\}}$, the vertex v is recolored red and $u \in B_1$. Moreover, in G^D , v and u has at least $d - 1$ and $d - 3$ blue neighbors, respectively. By condition (19),

$$w(G^D) - w(G^{D \cup \{v\}}) \geq a + (d - 1)(b_3 - b_2) + (y_2 - b_1) + (d - 3)(b_3 - b_2) \geq s,$$

and G^D has the s -property. \square

Henceforth, we may assume that there are no edges between W and Y .

Claim G If G^D is a colored graph with $\max\{i \mid W_i \cup Y_{i+1} \cup B_{i+2} \neq \emptyset\} = 1$ and $Y_2 \neq \emptyset$, then G^D has the s -property.

Proof. Consider a vertex $v \in Y_2$ in G^D . As supposed, it has no white neighbors. Hence, v is adjacent to two vertices, say u_1 and u_2 , which are from $Y_2 \cup Y_1$. Then, in $G^{D \cup \{v\}}$, v is recolored red, u_1 and u_2 are recolored blue and belong to $B_1 \cup B_0$. The decrease in $\sum_{w \in B \cap (N(v) \cup N(u_1) \cup N(u_2))} \deg_{WY}(w)$ is at least $3(d-3)$. Then, also using (20),

$$w(G^D) - w(G^{D \cup \{v\}}) \geq y_2 + 2(y_2 - b_1) + 3(d-3)(b_3 - b_2) \geq s.$$

This proves the claim. \square

Claim H If G^D is a colored graph with $\max\{i \mid W_i \cup Y_i \cup B_{i+2} \neq \emptyset\} = 1$, it satisfies the s -property.

Proof. Suppose for a contradiction that there exists a colored graph G^D which satisfies the condition of our claim but does not have the s -property. First, let us assume $W_1 \neq \emptyset$ and recall that each white vertex with $\deg_{WY}(v) = 1$ has a white neighbor of the same type. We consider the following cases:

- (i) If there exists a vertex $v_1 \in W_1$ with a white neighbor v_2 , and with a blue neighbor u from B_3 such that u is not adjacent to v_2 , we assume that in two consecutive steps of the procedure v_2 and u are chosen. Then, v_2 and u are recolored red, and v_1 becomes blue with a WY-degree of 0. This contributes to the decrease of $w(G^D)$ with $2a + b_3$. The total weight of the further blue neighbors of v_1 and v_2 decreases by at least $((d-2) + (d-1))(b_3 - b_2)$. If u has a white neighbor w in G^D , w becomes yellow and contributes to the decrease of $w(G^D)$ with at least $a - y_1$. By (21), it is not smaller than $(2s - 2a - b_3 - (2d-3)(b_3 - b_2))/2$. If w' is a yellow neighbor of u in G^D , then it is recolored blue and $\deg_{WY}(w')$ is either 1 or 0 in $G^{D \cup \{v_2, u\}}$. Further, the weights of the at least $d-3$ blue neighbors of w' which are different from u are also decreased. In total, w' contributes to the decrease of $w(G^D)$ with at least

$$y_1 - b_1 + (d-3)(b_3 - b_2) \geq \frac{2s - 2a - b_3 - (2d-3)(b_3 - b_2)}{2},$$

where the last inequality is equivalent to (22). Therefore, we have

$$w(G^D) - w(G^{D \cup \{v_2, u\}}) \geq 2a + b_3 + (2d-3)(b_3 - b_2) + 2 \frac{2s - 2a - b_3 - (2d-3)(b_3 - b_2)}{2} = 2s,$$

and the s -property would be satisfied by G^D . This contradicts our assumption.

- (ii) Since G^D is supposed to be a counterexample, if a blue vertex $u \in B_3$ is adjacent to a white vertex then it is also adjacent to the white neighbor of it. If we have two adjacent white vertices v_1 and v_2 which have only one (common) neighbor u from B_3 , choose v_1 and u in the next two steps of the procedure. Then, v_1 and u are recolored

red, while v_2 is recolored blue and has a WY-degree of 0. Their weights are decreased by $2a + b_3$. All the further blue neighbors of v_1 and v_2 belong to $B_2 \cup B_1$ in G^D . The WY-degrees of these blue vertices are reduced, which contributes to the difference $w(G^D) - w(G^{D \cup \{v_1, u\}})$ with at least $2(d-2)(b_2 - b_1)$. The blue vertex u has one white or yellow neighbor w which is different from v_1 and v_2 . If w is white, it is from W_0 , as otherwise w , its white neighbor, and u would satisfy the assumption in case (i). Hence, when w is recolored yellow, $w(w)$ decreases by $a - y_0$, and

