On Robertson-type Uncertainty Principles *

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Abstract

Uncertainty principles are one of the basic relations of Quantum Mechanics. Robertson has discovered first that the Schrödinger uncertainty principle can be interpreted as a determinant inequality. Generalized quantum covariance has been previously presented by Gibilisco, Hiai and Petz. Gibilisco and Isola have proved that among these covariances the usual quantum covariance introduced by Schrödinger gives the sharpest inequalities for the determinants of covariance matrices. We have introduced the concept of symmetric and antisymmetric quantum f-covariances which give better uncertainty inequalities. Furthermore, they have a direct geometric interpretation. Using a simple matrix analytical framework, we present here a short and tractable proof for the celebrated Robertson uncertainty principle.

Introduction

In quantum information theory, the most popular model of the quantum event algebra associated to an n-level system is the projection lattice of an n-dimensional Hilbert space ($\mathcal{L}(\mathbb{C}^n)$). According to Gleason's theorem [2], for $n > 2$ the states are of the form

$$(\forall P \in \mathcal{L}(\mathbb{C}^n)) \quad P \mapsto \operatorname{Tr}(DP),$$

where D is a positive semidefinite matrix with trace 1 hence the quantum mechanical state space arises as the intersection of the standard cone of positive semidefinite matrices and the hyperplane of trace one matrices. Let us denote by $\mathcal{M}_n$ the set of $n \times n$ positive definite matrices and by $\mathcal{M}_n^1$ the interior of the n-level quantum mechanical state space, namely $\mathcal{M}_n^1 = \{D \in \mathcal{M}_n \mid \operatorname{Tr} D = 1,\, D > 0\}$. Let $M_{n,\mathrm{sa}}$ be the set of observables of the n-level quantum system, in other words the set of $n \times n$ self adjoint matrices, and $M_{n,\mathrm{sa}}^{(0)}$ stands for the set observables with zero trace. Observables are non commutative analogues of random variables known from Kolmogorovian probability. If A is an observable, then the expectation of A in $D \in \mathcal{M}_n^1$ is defined by $\mathbb{E}_D(A) = \operatorname{Tr}(AD)$.

Spaces $\mathcal{M}_n$ and $\mathcal{M}_n^1$ are form convex sets in the space of self adjoint matrices, and they are obviously differentiable manifolds [7]. The tangent space of $\mathcal{M}_n$ at a given state D can be identified with $M_{n,\mathrm{sa}}$ and the tangent space of $\mathcal{M}_n^1$ with $M_{n,\mathrm{sa}}^{(0)}$. Monotone metrics are the quantum analogues of the Fisher information matrix known from classical information geometry. Petz's classification

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theorem [11] establishes a connection between monotone metrics and the set of symmetric and normalized operator monotone functions $\mathcal{F}_{\mathrm{op}}$. For the mean induced by the operator monotone function $f \in \mathcal{F}_{\mathrm{op}}$ we also introduce the notation

$$m_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \qquad (x,y) \mapsto y f\left(\frac{x}{y}\right).$$

The monotone metric associated to $f \in \mathcal{F}_{\mathrm{op}}$ is given by

$$K_D^{(n)}(X,Y) = \mathrm{Tr}\left(X m_f\left(L_{n,D}, R_{n,D}\right)^{-1}(Y)\right)$$

for all $n \in \mathbb{N}$ where $L_{n,D}(X) = DX$, $R_{n,D}(X) = XD$ for all $D, X \in M_n(\mathbb{C})$. The metric $K_D^{(n)}$ can be extended to the space $\mathcal{M}_n$. For every $D \in \mathcal{M}_n$ and matrices $A, B \in M_{n,\mathrm{sa}}$ let us define

$$\langle A, B\rangle_{D,f} = \mathrm{Tr}\left(A m_f\left(L_{n,D}, R_{n,D}\right)^{-1}(B)\right),$$

with this notion the pair $(\mathcal{M}_n, \langle\cdot,\cdot\rangle_{\cdot,f})$ will be a Riemannian manifold for every operator monotone function $f \in \mathcal{F}_{\mathrm{op}}$.

