

# Best possible bounds concerning the set-wise union of families

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## Abstract

For two families of sets  $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$  we define their set-wise union,  $\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\}$  and establish several – hopefully useful – inequalities concerning  $|\mathcal{F} \vee \mathcal{G}|$ . Some applications are provided as well.

## 1 Introduction

For a non-negative integer  $n$  let  $[n] = \{1, \dots, n\}$  be the standard  $n$ -element set and  $2^{[n]}$  its power set. A subset  $\mathcal{F} \subset 2^{[n]}$  is called a *family*. If  $G \subset F \in \mathcal{F}$  implies  $G \in \mathcal{F}$  for all  $G, F \subset [n]$  then  $\mathcal{F}$  is called a *complex* (*down-set*). Let  $F^c$  denote the complement,  $[n] \setminus F$  of  $F$ . Also let  $\mathcal{F}^c = \{F^c : F \in \mathcal{F}\}$  be the *complementary* family. One of the earliest and no doubt the easiest result in extremal set theory, contained in the seminal paper of Erdős, Ko and Rado can be formulated as follows.

**Theorem 0** ([EKR]). *Suppose that there are no  $F, G \in \mathcal{F}$  satisfying  $F \cup G = [n]$ . Then*

$$(1) \quad 2 \cdot |\mathcal{F}| \leq 2^n.$$

*Proof.* Just note that the condition implies  $\mathcal{F} \cap \mathcal{F}^c = \emptyset$ . □

This simple result was the starting point of a lot of research.

**Definition 1.** For a positive integer  $t$  let us say that  $\mathcal{F} \subset 2^{[n]}$  is *t-union* if  $|F \cup G| \leq n - t$  for all  $F, G \in \mathcal{F}$ .

An important result of Katona [Ka] was the determination of the maximum size of  $t$ -union families.

In the present paper we mostly deal with problems concerning several families.

**Definition 2.** For positive integers  $t$  and  $r$ ,  $r \geq 2$  and non-empty families  $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$ , we say that they are *cross  $t$ -union* if  $|F_1 \cup \dots \cup F_r| \leq n - t$  for all  $F_1 \in \mathcal{F}_1, \dots, F_r \in \mathcal{F}_r$ .

**Definition 3.** For families  $\mathcal{F}, \mathcal{G}$  let  $\mathcal{F} \vee \mathcal{G}$  denote their *set-wise union*,

$$\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\}.$$

To state our main results we need one more definition. A family  $\mathcal{F} \subset 2^{[n]}$  is said to be *covering* if  $\{i\} \in \mathcal{F}$  for all  $i \in [n]$ . If  $\mathcal{F}$  is a complex, it is equivalent to saying that  $\bigcup_{F \in \mathcal{F}} F = [n]$ .

Let us use the term *cross-union* for cross 1-union.

**Theorem 1.** Suppose that  $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$  are cross-union and covering complexes. Then

$$(2) \quad |\mathcal{F} \vee \mathcal{G}| \geq \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|).$$

**Example 1.** Let  $n \geq 3$  and define  $\mathcal{A} = \{A \subset [n] : |A \cap [3]| \leq 1\}$ . Then  $|\mathcal{A}| = 2^{n-1}$  and  $|\mathcal{A} \vee \mathcal{A}| = \frac{7}{8}2^n$  hold.

The above example shows that (2) is best possible.

**Theorem 2.** Suppose that  $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$  are non-empty cross-union complexes and  $\mathcal{F}$  is covering. Then

$$(3) \quad |\mathcal{F} \vee \mathcal{G}| \geq \frac{3}{4}(|\mathcal{F}| + |\mathcal{G}|).$$

The bound (3) is best possible as shown by the next example.

**Example 2.** Let  $n \geq 2$  and define  $\mathcal{A} = \{A \subset [n] : |A \cap [2]| \leq 1\}$ ,  $\mathcal{B} = \{B \subset [n] : B \cap [2] = \emptyset\}$ .

**Theorem 3.** Suppose that  $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$  are cross 2-union and covering complexes. Then

$$(4) \quad |\mathcal{F} \vee \mathcal{G}| > |\mathcal{F}| + |\mathcal{G}|.$$

## 2 The proof of Theorems 1 and 2

Let us first note that if  $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$  are cross-union then

$$(2.1) \quad |\mathcal{F}| + |\mathcal{G}| \leq 2^n.$$

Indeed the cross-union property guarantees  $\mathcal{F} \cap \mathcal{G}^c = \emptyset$  and thereby  $|\mathcal{F}| + |\mathcal{G}| = |\mathcal{F}| + |\mathcal{G}^c| \leq |2^{[n]}| = 2^n$ .

In view of (2.1) the following statement easily implies Theorem 1.