$$w(G^D) - w(G^{D \cup \{v_1, u\}}) \geq 2a + b_3 + 2(d-2)(b_2 - b_1) + a - y_0,$$

which is at least $2s$ by condition (23). If w is yellow then $w \in Y_1 \cup Y_0$. When w is recolored blue, the WY-degrees of its blue neighbors are also reduced. These contribute to the difference $w(G^D) - w(G^{D \cup \{v_1, u\}})$ with at least $y_1 - b_1 + (d-3)(b_3 - b_2)$. Therefore, referring to (24),

$$w(G^D) - w(G^{D \cup \{v_1, u\}}) \geq 2a + b_3 + 2(d-2)(b_2 - b_1) + y_1 - b_1 + (d-3)(b_3 - b_2) \geq 2s.$$

We infer that in the counterexample G^D we cannot have a white vertex in W_1 that has exactly one neighbor from B_3 .

(iii) Now assume that $v_1, v_2 \in W_1$ and their neighbors u_1 and u_2 are from B_3 in G^D . Choose u_1 and u_2 and consider $G^{D \cup \{u_1, u_2\}}$. Here, v_1 and v_2 are blue vertices of WY-degree 0, while u_1 and u_2 are red. In G^D , each blue neighbor of v_1 and v_2 which is different from u_1 and u_2 is either from B_2 or it is a further common neighbor of v_1 and v_2 from B_3 . In the worst case, the decrease in their weights contributes to $w(G^D) - w(G^{D \cup \{u_1, u_2\}})$ with $(d-3)(b_3 - b_1)$. Finally, u_1 and u_2 have neighbors from $W_0 \cup Y_1 \cup Y_0$. It is enough to consider the following cases.

– u_1 and u_2 have a common neighbor $w \in W_0$. Then, w is recolored blue. The weight of w and that of its blue neighbors (different from u_1 and u_2) decrease by at least $a + (d-2)(b_3 - b_2)$. Then, by (25) and by our earlier observations

$$w(G^D) - w(G^{D \cup \{u_1, u_2\}}) \geq 2a + 2b_3 + (d-3)(b_3 - b_1) + a + (d-2)(b_3 - b_2) \geq 2s.$$

Hence, in a counterexample we cannot have this case.

– u_1 and u_2 have a common neighbor $w \in Y_1$. Then, w is recolored blue and moved to B_1 in $G^{D \cup \{u_1, u_2\}}$. Also, the weights of its blue neighbors decrease. These contribute to the difference $w(G^D) - w(G^{D \cup \{u_1, u_2\}})$ with at least $y_1 - b_1 + (d-4)(b_3 - b_2)$, and we have

$$w(G^D) - w(G^{D \cup \{u_1, u_2\}}) \geq 2a + 2b_3 + (d-3)(b_3 - b_1) + y_1 - b_1 + (d-4)(b_3 - b_2) \geq 2s,$$

where the last inequality follows from (26). Again, this case is not possible in a counterexample.

– u_1 and u_2 have two different neighbors, namely w_1 and w_2 , from W_0 . Then, w_1 and w_2 are recolored yellow and we have

$$\begin{aligned} w(G^D) - w(G^{D \cup \{u_1, u_2\}}) &\geq 2a + 2b_3 + (d-3)(b_3 - b_1) + 2(a - y_0) \\ &\geq 2a + b_3 + 3(b_3 - b_2) + 2(d-3)(b_3 - b_2) + 2(a - y_1) \geq 2s. \end{aligned}$$

Here, we used (21) and the inequalities $b_3 \geq 3(b_3 - b_2)$ and $b_3 - b_1 \geq 2(b_3 - b_2)$ which follow from (2).

We have shown that there are no edges between W_1 and B_3 if G^D is a counterexample to Claim F. In what follows we prove that $B_3 = \emptyset$ and $Y_1 = \emptyset$.

Suppose that $B_3 \neq \emptyset$ and choose a vertex v from B_3 . As it has been shown, all white and yellow neighbors of v belong to $W_0 \cup Y_1 \cup Y_0$. If u is a white neighbor, $w(u)$ decreases by $a - y_0$, and if u' is yellow, its recoloring contributes to the decrease of G^D by at least $y_1 - b_1 + (d - 3)(b_3 - b_2)$. By conditions (27) and (28),

$$w(G^D) - w(G^{D \cup \{v\}}) \geq b_3 + 3 \frac{s - b_3}{3} = s.$$

Hence, in the counterexample each blue vertex is of a WY-degree of at most 2.

