An exact result for $(0, \pm 1)$-vector

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Abstract

After reviewing some memories and results of a dear friend and great collaborator, Michel-Marie Deza, a result is proven that could have very well been a joint paper, should not he have departed under tragical circumstances.

1 Introduction

Let me begin with some reminiscences about Michel Deza who was one of the handful of people having had a huge influence on my life.

It was the beginning of October 1975. I was still a student in Budapest and I was in the middle of preparing my first ever trip to the West with a grant from the French government. My advisor, G. O. H. Katona showed me a letter that he just received from Michel Deza. In the letter Michel wrote that he just read my paper (my first paper!) proving a conjecture of Katona. Katona told me that Deza was a very interesting person and suggested me that I should visit him. There was no e-mail at the time and we did not know his phone number. However, '3 rue de Duras' was marked as his address on the envelop.

I was hoping to bump into him at some seminar at the University of Paris, but it did not happen. After a few weeks of hesitation I made up my mind. Looked up his address on the map of Paris and on a sunny day gathered enough courage and went to his place. I felt very awkward ringing the doorbell of a person I had never met.

However, once I told him that I was a mathematician from Hungary, he let me come in and offered me tea in his small apartment where there was hardly enough room for two people to sit.

After we finished tea he invited me for dinner to the famous Parisian café 'La Coupole'. That is where we started our first mathematical discussion which eventually lead to our first joint paper.

Let X be an n-element set, e.g., $X = \{1, \dots, n\}$. Let 2^X $\left(\binom{X}{k}\right)$ denote the collection of all subsets (all k-element subsets) of X, respectively.

Definition 1. A family $\mathcal{F}$ of subsets of X is called t-intersecting (t a fixed positive integer) if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$.

One of the most important results in extremal set theory is the following.

Erdős–Ko–Rado Theorem ([EKR]). *Suppose that $n \geq n_0(k,t)$, $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting. Then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}. \tag{1}$$

Let us note that considering all k-subsets containing a *fixed* t-subset shows that (1) is best possible. It is known (cf. [F] and [W] that the correct value of $n_0(k,t)$ is $(k-t+1)(t+1)$ for $k > t > 0$.

Let us also mention the following, much simpler result.

Proposition 2 ([EKR]). *Suppose that $\mathcal{F} \subset 2^X$ is intersecting. Then*

$$|\mathcal{F}| \leq 2^{n-1}. \tag{2}$$

Remark 3. To prove (2) one simply notes that out of a set S and its complement $X \setminus S$ at most one can belong to an intersecting family. Therefore $|\mathcal{F}| \leq \frac{1}{2} \cdot 2^n = 2^{n-1}$.

During our dinner Michel suggested that we try and generalize the Erdős–Ko–Rado Theorem to other situations, in particular to permutations.

2 Permutations

Let S_n denote the full symmetric group, that is, S_n consists of all $n!$ permutations of $X = \{1, \dots, n\}$.

For a permutation π let π^{-1} denote its inverse. Also for $\pi \in S_n$ let $F(\pi)$ denote the set of fixed points of π, that is,

$$F(\pi) = \{i : \pi(i) = i\}.$$

Definition 4. For integers $n > t > 0$ a family (of permutations) $\mathcal{R} \subset S_n$ is called t-intersecting if

$$|F(\varrho^{-1}\pi)| \geq t \quad \text{for all} \quad \pi, \varrho \in \mathcal{R}.$$

Let $p(n,t)$ denote the maximum size of $\mathcal{R} \subset S_n$ over all t-intersecting families.

With Michel we found two natural constructions for t-intersecting families of permutations. For the first one Michel coined the name *stabilizer family.* Fix a t-element subset $T \subset X$ and define $\mathcal{S}(T) = \{\pi \in S_n : \pi(i) = i$ for all $i \in T\}$. It is easy to see that $|\mathcal{S}(T)| = (n-t)!$ and $\mathcal{S}(T)$ is t-intersecting.

