Best bounds for dispersion of ratio block sequences for certain subsets of integers

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Abstract
In this paper, we study the behavior of dispersion of special types of sequences which block sequence is dense.

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1. Introduction
Denote by \( \mathbb{N} \) and \( \mathbb{R}^+ \) the set of all positive integers and positive real numbers, respectively. Let \( X = \{x_1 < x_2 < x_3 < \cdots \} \) be an infinite subset of \( \mathbb{N} \). Denote by \( R(X) = \left\{ \frac{x_i}{x_j} : i, j \in \mathbb{N} \right\} \) the ratio set of \( X \), and say that a set \( X \) is \((R)-dense\) if \( R(X) \) is (topologically) dense in the set \( \mathbb{R}^+ \). The concept of \((R)-density\) was introduced by T. Šalát [7].

The following sequence of finite sequences derived from \( X \)
\[
\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \ldots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_n}{x_n}, \ldots
\]
(1.1)
is called the block sequence of the sequence \( X \).

It is formed by the blocks \( X_1, X_2, \ldots, X_n, \ldots \) where
\[
X_n = \left( \frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_n}{x_n} \right), \quad n = 1, 2, \ldots
\]
is called the \( n \)-th block. This kind of block sequences was introduced by O. Strauch and J. T. Tóth [9].

For each \( n \in \mathbb{N} \) consider the \textit{step distribution function}

\[
F(X_n, x) = \frac{\#\{i \leq n; \frac{x_i}{x_n} < x\}}{n},
\]

and define the \textit{set of distribution functions of the ratio block sequence}

\[
G(X_n) = \left\{ \lim_{k \to \infty} F(X_{nk}, x) \right\}.
\]

The set of distribution functions of ratio block sequences was studied in [1, 2, 5, 6, 8, 12].

For every \( n \in \mathbb{N} \) let

\[
D(X_n) = \max \left\{ \frac{x_1}{x_n}, \frac{x_2 - x_1}{x_n}, \ldots, \frac{x_{i+1} - x_i}{x_n}, \ldots, \frac{x_n - x_{n-1}}{x_n} \right\},
\]

the maximum distance between two consecutive terms in the \( n \)-th block. We will consider the quantity

\[
\overline{D}(X) = \liminf_{n \to \infty} D(X_n),
\]

(see [10]) called the \textit{dispersion} of the block sequence (1.1) derived from \( X \). Relations between asymptotic density and dispersion were studied in [11].

The aim of this paper is to study the behavior of dispersion of the block sequence derived from \( X \) under the assumption that \( X = \bigcup_{n=1}^{\infty} (c_n, d_n) \cap \mathbb{N} \) is \((R)\)-dense and the limit \( \lim_{n \to \infty} \frac{d_n}{c_n} = s \) exists. In this case

\[
\overline{D}(X) \leq \begin{cases} 
\frac{1}{s+1}, & \text{if } s \in \left( 1, \frac{1+\sqrt{5}}{2} \right), \\
\frac{1}{s^2}, & \text{if } s \in \left( \frac{1+\sqrt{5}}{2}, 2 \right), \\
\frac{s-1}{s^2}, & \text{if } s \in (2, \infty)
\end{cases}
\]

(see [10, Theorem 10]). This upper bound for \( \overline{D}(X) \) is the best possible if \( s \geq 2 \) (see [4]) and in the case \( \frac{1+\sqrt{5}}{2} \leq s \leq 2 \) (see [3]). We prove that the above upper bound for \( \overline{D}(X) \) is also optimal in the remaining case \( 1 \leq s < \frac{1+\sqrt{5}}{2} \), i.e. \( \overline{D}(X) \) can be any number in the interval \( (0, \frac{1}{s+1}) \).

2. Results

First, we show that there is a connection between the dispersion and the distribution functions of a ratio block sequence.

**Theorem 2.1.** Let \( X \subset \mathbb{N} \), and assume that the dispersion \( \overline{D}(X) \) of the related block sequence is positive. Let \( g \in G(X_n) \). Then \( g \) is constant on an interval of length \( \overline{D}(X) \).
Proof. Let \( \varepsilon < D(X) \) be an arbitrary positive real number. By the definition of dispersion it follows that for sufficiently large \( n \) the step distribution function \( F(X_n, x) \) is constant on some interval \( (\frac{x_k}{x_n}, \frac{x_{k+1}}{x_n}) \) of length \( D(X) - \varepsilon \). A simple compactness argument yields that there exist

