

On identities with multinomial coefficients for Fibonacci-Narayana sequence

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Abstract

In this paper we study some families of Toeplitz-Hessenberg determinants the entries of which are Fibonacci-Narayana (or Narayana's cows) numbers. This leads to discover some identities for these numbers. In particular, we establish connection between Fibonacci-Narayana numbers with Fibonacci and tribonacci numbers. We also present new formulas for Fibonacci-Narayana numbers via recurrent determinants of four-diagonal matrix.

Keywords: Fibonacci-Narayana sequence, Narayana's cow sequence, Toeplitz-Hessenberg matrix, Fibonacci sequence, tribonacci sequence.

MSC: AMS classification 11B39, 15B05.

1. Introduction

The *Fibonacci sequence* $\{F_n\}_{n \geq 0}$ is defined by the initial values $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $n \geq 2$.

Among the several generalizations of Fibonacci numbers, some of the best known are the *tribonacci sequence* $\{t_n\}_{n \geq 0}$ and the *Fibonacci-Narayana sequence* (or *Narayana's cows sequence*) $\{b_n\}_{n \geq 0}$, which are defined by the following third-order recurrence relations:

$$\begin{aligned}t_n &= t_{n-1} + t_{n-2} + t_{n-3}, & t_0 &= t_1 = 0, \quad t_2 = 1, \\b_n &= b_{n-1} + b_{n-3}, & b_0 &= 0, \quad b_1 = b_2 = 1,\end{aligned}$$

for $n \geq 3$.

There are large number of sequences indexed in OEIS [10], being in this case

$$\begin{aligned} \{F_n\}_{n \geq 0} &= \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots\} : & A000045 \\ \{t_n\}_{n \geq 0} &= \{0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, \dots\} : & A000073 \\ \{b_n\}_{n \geq 0} &= \{0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, \dots\} : & A000930 \end{aligned}$$

The Fibonacci-Narayana sequence was introduced by the Indian mathematician Narayana in the 14th century, while studying the following problem: *A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many cows are there altogether after, for example, 20 years?* This problem can be solved in the same way that Fibonacci solved its problem about rabbits [6].

There has been considerable recent interest in the Fibonacci-Narayana sequence and its generalizations (see [1, 2, 3, 4, 5, 9, 11] for more details). For instance, Didkivska and St'opochkina [4] proved some basic properties of Fibonacci-Narayana numbers. Biglici [3] defined a generalized order- k Fibonacci-Narayana sequence and by using this generalization and some matrix properties, established some identities related to Fibonacci-Narayana numbers. Flaut and Shpakivskiy [5] studied some properties of generalized and Fibonacci quaternions and Fibonacci-Narayana quaternions. Ramírez and Sirvent [9] defined the k -Narayana sequence of integer numbers and studied recurrence relations and some combinatorial properties of these numbers, and of the sum of their first n terms. These authors also established some relations between the k -Narayana sequence and determinants of one type of Hessenberg matrix.

The purpose of this paper is to study Fibonacci-Narayana numbers. We investigate some families of Toeplitz-Hessenberg determinants the entries of which are Fibonacci-Narayana numbers. This leads to discover some identities for these numbers. In particular, we establish connection between Fibonacci-Narayana numbers with Fibonacci and tribonacci numbers. We also present new formulas for Fibonacci-Narayana numbers via recurrent determinants of four-diagonal matrix.

2. Toeplitz-Hessenberg matrices and determinants

A *Toeplitz-Hessenberg matrix* is an $n \times n$ matrix of the form

$$M_n(a_0; a_1, a_2, \dots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}, \quad (2.1)$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$. So $a_{ij} = 0$ for $j > i + 1$.

Several mathematical objects may be represented as determinant of such matrices (see, for example, [7] and the references given there).

Expanding the determinant $\det(M_n)$ according to the first row repeatedly, we obtain the recurrence

$$\det(M_n) = \sum_{k=1}^n (-a_0)^{k-1} a_k \det(M_{n-k}), \tag{2.2}$$

where, by definition, $\det(M_0) \equiv 1$.

The following result is known as Trudi’s formula [8]. This gives the multinomial extension for $\det(M_n)$.