For simplicity let us define the other one only in the case $n-t$ is even. Set $d = (n-t)/2$. One defines

$$\mathcal{P}(n,t) = \bigl\{\pi \in S_n : |X \setminus F(\pi)| \leq d\bigr\}.$$

It is not hard to check that $\mathcal{P}(n,t)$ is t-intersecting.

Comparing the size s of $\mathcal{S}(T)$ and $\mathcal{P}(n,t)$ one realizes that for $n > n_0(t)$ the first one is larger. However, if $d = \frac{n-t}{2}$ is fixed and n together with t tend to infinity then $|\mathcal{P}(n,t)|$ is larger. For the latter case we established

$$p(n,t) = |\mathcal{P}(n,t)|$$

and a similar best possible result for the case $n - t = 2d + 1 \geq 3$.

For the opposite case we made the following.

Conjecture 5. *For every $t \geq 1$ and $n \geq n_0(t)$*

$$p(n,t) = (n-t)! \tag{3}$$

We proved this in some special cases.

Proposition 6. (3) *holds in each of the following cases.*

(i) $t = 1$, *n arbitrary;*

(ii) $t = 2$, *n is a prime power;*

(iii) $t = 3$, *$n - 1$ is a prime power.*

Cameron and Ku [CK] strengthened (i) by showing that the only one-intersecting families of size $(n-1)!$ are stabilizer families and their cosets.

More recently Ellis et al. [EFP] proved that Conjecture 5 is true for all t, $n \geq n_0(t)$.

3 Sunflowers

A family $\mathcal{A} = \{A_1, \dots, A_q\}$ of distinct subsets is called a *sunflower* if $A_i \cap A_j$ is constant (the same set) for all $1 \le i < j \le q$. It is called a *weak sunflower* if $|A_i \cap A_j|$ is constant (the same size) for all $1 \le i < j \le q$.

In extremal set theory the most beautiful result of Deza was the proof of a conjecture of Erdős and Lovász [EL].

Deza Theorem ([D])**.** *Suppose that $\mathcal{A} = \{A_1, \dots, A_q\}$ is a weak sunflower consisting of k-element sets, $q > k^2 - k + 2$. Then $\mathcal{A}$ is a sunflower.*

With the help of Michel in 1979 I moved to France and got a job at CNRS. The next summer he proposed me to work on extending his result to the more general setting of $(0, \pm 1)$-vectors.

Fixing the integers $n \ge k > 0$, there is a 1–1 correspondence between sets $A \in \binom{X}{k}$ and $(0,1)$-vectors $\overrightarrow{v}(A) = (v_1, \dots, v_n)$. Namely, $v_i = 1$ iff $i \in A$.

Using the ordinary scalar product

$$\big(\overrightarrow{v}, \overrightarrow{w}\big) = \sum_{1 \le i \le n} v_i w_i, \ |A| = k \ \text{ is equivalent to } \ \big(\overrightarrow{v}(A), \overrightarrow{v}(A)\big) = k.$$

Allowing -1's makes the situation more complicated. One can still define sunflowers and weak sunflowers. A family $W = \big\{\overrightarrow{w}(1), \dots, \overrightarrow{w}(q)\big\}$ of $(0, \pm 1)$-vectors of length n is called a *weak sunflower* if $\big(\overrightarrow{w}(i), \overrightarrow{w}(j)\big)$ is constant over all choices of $1 \le i < j \le q$.

For a vector $\overrightarrow{v} = (v_1, \dots, v_n)$ let us define its *support*, $S(\overrightarrow{v}) = \{i : v_i \ne 0\}$.

Definition 7. A family $W = \big\{\overrightarrow{w}(1), \dots, \overrightarrow{w}(q)\big\}$ of $(0, \pm 1)$-vectors is called a *sunflower* if (i) and (ii) hold.

(i) The sets $S\big(\overrightarrow{w}(1)\big), \dots, S\big(\overrightarrow{w}(q)\big)$ form a sunflower.