**Theorem 2.1.** *Let  $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$  be covering complexes. Then*

$$(2.2) \quad |\mathcal{F} \vee \mathcal{G}| \geq \min \left\{ 2^n, \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|) \right\}.$$

*Proof.* First we consider the case that  $\mathcal{F}, \mathcal{G}$  are *not* cross-union. It is easy. If  $\mathcal{F}$  and  $\mathcal{G}$  are not cross-union then there exist  $F \in \mathcal{F}$ ,  $G \in \mathcal{G}$  satisfying  $F \cup G = [n]$ . Since  $\mathcal{F}$  and  $\mathcal{G}$  are complexes for all  $H \subset [n]$ ,  $F \cap H \in \mathcal{F}$ ,  $G \cap H \in \mathcal{G}$ , implying  $H \in \mathcal{F} \vee \mathcal{G}$ . Thus  $\mathcal{F} \vee \mathcal{G} = 2^{[n]}$ , proving (2.2). In view of (2.1), while proving (2.2) we may assume that  $|\mathcal{F}| + |\mathcal{G}| \leq 2^n$ .

Note that a covering complex  $\mathcal{H}$  satisfies  $|\mathcal{H}| \geq n+1$ . Thus  $|\mathcal{F}| + |\mathcal{G}| \leq 2^n$  cannot hold for  $n < 3$  and even for  $n = 3$  the only possibility is  $\mathcal{F} = \mathcal{G} = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ . In this case  $\mathcal{F} \vee \mathcal{G} = 2^{[3]} \setminus \{\{3\}\}$ , proving (2.2).

Suppose  $n > 3$  and apply induction. We distinguish two cases.

(a)  $|\mathcal{F}(\bar{i})| + |\mathcal{G}(\bar{i})| > 2^{n-1}$  for all  $1 \leq i \leq n$ .

Now (2.1) implies  $([n] \setminus \{i\}) \in \mathcal{F}(\bar{i}) \vee \mathcal{G}(\bar{i})$ . Since  $\mathcal{F}(\bar{i}) \subset \mathcal{F}$ ,  $\mathcal{G}(\bar{i}) \subset \mathcal{G}$ ,  $H \in \mathcal{F} \vee \mathcal{G}$  follows for all  $H \subsetneq [n]$ . Thus  $|\mathcal{F} \vee \mathcal{G}| \geq 2^n - 1 > \frac{7}{8}2^n$  for  $n > 3$ .

(b) There exists  $j \in [n]$  satisfying  $|\mathcal{F}(\bar{j})| + |\mathcal{G}(\bar{j})| \leq 2^{n-1}$ .

Since  $\mathcal{F}(\bar{j})$  and  $\mathcal{G}(\bar{j})$  are covering the induction hypothesis yields

$$(2.3) \quad |\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| \geq \frac{7}{8}(|\mathcal{F}(\bar{j})| + |\mathcal{G}(\bar{j})|).$$

Assume by symmetry that  $|\mathcal{G}(j)| \geq |\mathcal{F}(j)|$  holds. If  $\mathcal{G}(j)$  is not covering, i.e., for some  $i \in ([n] \setminus \{j\})$ ,  $\{i\} \notin \mathcal{G}(j)$  then  $\{i\} \in \mathcal{F}(\bar{j})$  implies

$$|\mathcal{G}(j) \vee \mathcal{F}(\bar{j})| \geq 2|\mathcal{G}(j)| \geq |\mathcal{F}(j)| + |\mathcal{G}(j)| > \frac{7}{8}(|\mathcal{F}(j)| + |\mathcal{G}(j)|).$$

In this way we obtain

$$|\mathcal{F} \vee \mathcal{G}| \geq |\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| + |\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| > \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|).$$

On the other hand, if  $\mathcal{G}(j)$  is covering then we first observe that it is a complex. Also,  $|\mathcal{F}(\bar{j})| \geq |\mathcal{F}(j)|$  follows from the fact  $\mathcal{F}$  is a complex. Using the induction hypothesis these yield

$$|\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| \geq \frac{7}{8}(|\mathcal{F}(\bar{j})| + |\mathcal{G}(j)|) \geq \frac{7}{8}(|\mathcal{F}(j)| + |\mathcal{G}(j)|).$$

Using (2.3) we infer (2.2) again

$$|\mathcal{F} \vee \mathcal{G}| \geq |\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| + |\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| \geq \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|). \quad \square$$

Let us now prove Theorem 2. For  $n = 1$  the statement is void. For  $n = 2$  the only possibilities are  $\mathcal{F} = \{\emptyset, \{1\}, \{2\}\}$  and  $\mathcal{G} = \{\emptyset\}$  which satisfy (2).