Suppose now that $Y_1 \neq \emptyset$ and choose a vertex v from it. Since v cannot have a neighbor from W , it must have a neighbor u from Y_1 . In $G^{D \cup \{v\}}$, v is recolored red, u is recolored blue with a WY-degree 0, and each of their at least $2(d - 2)$ blue neighbors has a decrease of at least $b_2 - b_1$ in its weight. Hence, we have

$$w(G^D) - w(G^{D \cup \{v\}}) \geq 2y_1 + 2(d - 2)(b_2 - b_1),$$

which is at least s by (29). We may conclude that $Y_1 = \emptyset$ holds in our counterexample.

Assume that W_1 is not empty. Then, W_1 consists of pairs of adjacent vertices, we refer to which as “white pairs”.

First, suppose that there exists a white pair v_1, v_2 and a vertex $u \in B_2$ such that u is adjacent to v_1 and nonadjacent to v_2 . In the next two steps of the procedure we choose v_2 and u . Then, v_2 and u are recolored red, v_1 becomes a blue vertex of WY-degree 0. The WY-degrees of blue neighbors of v_1 and v_2 are also reduced. In total, these result in a decrease of at least $2a + b_2 + (2d - 3)(b_2 - b_1)$ in $w(G^D)$. Moreover, u has a white or a yellow neighbor w different from v_1 . For the cases $w \in W_1$ and $w \in Y_1 \cup Y_0$ we have the following inequalities by (30) and (31), respectively.

$$w(G^D) - w(G^{D \cup \{v_2, u\}}) \geq 2a + b_2 + (2d - 3)(b_2 - b_1) + (a - y_1) \geq 2s$$

$$w(G^D) - w(G^{D \cup \{v_2, u\}}) \geq 2a + b_2 + (2d - 3)(b_2 - b_1) + (y_1 - b_1 + (d - 3)(b_2 - b_1)) \geq 2s$$

We may infer that G^D has the s -property, which is a contradiction. Hence, if a blue vertex from B_2 is adjacent to a vertex from W_1 , then it is also adjacent to the other vertex from that white pair.

Now, consider any white pair v_1, v_2 and choose these two vertices in two consecutive steps of the procedure. As a result, v_1 and v_2 are recolored red and all their blue neighbors are of WY-degree 0. Since $b_2 - b_1 \leq b_1 - b_0 = b_1$, the worst case is when v_1 and v_2 share $d - 1$ blue neighbors from B_2 in G^D . By (32), we have

$$w(G^D) - w(G^{D \cup \{v_1, v_2\}}) \geq 2a + (d - 1)b_2 \geq 2s,$$

contradicting our assumption that G^D is counterexample.

Consequently, if $\max\{i \mid W_i \cup Y_i \cup B_{i+2} \neq \emptyset\} = 1$ then G^D has the s -property, as stated in Claim H. \square

What remains to consider after Claims A-H is the case when D is not a 2-dominating set that is $W \cup Y \neq \emptyset$ but all white and yellow vertices are of WY-degree 0 and all blue vertices have a WY-degree of at most 2.

First, suppose that we have an edge between B_2 and Y_0 . Then, choose a blue vertex $v \in B_2$ which has a yellow neighbor u . Vertex v has a further neighbor u' from $W_0 \cup Y_0$. Depending on the color of u' , we can use either (33) or (34) and obtain the following inequalities. If u' is yellow,

$$w(G^D) - w(G^{D \cup \{v\}}) \geq b_2 + 2y_0 + 2(d-2)(b_2 - b_1) \geq s.$$

If u' is white

$$w(G^D) - w(G^{D \cup \{v\}}) \geq b_2 + y_0 + (d-2)(b_2 - b_1) + a - y_0 \geq s.$$

Thus, in these cases G^D has the s -property.

Now assume that $Y_0 \neq \emptyset$ and choose a vertex v from Y_0 . We have just shown that v has no neighbors from B_2 . Hence, v has at least $d-1$ blue neighbors from B_1 . Together with (35), these imply

$$w(G^D) - w(G^{D \cup \{v\}}) \geq y_0 + (d-1)b_1 \geq s,$$

and G^D has the s -property.