(ii) Setting $T = S\big(\overrightarrow{w}(1)\big) \cap S\big(\overrightarrow{w}(2)\big)$, for every $\ell \in T$ all q of the vectors have the same non-zero ℓ'th coordinate.

Let us note that (ii) implies that every sunflower is a weak sunflower.

With Michel we extended and sharpened his theorem to this case.

4 More on $(0, \pm 1)$-vectors

To the reader $(0, \pm 1)$-vectors may look to be an awkward extension of subsets. However, it is not only a rich subject, it has proved to be quite useful. For example, currently the best lower bounds for the chromatic number of the n-space and Borsuk's problem were established by extending theorems for families of subsets to the more general setting of $(0, \pm 1)$-vectors (cf. [R], [PR], [K]).

In this section we would like to prove a new result for $(0, \pm 1)$-vectors. Let $(0, \pm 1)^n$ be the set of all $(0, \pm 1)$-vectors of length n.

Theorem 8. *Suppose that $W \subset (0, \pm 1)^n$ and $|W| > 2 \cdot 3^{n-1}$. Then there exist three distinct vectors $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in W$ such that $\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w} = (0, 0, \ldots, 0)$, the all-zero vector.*

Proof. Let us define the operation of cyclic addition on $(0, \pm 1)$. We simply use regular addition if the result is in $(0, \pm 1)$ and set $1 + 1 = -1$, $(-1) + (-1) = 1$. For vectors $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ their sum is $(v_1 + w_1, \ldots, v_n + w_n) \in (0, \pm 1)^n$.

Set $\overrightarrow{1} = (1, \ldots, 1)$, $\overrightarrow{0} = (0, \ldots, 0)$. With this definition for an arbitrary vector $\overrightarrow{u} \in (0, \pm 1)^n$ the sum of the three vectors $\overrightarrow{u}$, $\overrightarrow{u} + \overrightarrow{1}$, $\overrightarrow{u} + \overrightarrow{1} + \overrightarrow{1}$ is $\overrightarrow{0}$. Therefore if these three vectors are all in W, we are done.

Let us partition $(0, \pm 1)^n$ according to the first coordinate. Set $Z(0) = \big\{(v_1, \ldots, v_n) : v_1 = 0\big\}$ and similarly for $Z(1)$ and $Z(-1)$.

If $\overrightarrow{u} \in Z(0)$ then $\overrightarrow{u} + \overrightarrow{1}$ is in $Z(1)$ while $\overrightarrow{u} + \overrightarrow{1} + \overrightarrow{1}$ is in $Z(-1)$.

This way we obtain a partition of $(0, \pm 1)^n$ into 3^{n-1} partition classes each consisting of three vectors. Moreover, in each class the sum of the three vectors is $\overrightarrow{0}$.

Since $|W| > 2 \cdot 3^{n-1}$, there must be a partition class so that all three vectors are in W. Thus we found three distinct vectors whose sum is the all-zero vector. $\square$

Remark. Let us note that the family $\mathcal{B}(\ell) = \big\{(b_1, \ldots, b_n) \in (0, \pm 1)^n : b_\ell = 1 \text{ or } -1\big\}$ satisfies $|\mathcal{B}(\ell)| = 2 \cdot 3^{n-1}$ for all $1 \leq \ell \leq n$. Moreover, the sum of three vectors from $\mathcal{B}(\ell)$ has non-zero in the ℓ'th coordinate. This shows that Theorem 8 is best possible.

In the case of Proposition 2 there are doubly exponentially many ways to attain equality in (2). However, for the case of $(0, \pm 1)$-vectors one can prove uniqueness.

Theorem 9. *Suppose that $W \subset (0, \pm 1)^n$, $n \geq 2$, $|W| = 2 \cdot 3^{n-1}$ and there are no distinct $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in W$ satisfying $\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w} = \overrightarrow{0}$. Then $W = \mathcal{B}(\ell)$ for some $1 \leq \ell \leq n$.*

Proof. Define $V = (0, \pm 1)^n \big/ W$, the complement of W.