Although the generalization of expectation and variance to the non commutative case is straightforward, covariance has many different possible generalization to the quantum case. Schrödinger has defined the (symmetric) covariance of the observables for a given state D as

$$\mathrm{Cov}_D(A,B) = \frac{1}{2}\left(\mathrm{Tr}(DAB) + \mathrm{Tr}(DBA)\right) - \mathrm{Tr}(DA)\,\mathrm{Tr}(DB).$$

In a recent paper [9], we have introduced the concept of *symmetric f-covariance* as the scalar product of anti-commutators

$$\mathrm{qCov}^s_{D,f}(A,B) = \frac{f(0)}{2}\left\langle \{D,A\}, \{D,B\}\right\rangle_{D,f}.$$

Note that, $\mathrm{qCov}^s_{D,f}(A,B)$ coincides with $\mathrm{Cov}_D(A,B)$ whenever $f(x) = \frac{1+x}{2}$. We have proved that for any f symmetric and normalized operator monotone function

$$\det\left(\left[\frac{f(0)}{2}\left\langle \{D,A_h\}, \{D,A_j\}\right\rangle_{D,f}\right]_{h,j=1,\dots,N}\right) \geq \det\left(\left[\frac{f(0)}{2}\left\langle \mathrm{i}\,[D,A_h], \mathrm{i}\,[D,A_j]\right\rangle_{D,f}\right]_{h,j=1,\dots,N}\right) \tag{1}$$

eq:introfirstresult

holds. Moreover we showed that the function $f_0(x) = \frac{1}{2}\left(\frac{1+x}{2} + \frac{2x}{1+x}\right)$ gives the smallest universal upper bound for the right-hand side, that is, for every symmetric and normalized operator monotone function g

$$\det\left(\left[\frac{f_0(0)}{2}\left\langle \{D,A_h\}, \{D,A_j\}\right\rangle_{D,f_0}\right]_{h,j=1,\dots,N}\right) \geq \det\left(\left[\frac{g(0)}{2}\left\langle \mathrm{i}\,[D,A_h], \mathrm{i}\,[D,A_j]\right\rangle_{D,g}\right]_{h,j=1,\dots,N}\right) \tag{2}$$

eq:introsecondresul

holds.

The Equations (1) and (2) are Robertson-type uncertainty principles with clear geometric meaning, namely, they can be viewed as a kind of volume inequalities. The volume of the N-parallelotope spanned by the vectors X_k $(k = 1, \dots, N)$ with respect to the inner product $\langle\cdot,\cdot\rangle$ is

$$V_f(X_1,\dots,X_N) = \sqrt{\det\left(\left[\langle X_h, X_j\rangle_{D,f}\right]_{h,j=1,\dots,N}\right)}.$$

In this setting Equation (1) and (2) can be written as

$$V_f(\{D,A_1\},\dots,\{D,A_N\}) \geq V_f(\mathrm{i}\,[D,A_1],\dots,\mathrm{i}\,[D,A_N])$$
$$V_{f_0}(\{D,A_1\},\dots,\{D,A_N\}) \geq V_g(\mathrm{i}\,[D,A_1],\dots,\mathrm{i}\,[D,A_N]).$$

To make the explanation self contained and understandable for the largest possible audience, in Section 1, we briefly outline the origin and development of Robertson-type uncertainty principles. In Section 2, we present a simple and rather understandable proof for the original Robertson uncertainty principle.

1 Overview

The concept of uncertainty was introduced by Heisenberg in 1927 [6], who demonstrated the impossibility of simultaneous measurement of position (q) and momentum (p). He considered Gaussian distributions ($f(q)$), and defined uncertainty of f as its width D_f. If the width of the Fourier transform of f is denoted by $D_{\mathcal{F}(f)}$, then the first formalisation of the uncertainty principle can be written as

$$D_f D_{\mathcal{F}(f)} = \text{constant}.$$

In 1927 Kennard generalised Heisenberg's result [8], he proved the inequality

$$\mathrm{Var}_D(A)\,\mathrm{Var}_D(B) \geq \frac{1}{4}$$

for observables A,B which satisfy the relation $[A,B]=-\mathrm{i}$, for every state D, where $\mathrm{Var}_D(A) = \mathrm{Tr}(DA^2)-(\mathrm{Tr}(DA))^2$.