Finally, we assume that $Y = \emptyset$, but we have x vertices in W_0 , z_2 vertices in B_2 and z_1 vertices in B_1 . Thus, $w(G^D) = xa + z_2b_2 + z_1b_1$. On the other hand, counting the number of edges between W_0 and $B_2 \cup B_1$ in two different ways, $dx \leq 2z_2 + z_1$. Consider $G^{D \cup Y_0}$, that is assume that in x consecutive steps we select all white vertices. Clearly, in $G^{D \cup Y_0}$ every vertex has a weight of 0. Hence,

$$\begin{aligned} w(G^D) - w(G^{D \cup Y_0}) &= xa + z_2b_2 + z_1b_1 \geq xa + (2z_2 + z_1) \min \left\{ \frac{b_2}{2}, b_1 \right\} \\ &\geq xa + dx \frac{b_2}{2} \geq xs. \end{aligned}$$

The last inequality is a consequence of (32), and $b_2/2 \leq b_1$ follows from $b_2 - b_1 \leq b_1$.

The cases discussed in our proof together cover all possibilities, hence every colored graph G^D satisfies the s -property under the conditions of Lemma 3. \square

As we discussed it at the beginning of this section, Theorem 1 is an immediate consequence of Lemma 3.

4 Concluding remarks

Finally, we make some remarks on the algorithmic aspects of our proof. In Table 1, we compared the upper bounds obtained by our Theorem 1 and those proved in [12] with probabilistic method. Our upper bounds on $\gamma_2(G)$ improve the earlier best results if the minimum degree δ is between 6 and 21. Nevertheless the algorithm, which is behind our proof, can also be useful for $\delta \geq 22$, as we can guarantee the determination of a 2-dominating set of bounded size for each input graph.

We can identify two different algorithms based on the proof in Section 3. For the first version, we do not need to count the weights assigned to the vertices. We just consider the list of instructions below and in each step of the algorithm we follow the first one which is applicable.

1. If $k = \max\{i \mid W_i \cup Y_{i+1} \neq \emptyset\} \geq d - 1$ and $W_k \neq \emptyset$, choose a vertex from W_k .
2. If $k = \max\{i \mid W_i \cup Y_{i+1} \neq \emptyset\} \geq d - 1$, choose a vertex from Y_{k+1} .
3. If $k = \max\{i \mid B_i \neq \emptyset\} \geq d + 1$, choose a vertex from B_k .
4. If $2 \leq k = \max\{i \mid W_i \cup Y_{i+1} \cup B_{i+2} \neq \emptyset\} \leq d - 2$ and $W_k \neq \emptyset$, choose a vertex from W_k .
5. If $2 \leq k = \max\{i \mid W_i \cup Y_{i+1} \cup B_{i+2} \neq \emptyset\} \leq d - 2$ and $Y_{k+1} \neq \emptyset$, choose a vertex from Y_{k+1} .
6. If $2 \leq k = \max\{i \mid W_i \cup Y_{i+1} \cup B_{i+2} \neq \emptyset\} \leq d - 2$, choose a vertex from B_{k+2} .
7. If there exists a white vertex v with a yellow neighbor, choose v .
8. If $Y_2 \neq \emptyset$, choose a vertex from it.
9. If there exist two adjacent white vertices v_1 and v_2 such that v_1 has a neighbor u from B_3 which is not adjacent to v_2 , choose v_2 and u .
10. If there exists a vertex v in W_1 , which has exactly one neighbor, say u , in B_3 , choose v and u .
11. If there exists a vertex v in W_1 , which has at least two neighbors in B_3 , choose two vertices from $N(v) \cap B_3$.
12. If $B_3 \neq \emptyset$, choose a vertex from it.
13. If $Y_1 \neq \emptyset$, choose a vertex from it.
14. If there exist two adjacent white vertices v_1 and v_2 such that v_1 has a neighbor u from B_2 which is not adjacent to v_2 , choose v_2 and u .
15. If there exist two adjacent white vertices, choose such two vertices.

16. If there exists a blue vertex $v \in B_2$ which has at least one yellow neighbor, choose v .
17. If $Y \neq \emptyset$, choose a yellow vertex.
18. Choose all the white vertices.

By a slightly different interpretation, we can define a 2-domination algorithm based on the weight assignment introduced in Section 3. Then, in each step, we choose a vertex v such that the decrease $w(G^D) - w(G^{D \cup \{v\}})$ is the possible largest. The exceptions are those steps where G^D would be treated by instructions 9, 10, 11, 14, 15 or 18 of the previous algorithm. In these cases, the greedy choice concerns the maximum decrease of $w(G^D)$ in two (or more) consecutive steps.

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