Lemma 2.1. *Let n be a positive integer. Then*

$$\det(M_n) = \sum_{(s_1, \dots, s_n)} (-a_0)^{n-(s_1+\dots+s_n)} \binom{s_1 + \dots + s_n}{s_1, \dots, s_n} a_1^{s_1} a_2^{s_2} \dots a_n^{s_n}, \tag{2.3}$$

where the summation is over integers $s_i \geq 0$ satisfying $s_1 + 2s_2 + \dots + ns_n = n$, and

$$\binom{s_1 + \dots + s_n}{s_1, \dots, s_n} = \frac{(s_1 + \dots + s_n)!}{s_1! \dots s_n!}$$

is the multinomial coefficient.

Example 2.2. It follows from (2.3) that

$$\begin{aligned} \det(M_3) &= (-a_0)^0 \binom{3}{3, 0, 0} a_1^3 + (-a_0)^1 \binom{2}{1, 1, 0} a_1 a_2 + (-a_0)^2 \binom{1}{0, 0, 1} a_3 \\ &= a_1^3 - 2a_0 a_1 a_2 + a_0^2 a_3; \\ \det(M_4) &= (-a_0)^0 \binom{4}{4, 0, 0, 0} a_1^4 + (-a_0)^1 \binom{3}{2, 1, 0, 0} a_1^2 a_2 + (-a_0)^2 \binom{2}{1, 0, 1, 0} a_1 a_3 \\ &\quad + (-a_0)^2 \binom{2}{0, 2, 0, 0} a_2^2 + (-a_0)^3 \binom{1}{0, 0, 0, 1} a_4 \\ &= a_1^4 - 3a_0 a_1^2 a_2 + 2a_0^2 a_1 a_3 + a_0^2 a_2^2 - a_0^3 a_4. \end{aligned}$$

Throughout this paper, we denote

$$\det(\pm 1; a_1, a_2, \dots, a_n) = \det(M_n(\pm 1; a_1, a_2, \dots, a_n)).$$

3. Connection formulas between the Fibonacci-Narayana numbers and Fibonacci numbers

The next theorem gives relationship between Fibonacci-Narayana numbers and Fibonacci numbers via the Toeplitz-Hessenberg determinants.

Theorem 3.1. For all $n \geq 1$,

$$\begin{aligned} \det(1; b_1, b_3, \dots, b_{2n-1}) &= 1 - (-1)^n F_{n-1}, \\ \det(1; b_0, b_2, \dots, b_{2n-2}) &= (-1)^{n-1} F_n. \end{aligned} \quad (3.1)$$

Proof. We will prove formula (3.1) using induction on n . The other proof follow similarly, so we omit it for interest of brevity. For simplicity of notation, we write D_n instead of $\det(1; b_1, b_3, \dots, b_{2n-1})$.

Clearly, formula (3.1) works, when $n = 1$ and $n = 2$. Suppose it is true for all positive integers $k \leq n - 1$, where $n \geq 3$.

Using recurrence (2.2), we have

$$\begin{aligned} D_n &= \sum_{i=1}^n (-1)^{i-1} b_{2i-1} D_{n-i} \\ &= b_1 D_{n-1} - b_3 D_{n-2} + b_5 D_{n-3} + \sum_{i=4}^n (-1)^{i-1} (b_{2i-2} + b_{2i-4}) D_{n-i} \\ &= D_{n-1} - D_{n-2} + 3D_{n-3} + \sum_{i=4}^n (-1)^{i-1} (b_{2i-3} + 2b_{2i-5} + b_{2i-7}) D_{n-i} \\ &= D_{n-1} - D_{n-2} + 3D_{n-3} + \sum_{i=3}^{n-1} (-1)^i b_{2i-1} D_{n-i-1} \\ &\quad + 2 \sum_{i=2}^{n-2} (-1)^{i+1} b_{2i-1} D_{n-i-2} + \sum_{i=1}^{n-3} (-1)^{i+2} b_{2i-1} D_{n-i-3} \\ &= D_{n-1} - D_{n-2} + 3D_{n-3} + \left(\sum_{i=1}^{n-1} (-1)^i b_{2i-1} D_{n-i-1} + b_1 D_{n-2} - b_3 D_{n-3} \right) \\ &\quad + \left(2 \sum_{i=1}^{n-2} (-1)^i b_{2i-1} D_{n-i-2} - 2b_1 D_{n-3} \right) - D_{n-3} \\ &= D_{n-1} - D_{n-2} + 3D_{n-3} - D_{n-1} + D_{n-2} - D_{n-3} + 2D_{n-2} - 2D_{n-3} - D_{n-3} \\ &= 2D_{n-2} - D_{n-3}. \end{aligned}$$