In 1929 Robertson [12] extended Kennard's result for arbitrary two observables A,B

$$\mathrm{Var}_D(A)\,\mathrm{Var}_D(B) \geq \frac{1}{4}\left|\mathrm{Tr}(D\,[A,B])\right|^2.$$

In 1930 Scrödinger [14] improved this relation including the correlation between observables A,B

$$\mathrm{Var}_D(A)\,\mathrm{Var}_D(B) - \mathrm{Cov}_D(A,B)^2 \geq \frac{1}{4}\left|\mathrm{Tr}(D\,[A,B])\right|^2.$$

The Schrödinger uncertainty principle can be formulated as

$$\det\begin{pmatrix}\mathrm{Cov}_D(A,A) & \mathrm{Cov}_D(A,B)\\ \mathrm{Cov}_D(B,A) & \mathrm{Cov}_D(B,B)\end{pmatrix} \geq \det\left(-\frac{\mathrm{i}}{2}\begin{pmatrix}\mathrm{Tr}(D\,[A,A]) & \mathrm{Tr}(D\,[A,B])\\ \mathrm{Tr}(D\,[B,A]) & \mathrm{Tr}(D\,[B,B])\end{pmatrix}\right).$$

For the set of observables $(A_i)_{1,\dots,N}$ this inequality was generalised by Robertson in 1934 [13] as

$$\det\left(\left[\mathrm{Cov}_D(A_h,A_j)\right]_{h,j=1,\dots,N}\right) \geq \det\left(\left[-\frac{\mathrm{i}}{2}\,\mathrm{Tr}(D\,[A_h,A_j])\right]_{h,j=1,\dots,N}\right).$$

The main drawback of this inequality is that the right-hand side is identical to zero whenever N is odd.

Gibilisco and Isola in 2006 conjectured that

$$\det\left(\left[\mathrm{Cov}_D(A_h,A_j)\right]_{h,j=1,\dots,N}\right) \geq \det\left(\left[\frac{f(0)}{2}\left\langle \mathrm{i}\,[D,A_h]\,,\mathrm{i}\,[D,A_j]\right\rangle_{D,f}\right]_{h,j=1,\dots,N}\right), \qquad (3)$$ eq:GibiliscoConject

holds [5], where the scalar product $\langle \cdot, \cdot \rangle_{D,f}$ is induced by an operator monotone function f, according to Petz classification theorem [11]. We note that if the density matrix is not strictly positive, then the scalar product $\langle \cdot, \cdot \rangle_{D,f}$ is not defined. For arbitrary N the conjecture was proved by Andai [1] and Gibilisco, Imparato and Isola [4]. The inequality (3) is called *dynamical uncertainty principle* [3] because the right-hand side can be interpreted as the volume of a parallelepiped determined by the tangent vectors of the time-dependent observables $A_k(t) = \mathrm{e}^{\mathrm{i}\, tD} A_k \, \mathrm{e}^{-\mathrm{i}\, tD}$.

Gibilisco, Hiai and Petz studied the behaviour of a possible generalization of the covariance under coarse graining and they deduced that the covariance must have the following form for traceless observables A, B

$$\mathrm{Cov}_D^f(A,B) = \mathrm{Tr}\left(A f(L_{n,D} R_{n,D}^{-1}) R_{n,D}(B) \right), \tag{4}$$ petzcov

where $L_{n,D}$ and $R_{n,D}$ are superoperators acting on $n \times n$ matrices like $L_{n,D}(A) = DA$, $R_{n,D}(A) = AD$ and f is a symmetric and normalized operator monotone function [3]. Quantum covariances of the form (4) are called *quantum f-covariance*, which has been introduced for the first time by D. Petz [10]. It has been proved [3] that the generalized form of dynamical uncertainty principle holds true for an arbitrary quantum f-covariance

$$\det\left([\mathrm{Cov}_D^g(A_h, A_j)]_{h,j=1,\ldots,N} \right) \geq \det\left(\left[f(0) g(0) \left\langle \mathrm{i}\,[D, A_h]\,, \mathrm{i}\,[D, A_j] \right\rangle_{D,f} \right]_{h,j=1,\ldots,N} \right)$$

and for all g symmetric and normalized operator monotone function. If $g(x) = \frac{1+x}{2}$ is chosen, then we get the sharpest form of the inequality. Uncertainty relations involving different type of covariances are illustrated in Figure 1.