- real numbers \( \gamma, \delta \in (0, 1) \) such that \( \delta - \gamma \geq D(X) - \varepsilon \),
- an increasing sequence \( (n_k) \) and a sequence \( (m_k) \) of positive integers such that \( m_k < n_k \),

\[
\lim_{k \to \infty} \frac{x_{m_k}}{x_{n_k}} = \gamma, \quad \lim_{k \to \infty} \frac{x_{m_k+1}}{x_{n_k}} = \delta \quad \text{and} \quad \lim_{k \to \infty} F(X_{n_k}, x) = g(x) \text{ a.e. on } (0, 1).
\]

Hence \( g \) is constant on the interval \( (\gamma, \delta) \) of length \( D(X) - \varepsilon \). Since \( \varepsilon \) can be chosen arbitrary small, and the assertion of the theorem follows. \( \square \)

The next lemma is useful for the determination of the value of the dispersion \( D(X) \) (see [10, Theorem 1]).

**Lemma 2.2.** Let

\[
X = \bigcup_{n=1}^{\infty} (c_n, d_n) \cap \mathbb{N},
\]

and for \( n \in \mathbb{N} \) let \( c_n < d_n < c_{n+1} \) be positive integers. Then

\[
D(X) = \liminf_{n \to \infty} \max\{ c_{i+1} - d_i : i = 1, \ldots, n \}.
\]

For the proof of \((R)\)-density we shall use the following lemma.

**Lemma 2.3.** Denote by \( (p_n), (q_n), (u_n), (v_n), (w_n) \) and \( (z_n) \) be strictly increasing sequences of positive integers satisfying

\[
p_n < q_n < u_n < v_n \quad \text{and} \quad w_n < z_n, \quad (n = 1, 2, 3, \ldots).
\]

Further, let

\[
\left( \frac{q_n}{p_n} \right), \left( \frac{u_n}{p_n} \right), \left( \frac{v_n}{u_n} \right), \left( \frac{w_n}{u_n} \right) \quad \text{and} \quad \left( \frac{z_n}{w_n} \right)
\]

converge to real numbers greater than 1, moreover

\[
\lim_{n \to \infty} \frac{z_n}{w_n} \geq \lim_{n \to \infty} \frac{u_n}{q_n}.
\]

Then the ratio set of

\[
\bigcup_n \left( (p_n, q_n) \cup (u_n, v_n) \cup (w_n, z_n) \right) \cap \mathbb{N}
\]

is dense on the interval

\[
\left( \lim_{n \to \infty} \frac{w_n}{v_n}, \lim_{n \to \infty} \frac{z_n}{p_n} \right).
\]
The proof is elementary and we leave it to the reader. Let us suppose that\( k \in \mathbb{N} \) is a constant. Note that the assertion of the lemma remains still true if one removes \( k \) elements from the sets \( (u_n, v_n) \cap \mathbb{N} \) for all sufficiently large \( n \).

The main result of this paper is the following.

**Theorem 2.4.** Let \( s \in (1, \frac{1+\sqrt{5}}{2}) \) be an arbitrary real number. Then for any \( \alpha \in \left(0, \frac{1}{s+1}\right) \) there is an \((R)\)-dense set

\[
X = \bigcup_{n=1}^{\infty} (c_n, d_n) \cap \mathbb{N},
\]

where \( c_n < d_n < c_{n+1} \) are positive integers for that \( \lim_{n \to \infty} \frac{d_n}{c_n} = s \) and \( D(X) = \alpha \).

**Proof.** It was shown in [4, Theorem 2] that the dispersion \( D(X) \) can take any number in the interval \( (0, \frac{s-1}{s^2}) \). In what follows we suppose \( \frac{s-1}{s^2} < \alpha \leq \frac{1}{s+1} \).

Let us consider the function \( f(x) = \frac{x-1}{sx} \). Clearly, this function is continuous and increasing on the interval \((1, \infty)\). Moreover

\[
f(s) = \frac{s-1}{s^2} \quad \text{and} \quad f(s+1) = \frac{1}{s+1}.
\]

Thus, there exists a real number \( t \in (s, s+1) \) with the property

\[
\frac{t-1}{st} = \alpha. \tag{2.1}
\]

Write \( \frac{1}{\alpha} \) in the form \( s^k + \delta \), where \( k \) is an integer and \( 0 \leq \delta < 1 \). The lower bound \( k \geq 2 \) follows from the facts that \( s+1 \leq \frac{1}{\alpha} \) and \( s+1 \geq s^2 \) whenever \( 1 < s \leq \frac{1+\sqrt{5}}{2} \).