Using the induction hypothesis and the definition of the Fibonacci sequence, we obtain

$$\begin{aligned} D_n &= 2(1 - (-1)^{n-2} F_{n-3}) - (1 - (-1)^{n-3} F_{n-4}) \\ &= 1 - (-1)^n (2F_{n-3} + F_{n-4}) \\ &= 1 - (-1)^n F_{n-1}. \end{aligned}$$

Consequently, the formula (3.1) is true for n . Therefore, by induction, the formula works for all positive integers n . \square

4. Connection formula between the Fibonacci-Narayana numbers and tribonacci numbers

The next theorem gives relationship between Fibonacci-Narayana numbers and tribonacci numbers via the Toeplitz-Hessenberg determinants.

Theorem 4.1. For all $n \geq 1$,

$$\det(-1; b_0, b_1, \dots, b_{n-1}) = t_n. \tag{4.1}$$

Proof. We will prove formula (4.1) using induction on n . It is easily seen that $\det(-1; b_0, b_1, \dots, b_{n-1}) = (-1)^n D_n$, where

$$D_n = \det(1; -b_0, -b_1, \dots, -b_{n-1}).$$

Clearly, formula (4.1) works, when $n = 1$ and $n = 2$. Suppose it is true for all $k \leq n - 1$, where $n \geq 2$. Using recurrence (2.2), we have

$$\begin{aligned} D_n &= \sum_{i=1}^n (-1)^i b_{i-1} D_{n-i} \\ &= -b_0 D_{n-1} + b_1 D_{n-2} - b_2 D_{n-3} + \sum_{i=4}^n (-1)^i (b_{i-2} + b_{i-4}) D_{n-i} \\ &= D_{n-2} - D_{n-3} + \sum_{i=3}^{n-1} (-1)^{i+1} b_{i-1} D_{n-i-1} + \sum_{i=1}^{n-3} (-1)^{i+3} b_{i-1} D_{n-i-3} \\ &= D_{n-2} - D_{n-3} + \left(\sum_{i=1}^{n-1} (-1)^{i+1} b_{i-1} D_{n-i-1} - b_0 D_{n-2} + b_1 D_{n-3} \right) - D_{n-3} \\ &= D_{n-2} - D_{n-3} - D_{n-1} + D_{n-3} - D_{n-3} \\ &= -D_{n-1} + D_{n-2} - D_{n-3}. \end{aligned}$$

Thus,

$$\begin{aligned} \det(1; -b_0, -b_1, \dots, -b_{n-1}) &= (-1)^n \left(-\frac{t_{n-1}}{(-1)^{n-1}} + \frac{t_{n-2}}{(-1)^{n-2}} - \frac{t_{n-3}}{(-1)^{n-3}} \right) \\ &= t_{n-1} + t_{n-2} + t_{n-3} = t_n. \end{aligned}$$

Consequently, the formula (4.1) is true for n . Therefore, by induction, the formula works for all positive integers n . \square

5. Some Toeplitz-Hessenberg determinants with Fibonacci-Narayana entries

In this section, we evaluate $\det(\pm 1; a_1, a_2, \dots, a_n)$ with special Fibonacci-Narayana entries a_i .

Theorem 5.1. *Let $n \geq 1$, except when noted otherwise. Then*

$$\begin{aligned}
\det(1; b_0, b_1, \dots, b_{n-1}) &= \frac{(-1)^{n-1} + (-1)^{\lfloor \frac{n+1}{2} \rfloor}}{2}, \\
\det(1; b_1, b_2, \dots, b_n) &= \frac{(-1)^{n-1} + (-1)^{\lfloor n/3 \rfloor}}{2}, \\
\det(-1; b_1, b_2, \dots, b_n) &= 2^{n-1} \sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^i} \binom{n-1-2i}{i}, \quad (5.1) \\
\det(1; b_2, b_3, \dots, b_{n+1}) &= 0, \quad n \geq 4, \\
\det(1; b_2, b_4, \dots, b_{2n}) &= (-1)^{n-1} \sum_{i=0}^n \binom{i+1}{n-1-2i}, \\
\det(-1; b_2, b_4, \dots, b_{2n}) &= \sum_{i=0}^{2n-1} \binom{i}{4n-1-2i}, \\
\det(1; b_3, b_4, \dots, b_{n+2}) &= (-1)^{\frac{n}{2}} (1 + (-1)^n) / 2, \quad n \geq 2, \\
\det(1; b_3, b_5, \dots, b_{2n+1}) &= 0, \quad n \geq 4, \\
\det(1; b_4, b_5, \dots, b_{n+3}) &= \frac{(-1)^{\lfloor \frac{n-1}{3} \rfloor} + (-1)^{\lfloor \frac{n}{3} \rfloor}}{2}, \\
\det(1; b_4, b_6, \dots, b_{2n+2}) &= 1, \quad n \geq 3, \\
\det(1; b_5, b_6, \dots, b_{n+4}) &= \sum_{i=0}^{n+2} \binom{n+2-i}{1+2i}, \\
\det(1; b_5, b_7, \dots, b_{2n+3}) &= n+1, \quad n \geq 2, \\
\det(1; b_6, b_8, \dots, b_{2n+4}) &= (n^2 + 3n + 4) / 2,
\end{aligned}$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ is the binomial coefficient and $\lfloor \cdot \rfloor$ is the floor function.