Lemma 10. *If $\overrightarrow{v} = (v_1, \ldots, v_n)$ and $\overrightarrow{w} = (w_1, \ldots, w_n)$ are in V then $v_i = w_i$ holds for some $1 \leq i \leq n$.*

Proof of the lemma. Suppose for contradiction that the lemma fails for $\overrightarrow{v}, \overrightarrow{w} \in V$. For $1 \leq i \leq n$ define u_i by $\{u_i\} = (0, \pm 1) \setminus \{v_i, w_i\}$. In words, u_i is the remaining of the three possible coordinates. Now $\overrightarrow{u} = (u_1, \ldots, u_n)$ satisfies $\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w} = \overrightarrow{0}$.

Our aim is to find a partition of $(0, \pm 1)^n$ into 3^{n-1} triples $\left\{\overrightarrow{u}^{(j)}, \overrightarrow{v}^{(j)}, \overrightarrow{w}^{(j)}\right\}$, satisfying $\overrightarrow{u}^{(j)} + \overrightarrow{v}^{(j)} + \overrightarrow{w}^{(j)} = \overrightarrow{0}$, $1 \leq j \leq 3^{n-1}$ where $\left\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\right\}$ is one of these triples. If we can achieve this, we are done. Indeed, at least one of the triples must be in V and for the triple $\left\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\right\}$ at least two of them are in V by our original indirect assumption. These show $|V| \geq 3^{n-1} + 1$, i.e., $|W| < 2 \cdot 3^n$, the desired contradiction.

To find a required partition let $\mathcal{Z} = \left\{\overrightarrow{z} = (z_1, \ldots, z_n) : z_n = 0\right\}$. Then $|\mathcal{Z}| = 3^{n-1}$. Number the members of $\mathcal{Z}$ to have $\mathcal{Z} = \left\{\overrightarrow{z}^{(j)} : 1 \leq j \leq 3^{n-1}\right\}$ and define $\overrightarrow{u}^{(j)} = \overrightarrow{u} + \overrightarrow{z}^{(j)}$, $\overrightarrow{v}^{(j)} = \overrightarrow{v} + \overrightarrow{z}^{(j)}$, $\overrightarrow{w}^{(j)} = \overrightarrow{w} + \overrightarrow{z}^{(j)}$ where addition is the componentwise cyclic addition defined above.

This way we obtain the desired partition and conclude the proof of the lemma. □

By Lemma 10 the family V is *intersecting*, i.e., any two of its members must coincide in at least one coordinate position.

The partition of $(0, \pm 1)^n$ defined above can be used to show that $|V| \leq 3^{n-1}$ for every intersecting family $V \subset (0, \pm 1)^n$. Moreover, it can be deduced from the results of Frankl and Füredi [FF] that in case of $|V| = 3^{n-1}$ one must have $V = \left\{(v_1, \ldots, v_n) : v_i = b\right\}$ for some fixed $1 \leq \ell \leq n$ and $b \in (0, \pm 1)$.

Let us mention that Borg [B] proved this in a stronger form.

Now, to conclude the proof of the theorem, we need to prove that $b = 0$. It is here that we use $n \geq 2$.

Indeed, if $b \neq 0$ then none of the following three vectors is in V:

$$\begin{aligned} \overrightarrow{u} &= (u_1, \ldots, u_n) \;\text{ with }\; u_\ell = 0 \;\text{ and }\; u_i = -1 \;\text{ for }\; i \neq \ell, \\ \overrightarrow{v} &= (v_1, \ldots, v_n) \;\text{ with }\; v_\ell = 0 \;\text{ and }\; v_i = 1 \;\text{ for }\; i \neq \ell, \end{aligned}$$

$\overrightarrow{0} = (0, \ldots, 0).$

That is, we found three distinct vectors in W whose sum is the all-zero vector, the final contradiction. □

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