Let now  $n \geq 3$  and let us apply induction. Replacing if necessary  $(\mathcal{F}, \mathcal{G})$  by  $(\mathcal{F} \cup \mathcal{G}, \mathcal{F} \cap \mathcal{G})$  we may assume that  $\mathcal{F} \supset \mathcal{G}$ ,  $\emptyset \in \mathcal{G}$ .

Just as above we may assume that for some  $j \in [n]$ ,  $\mathcal{F}(\bar{j})$  and  $\mathcal{G}(\bar{j})$  are cross-union (on  $[n] \setminus \{j\}$ ). By the induction hypothesis

$$(2.4) \quad |\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| \geq \frac{3}{4}(|\mathcal{F}(\bar{j})| + |\mathcal{G}(\bar{j})|).$$

There are two cases to consider according whether  $\mathcal{G}(j)$  is empty or not.

(i)  $\mathcal{G}(j) \neq \emptyset$

Since  $\mathcal{F}(\bar{j})$  is covering,

$$|\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| \geq \frac{3}{4}(|\mathcal{F}(\bar{j})| + |\mathcal{G}(j)|) \geq \frac{3}{4}(|\mathcal{F}(j)| + |\mathcal{G}(j)|)$$

follows from the induction hypothesis. Now (2.4) yields (2).

(ii)  $\mathcal{G}(j) = \emptyset$

Since  $\emptyset \in \mathcal{G}(\bar{j})$ ,

$$|\mathcal{F}(j) \vee \mathcal{G}(\bar{j})| \geq |\mathcal{F}(j)| > \frac{3}{4}|\mathcal{F}(j)|.$$

Adding this to (2.4) yields (2) with strict inequality.  $\square$

### 3 The deduction of Theorem 3

We could not prove Theorem 3 directly. We are going to deduce it from the following recent result of the author

**Theorem 3.1** ([F]). *Let  $\mathcal{F}, \mathcal{G}, \mathcal{H} \subset 2^{[n]}$  be covering complexes that are cross-union. Then*

$$(3.1) \quad |\mathcal{F}| + |\mathcal{G}| + |\mathcal{H}| < 2^n.$$

The proof of Theorem 3 using (3.1) is easy. First note that since  $\mathcal{F}$  and  $\mathcal{G}$  are cross 2-union  $\mathcal{F} \vee \mathcal{G}$  contains no  $(n-1)$ -element sets. Consequently  $\mathcal{H} \stackrel{\text{def}}{=} 2^{[n]} \setminus (\mathcal{F} \vee \mathcal{G})^c$  is covering. Since  $\mathcal{F}$  and  $\mathcal{G}$  are complexes,  $\mathcal{F} \vee \mathcal{G}$  and therefore  $\mathcal{H}$  also are complexes. Let us show that  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  are cross-union.

Since all three are complexes, the contrary means that there are  $F \in \mathcal{F}$ ,  $G \in \mathcal{G}$ ,  $H \in \mathcal{H}$  that partition  $[n]$ . Thus  $H = (F \cup G)^c \in (\mathcal{F} \vee \mathcal{G})^c$  contradicting  $\mathcal{H} = 2^{[n]} \setminus (\mathcal{F} \vee \mathcal{G})^c$ . Applying (3.1) gives

$$|\mathcal{F}| + |\mathcal{G}| + 2^n - |\mathcal{F} \vee \mathcal{G}| < 2^n.$$

Rearranging yields

$$|\mathcal{F}| + |\mathcal{G}| < |\mathcal{F} \vee \mathcal{G}| \quad \text{proving (4).} \quad \square$$

In [F] the following generalisation of Theorem 3.1 is established in a somewhat lengthy way. Here we provide a much simpler proof.

**Theorem 3.2.** *Suppose that  $r \geq 2$ ,  $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$  are cross-union and covering. Then*

$$(3.2) \quad \sum_{1 \leq i \leq r} |\mathcal{F}_i| \leq 2^n - (r-2).$$

*Proof.* The case  $r = 2$  follows from (2.1). We apply induction on  $r$  and use (3.2) to prove it for  $r$  replaced by  $r+1$ . Without loss of generality let  $\mathcal{F}_1, \dots, \mathcal{F}_{r+1}$  be complexes. Note that  $\mathcal{F}_r \vee \mathcal{F}_{r+1}$  is a covering complex and that the  $r$  families  $\mathcal{F}_1, \dots, \mathcal{F}_{r-1}, \mathcal{F}_r \vee \mathcal{F}_{r+1}$  are cross-union.