Define the set \( X \subset \mathbb{N} \) by

\[
X = \bigcup_{n=1}^{\infty} (A_n \cup B_n) \cap \mathbb{N},
\]

where

\[
A_n = \bigcup_{i=1}^{k} (a_{n,i}, b_{n,i}) \quad \text{and} \quad B_n = \bigcup_{j=1}^{n} (c_{n,j}, d_{n,j}).
\]

Put \( a_{1,1} = 1 \) and

\[
\begin{align*}
b_{n,i} &= [s,a_{n,i}] \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad i = 1, 2, \ldots, k, \\
an_{i} &= \begin{cases} 
\frac{d_{n-1,n-1}!}{s^0,b_{n,1}} + 1 & \text{for} \ n \geq 2, \ i = 1 \\
[s^0,b_{n,1}] + 1 & \text{for} \ n \in \mathbb{N}, \ i = 2 \\
b_{n,i-1} + 1 & \text{for} \ n \in \mathbb{N}, \ i = 3, \ldots, k,
\end{cases} \\
c_{n,j} &= \begin{cases} 
[t,b_{n,k}] + 1 & \text{for} \ n \in \mathbb{N}, \ j = 1 \\
[t.d_{n,j-1}] + 1 & \text{for} \ n \in \mathbb{N}, \ j = 2, \ldots, n,
\end{cases}
\end{align*}
\]
\[ d_{n,j} = \lfloor s \cdot c_{n,j} \rfloor \text{ for } n \in \mathbb{N} \text{ and } j = 1, 2, \ldots, n. \]

First we prove that \( D(X) = \alpha \). For sufficiently large \( n \), by the definition of the set \( X \) we have the inequalities
\[
a_{n+1,1} - d_{n,n} > c_{n,n} - d_{n,n-1} > c_{n,n-1} - d_{n,n-2} > \cdots > c_{n,3} - d_{n,2} > c_{n,2} - d_{n,1} > c_{n,1} - b_{n,k},
\]

further
\[
a_{n,1} - d_{n-1,n-1} < c_{n,1} - b_{n,k}
\]
and
\[
a_{n,2} - b_{n,1} < a_{n,1} - d_{n-1,n-1}.
\]

Observe that inequality (2.3) holds if \( \frac{1}{\alpha} (t - 1) > 1 \). In virtue of (2.1) this is equivalent with \( st > 1 \), which evidently holds. As
\[
s^{1+\delta} - s - 1 < s^2 - s - 1
\]
and \( s^2 - s - 1 \) is negative for \( s \in (1, \frac{1+\sqrt{5}}{2}) \), inequality (2.4) follows.

Now we use Lemma 1. From the inequalities (2.2–2.4) one can see that it is sufficient to study the quotients
\[ a) \frac{a_{n,1} - d_{n-1,n-1}}{b_{n,k}}, \quad b) \frac{c_{n,1} - b_{n,k}}{d_{n,1}}, \quad c) \frac{c_{n,k} - d_{n,k-1}}{d_{n,k}}. \]

In case a) we see
\[
\liminf_{n \to \infty} \frac{a_{n,1} - d_{n-1,n-1}}{b_{n,k}} = \liminf_{n \to \infty} \frac{a_{n,1}}{\alpha a_{n,1}} = \alpha,
\]
in case b)
\[
\liminf_{n \to \infty} \frac{c_{n,1} - b_{n,k}}{d_{n,1}} = \liminf_{n \to \infty} \frac{tb_{n,k} - b_{n,k}}{stb_{n,k}} = \frac{t - 1}{st} = \alpha,
\]
and the remaining case c) is analogous to case b).

It remains to prove that the set \( X \) is \((R)\)-dense. Using Lemma 2 we show that the ratio set of the set \( X \) is dense on the intervals
\[
\langle 1, \frac{1}{\alpha} \rangle \text{ (for } p_n = a_{n,1}, q_n = b_{n,1}, u_n = w_n = a_{n,2}, v_n = z_n = b_{n,k}),
\]
\[
\langle t^i s^{i-1}, \frac{1}{\alpha^i s^i t^i} \rangle \text{ (for } p_n = a_{n,1}, q_n = b_{n,1}, u_n = a_{n,2}, v_n = b_{n,k}, w_n = c_{n,i}, z_n = d_{n,i}).
\]

Hence, by \( \frac{1}{\alpha} \geq s + 1 \) and \( t < s + 1 \) we have
\[
\langle 1, \frac{1}{\alpha} \rangle \cup \bigcup_{i=1}^{\infty} \langle t^i s^{i-1}, \frac{1}{\alpha^i s^i t^i} \rangle = \langle 1, \infty \rangle,
\]
and therefore the \((R)\)-density of the set \( X \) follows. \( \square \)
References