Proof. We will prove only (5.1), the other ones can be proved in the same way. Obviously, $\det(-1; b_1, b_2, \dots, b_n) = (-1)^n D_n$, where

$$D_n = \det(1; -b_1, -b_2, \dots, -b_n),$$

When $n = 1$ and $n = 2$, the formula holds. Assuming (5.1) to hold for all $k \leq n-1$, we prove it for $n \geq 3$. Using (2.2), we have

$$\begin{aligned}
D_n &= \sum_{i=1}^n (-1)^i b_i D_{n-i} \\
&= -b_1 D_{n-1} + b_2 D_{n-2} - b_3 D_{n-3} + \sum_{i=4}^n (-1)^i (b_{i-1} + b_{i-4}) D_{n-i} \\
&= -D_{n-1} + D_{n-2} - D_{n-3} + \sum_{i=3}^{n-1} (-1)^{i+1} b_i D_{n-1-i} + \sum_{i=1}^{n-3} (-1)^{i+3} b_i D_{n-3-i}
\end{aligned}$$

$$\begin{aligned}
 &= -D_{n-1} + D_{n-2} - D_{n-3} + \sum_{i=1}^{n-1} (-1)^{i+1} b_i D_{n-1-i} - D_{n-2} + D_{n-3} - D_{n-3} \\
 &= -D_{n-1} + D_{n-2} - D_{n-3} - D_{n-1} - D_{n-2} + D_{n-3} - D_{n-3} \\
 &= -2D_{n-1} - D_{n-3} \\
 &= -(-2)^{n-1} \sum_{k=0}^{\lfloor \frac{n-2}{3} \rfloor} \frac{1}{8^k} \binom{n-2-2k}{k} - (-2)^{n-4} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor - 1} \frac{1}{8^k} \binom{n-4-2k}{k}.
 \end{aligned}$$

Thus,

$$D_n = -(-2)^{n-1} \sum_{k=0}^{\lfloor \frac{n-2}{3} \rfloor} \frac{1}{8^k} \binom{n-2-2k}{k} - (-2)^{n-1} \sum_{k=1}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-2-2k}{k-1}. \quad (5.2)$$

Let $n \neq 3m - 2$. Then $\lfloor \frac{n-2}{3} \rfloor = \lfloor \frac{n-1}{3} \rfloor$. From (5.2), using well-known formula

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}, \quad (5.3)$$

we have

$$\begin{aligned}
 D_n &= -(-2)^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-2-2k}{k} - (-2)^{n-1} \sum_{k=1}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-2-2k}{k-1}. \\
 &= -(-2)^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-1-2k}{k}.
 \end{aligned}$$

Let $n = 3m - 2$. Then $\lfloor \frac{n-1}{3} \rfloor - \lfloor \frac{n-2}{3} \rfloor = 1$. Now, using (5.3), we have

$$\begin{aligned}
 D_n &= -2^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor - 1} \frac{1}{8^k} \binom{n-2-2k}{k} - 2^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-2-2k}{k-1} \\
 &= -(-2)^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-1-2k}{k}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \det(-1; b_1, b_2, \dots, b_n) &= (-1)^{n+1} (-2)^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-1-2k}{k} \\
 &= 2^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{1}{8^k} \binom{n-1-2k}{k}.
 \end{aligned}$$

Since the formula (5.1) holds for n , it follows by induction that it is true for all positive integers n . □

The Trudi formula (2.3), taken together with Theorems 3.1 and 4.1 yields the following corollary. Similarly, one can obtain the multinomial extensions for formulas from Theorem 5.1.