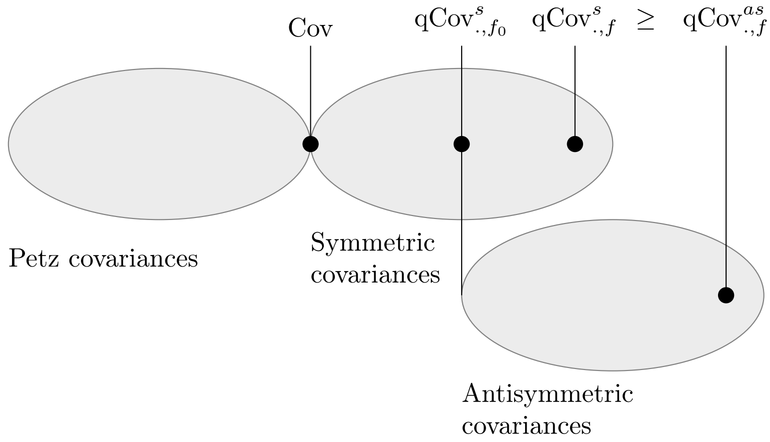

Figure 1: Robertson-type uncertainty principles.

⟨fig:1⟩

2 Robertson uncertainty principle

In this Section, we give a very simple and understandable proof for the Robertson uncertainty principle. It turns out to be that the uncertainty principle in question can be originated from a more general determinant inequality between real and imaginary part of positive definite matrices.

⟨lem:1⟩ **Lemma 1.** *Let $A \in \mathcal{M}_n$ be a positive definite invertible matrix. The real and imaginary part of A satisfy the following determinant inequality.*

$$\det(\Re(A)) \geq \det(\Im(A))$$

Proof. For odd n, the right-hand side is identically 0 because $\Im(A)$ is a real skew-symmetric matrix and thus we have nothing to prove.

Assume that n is even. The determinant of an even dimensional skew-symmetric matrix is obviously non-negative. The left-hand side is strictly positive because $\Re(A)$ arises as the convex combination of A and $\overline{A}$ that are positive definite invertible matrices, where $\overline{A}$ stands for the element-wise conjugate of A.

After some algebraic manipulation we get

$$1 \geq \det \left(\frac{1}{\mathrm{i}} \frac{I - A^{-1/2}\overline{A}A^{-1/2}}{I + A^{-1/2}\overline{A}A^{-1/2}} \right) = \left| \det \left(\frac{I - A^{-1/2}\overline{A}A^{-1/2}}{I + A^{-1/2}\overline{A}A^{-1/2}} \right) \right| \tag{5}$$?eq:equiv?

which is equivalent to the original inequality. The matrix $B := A^{-1/2}\overline{A}A^{-1/2}$ is positive definite hence its spectrum belongs to $(0, \infty)$. Consider the function $f : [0, \infty) \to \mathbb{R}$ $f(x) = \frac{1-x}{1+x}$ which is continuous and it maps $[0, \infty)$ onto $(-1, 1]$.

By the spectral mapping theorem, we can write $\sigma(f(B)) = f(\sigma(B)) \subset [-1, 1]$ that implies immediately $|\det(f(B))| = |\prod_{\lambda\in\sigma(f(B))} \lambda| \leq 1$. □

Now we are in the position to proof the Robertson uncertainty principle.

Theorem 1 (Robertson (1934))**.** *In every state $D \in \mathcal{M}_n^1$ and for arbitrary set of observables $(A_i)_{1,\dots,N}$*

$$\det \left([\mathrm{Cov}_D(A_h, A_j)]_{h,j=1,\dots,N} \right) \geq \det \left(\left[-\frac{\mathrm{i}}{2} \mathrm{Tr}(D\,[A_h, A_j]) \right]_{h,j=1,\dots,N} \right)$$

holds.

Proof. We may assume that the A_k-s are linearly independent and centered i.e. $\mathrm{Tr}(DA_k) = 0$ for $k = 1, \dots, N$. For any fixed $D \in \mathcal{M}_n^1$, the map $(A, B) \mapsto \mathrm{Tr}(DAB)$ defines a scalar product on the real vector space of complex $n \times n$ Hermitian matrices.

Consider the Gram matrix $G = [\mathrm{Tr}(DA_hA_j)]_{h,j=1,\dots,N}$. One can easily check that the following equalities hold.

$$\Re(G) = [\mathrm{Cov}_D(A_h, A_j)]_{h,j=1,\dots,N}$$
$$\Im(G) = \left[-\frac{\mathrm{i}}{2} \mathrm{Tr}(D\,[A_h, A_j]) \right]_{h,j=1,\dots,N}$$

By Lemma 1, $\det(\Re(G)) \geq \det(\Im(G))$ which completes the proof.

□

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