On the other hand the fact that  $\mathcal{F}_1$  is covering implies that  $\mathcal{F}_r$  and  $\mathcal{F}_{r+1}$  are cross 2-union. Applying the induction hypothesis and (4) yield

$$|\mathcal{F}_1| + \dots + |\mathcal{F}_{r+1}| \leq |\mathcal{F}_1| + \dots + |\mathcal{F}_{r-1}| + |\mathcal{F}_r \vee \mathcal{F}_{r+1}| - 1 \leq 2^n - (r-2) - 1 = 2^n - (r-1)$$

as desired.  $\square$

Actually in [F] only the slightly weaker statement,  $< 2^n$  is proved.

Especially for  $n > n_0(r)$  the bound (3.2) seems to be rather far from best possible.

**Example 3.3.** Let  $n > r \geq 3$ . Set  $\mathcal{G}_1 = \{G \subset [n] : |G| \leq n - r\}$ ,  $\mathcal{G}_2 = \dots = \mathcal{G}_r = \{G \subset [n] : |G| \leq 1\}$ . Then  $\mathcal{G}_1, \dots, \mathcal{G}_r$  are covering and cross-union. Define

$$g(n, r) = |\mathcal{G}_1| + \dots + |\mathcal{G}_r| = 2^n + (r - 1)(n + 1) - \sum_{0 \leq j < r} \binom{n}{j}.$$

Note that  $g(n, 2) = 2^n$ . For  $r \geq 3$  fixed and  $n \rightarrow \infty$  also  $g(n, r)/2^n$  tends to 1.

**Conjecture 3.1.** Suppose that  $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$  are covering and cross-union,  $r \geq 3$ . Then for  $n > n_0(r)$  one has

$$|\mathcal{F}_1| + \dots + |\mathcal{F}_r| \leq g(n, r).$$

## 4 Further applications

Let us use Theorems 1 and 2 to give a new proof for the following recent results from [F].

**Theorem 4.1.** Suppose that  $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$  are cross-union and  $\mathcal{A}, \mathcal{B}$  are covering. Then

$$(4.1) \quad |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| \leq \frac{9}{8} 2^n.$$

**Theorem 4.2.** Suppose that  $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$  are cross-intersecting and  $\mathcal{A}$  is covering. Then

$$(4.2) \quad |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| \leq \frac{5}{4} 2^n.$$

For a family  $\mathcal{H}$  let  $\mathcal{H}_*$  be the complex generated by  $\mathcal{H}$ :

$$\mathcal{H}_* = \{G : \exists H \in \mathcal{H}, G \subset H\}.$$

In both Theorems, replacing  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  by  $\mathcal{A}_*, \mathcal{B}_*, \mathcal{C}_*$  will not change the union and covering properties and can only increase the size of the families. Therefore in proving (4.1) and (4.2) we may assume that  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  are complexes.

*Proof of (4.1).* Apply (2) for  $\mathcal{A} = \mathcal{F}$ ,  $\mathcal{B} = \mathcal{G}$  to obtain

$$(4.3) \quad \frac{7}{8}|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A} \vee \mathcal{B}|.$$

Since  $\mathcal{A} \vee \mathcal{B}$  and  $\mathcal{C}$  are cross-union, we infer from (2.1):

$$|\mathcal{A} \vee \mathcal{B}| + |\mathcal{C}| \leq 2^n.$$

Combining with (4.3) yields

$$\frac{7}{8}|\mathcal{A}| + \frac{7}{8}|\mathcal{B}| + |\mathcal{C}| \leq 2^n.$$

Invoking (3.1) to  $\mathcal{A}$  and  $\mathcal{B}$  yields

$$\frac{1}{8}|\mathcal{A}| + \frac{1}{8}|\mathcal{B}| \leq \frac{1}{8}2^n.$$

Now adding these two inequalities gives (4.1).  $\square$

*Proof of (4.3).* It is very similar. Using (2.1) for the pairs  $(\mathcal{A}, \mathcal{B})$  and  $(\mathcal{A} \vee \mathcal{B}, \mathcal{C})$  yields

$$\begin{aligned} \frac{1}{4}|\mathcal{A}| + \frac{1}{4}|\mathcal{B}| &\leq \frac{1}{4}2^n, \\ |\mathcal{A} \vee \mathcal{B}| + |\mathcal{C}| &\leq 2^n. \end{aligned}$$

Adding these two inequalities and using

$$|\mathcal{A} \vee \mathcal{B}| \geq \frac{3}{4}(|\mathcal{A}| + |\mathcal{B}|)$$

gives (4.3).  $\square$

Let us mention that without covering assumptions (2.1) implies the bound  $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| \leq \frac{3}{2} \cdot 2^n$  which is best possible as shown by the choice  $\mathcal{A} = \mathcal{B} = \mathcal{C} = 2^{[n-1]}$ .

One can prove similar statements for  $r$  families,  $r > 3$  as well, cf. [F].

## References

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