Corollary 5.2. *The following formulas hold:*

$$\sum_{2s_1+3s_2+\cdots+ns_{n-1}=n} (-1)^{\sigma_{n-1}} p_n(s) b_2^{s_1} b_4^{s_2} \cdots b_{2n-2}^{s_{n-1}} = -F_n, \quad n \geq 2, \quad (5.4)$$

$$\sum_{s_1+2s_2+\cdots+ns_n=n} (-1)^{\sigma_n} p_n(s) b_1^{s_1} b_3^{s_2} \cdots b_{2n-1}^{s_n} = (-1)^n - F_{n-1}, \quad n \geq 1, \quad (5.5)$$

$$\sum_{2s_1+3s_2+\cdots+ns_{n-1}=n} p_n(s) b_1^{s_1} b_2^{s_2} \cdots b_{n-1}^{s_{n-1}} = t_n, \quad n \geq 2, \quad (5.6)$$

where $p_n(s) = \binom{s_1+\cdots+s_n}{s_1, \dots, s_n}$ is the multinomial coefficient, $\sigma_n = s_1 + \cdots + s_n$, $s_i \geq 0$, F_n and t_n are the n -th Fibonacci and tribonacci numbers, respectively.

Example 5.3. It follows from (5.4), (5.5), and (5.6), respectively, that

$$\begin{aligned} b_2^3 - 2b_2b_6 - b_4^2 + b_{10} &= F_6, \\ b_1^5 - 4b_1^3b_3 + 3b_1^2b_5 + 3b_1b_3^2 - 2b_1b_7 - 2b_3b_5 + b_9 - 1 &= F_4, \\ 3b_1^2 + 2b_1b_4 + 2b_2b_3 + b_6 &= t_7. \end{aligned}$$

6. Fibonacci-Narayana determinants

In this section, we prove two formulas expressing Fibonacci-Narayana numbers b_i with even (odd) subscripts via recurrent determinants of four-diagonal matrix of order n .

Let P_n and Q_n denote the $n \times n$ four-diagonal matrices

$$P_n = \begin{bmatrix} 1 & b_2 & & & & & & \\ -2 & 0 & b_4 & & & & & 0 \\ b_5 & -b_3 & 0 & b_6 & & & & \\ & b_7 & -b_5 & 0 & b_8 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & 0 & & b_{2n-3} & -b_{2n-5} & 0 & b_{2n-2} & \\ & & & & b_{2n-1} & -b_{2n-3} & 0 & \end{bmatrix}$$

and

$$Q_n = \begin{bmatrix} 1 & b_1 & & & & & & \\ -1 & 0 & b_3 & & & & & 0 \\ b_4 & -b_2 & 0 & b_5 & & & & \\ & b_6 & -b_4 & 0 & b_7 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & 0 & & b_{2n-4} & -b_{2n-6} & 0 & b_{2n-3} & \\ & & & & b_{2n-2} & -b_{2n-4} & 0 & \end{bmatrix}.$$

Theorem 6.1. For all $n \geq 1$,

$$b_{2n} = \frac{1}{\prod_{i=1}^{n-1} b_{2i}} \det(P_n), \tag{6.1}$$

$$b_{2n-1} = \frac{1}{\prod_{i=1}^{n-1} b_{2i-1}} \det(Q_n). \tag{6.2}$$

Proof. We prove only formula (6.2), the formula (6.1) one can be proved similarly. We use induction on n . Since $\det(P_1) = 1 = b_2$ and $\det(P_2) = 2 = b_4$, the result is true when $n = 1$ and $n = 2$. Assume it true for every positive integer $k < n$. Expanding $\det(P_n)$ by the last row, we have

$$\begin{aligned} b_{2n} &= \frac{1}{\prod_{i=1}^{n-1} b_{2i}} (b_{2n-3}b_{2n-2} \det(P_{n-2}) + b_{2n-1}b_{2n-2}b_{2n-4} \det(P_{n-3})) \\ &= \frac{1}{\prod_{i=1}^{n-1} b_{2i}} \left(b_{2n-3}b_{2n-2} \prod_{i=1}^{n-2} b_{2i} + b_{2n-1}b_{2n-2}b_{2n-4} \prod_{i=1}^{n-3} b_{2i} \right) \\ &= b_{2n-3} + b_{2n-1}. \end{aligned}$$

Therefore, the result is true for every $n \geq 1$. □

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